Computation and some new instances of Darmon points

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Number Theory Seminar, Institut de Mathématiques de Jussieu

Outline

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Computing algebraic points

- E elliptic curve with rational coefficients

$$E\colon\ y^2 + b_1xy + b_3y = x^3 + b_2x^2 + b_4x + b_6, \quad b_i \in \mathbb{Z}$$

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- Compute the rank r?
  - Related to the Birch and Swinnerton–Dyer Conjecture

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If $K =$ imaginary quadratic and $\mathrm{ord}_{s=1} L(E/K, s) = 1$, does there exist an efficient algorithm for computing a point of infinite order?

- Answer: yes, the Heegner points method
  - Fundamental ingredient in the Gross–Zagier–Kolyvagin theorem

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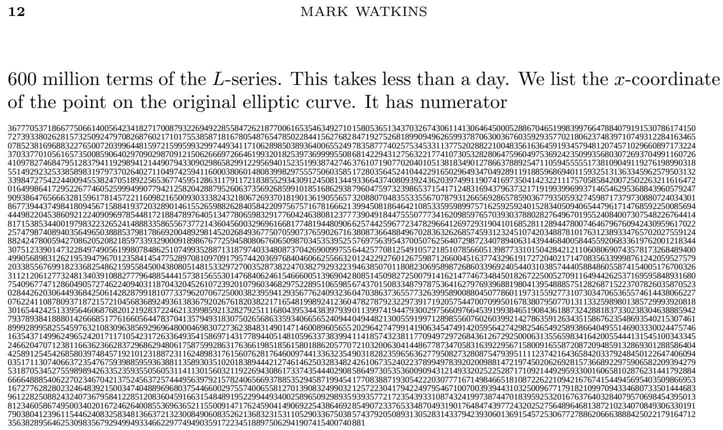

600 million terms of the L-series. This takes less than a day. We list the x-coordinate of the point on the original elliptic curve. It has numerator

36777053718667750661400564234182717008793226949228558472621877006165354634927101580536513437032674306114130646450005288670465199839976647884079191530786174150
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3563828956462530983567929499493346622977494903591722345188975062941907415400740881

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- Idea: look for a construction analogous to that of Heegner points
- Darmon Points (a.k.a. Stark–Heegner Points)

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  - Modular uniformization $X_0(N) \longrightarrow E$
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  - Modular uniformization $X_0(N) \longrightarrow E$
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Outline

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Conjecture (rationality)

$J_\tau \in E(K^{\mathrm{ab}})$ and $\mathrm{Tr}(J_\tau)$ of infinite order if $\operatorname{ord}_{s=1} L(E/K,s) = 1$

Numerical evidence for the conjecture

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  - Computing generators
  - Given $\gamma \in [\Gamma, \Gamma]$ write it explicitly as a product of commutators

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- How to compute effectively with rigid analytic differentials?

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- The isomorphism is explicit (it is essentially Shapiro's Lemma). So we can recover μ_E from φ_E.

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Outline

$p = 13$, $D = 6$, $\mathrm{prec} = 13^{60}$

$$E_{78}: \quad y^2 + xy = x^3 + x^2 - 19x + 685$$

d_K	P
5	$1 \cdot 48 \cdot \left(-2, 12\sqrt{5} + 1\right)$
149	$1 \cdot 48 \cdot \left(1558, -5040\sqrt{149} - 779\right)$
197	$1 \cdot 48 \cdot \left(\frac{310}{49}, \frac{720}{343}\sqrt{197} - \frac{155}{49}\right)$
293	$1 \cdot 48 \cdot \left(40, -15\sqrt{293} - 20\right)$
317	$1 \cdot 48 \cdot \left(382, -420\sqrt{317} - 191\right)$
437	$1 \cdot 48 \cdot \left(\frac{986}{23}, \frac{7200}{529}\sqrt{437} - \frac{493}{23}\right)$
461	$1 \cdot 48 \cdot \left(232, -165\sqrt{461} - 116\right)$
509	$1 \cdot 48 \cdot \left(-\frac{2}{289}, -\frac{5700}{4913}\sqrt{509} + \frac{1}{289}\right)$
557	$1 \cdot 48 \cdot \left(\frac{75622}{121}, \frac{882000}{1331}\sqrt{557} - \frac{37811}{121}\right)$

$p = 11$, $D = 10$, $\mathrm{prec} = 11^{60}$

$$E_{110}: \quad y^2 + xy + y = x^3 + x^2 + 10x - 45.$$

d_K	P
13	$2 \cdot 30 \cdot \left(\frac{1103}{81} - \frac{250}{81}\sqrt{13}, -\frac{52403}{729} + \frac{13750}{729}\sqrt{13}\right)$
173	$2 \cdot 30 \cdot \left(\frac{1532132}{9025}, -\frac{1541157}{18050} - \frac{289481483}{1714750}\sqrt{173}\right)$
237	$2 \cdot 30 \cdot \left(\frac{19096654883784 2073867}{40166486596584 12649} - \frac{10722443619184119320}{4016648659658412649}\sqrt{237}, -\frac{35055901930114371428532338 57149}{80499979138298454114237 56107} + \frac{235448460130564520991320372200}{8049997913829845411423756107}\sqrt{237}\right)$
277	$2 \cdot 30 \left(\frac{46317716623881}{12553387541776}, -\frac{58871104165657}{25106775083552} - \frac{2091276933523905 5243}{4447760611796554 2976}\sqrt{277}\right)$
293	$2 \cdot 30 \cdot \left(\frac{7088486530742}{2971834657801}, -\frac{10060321188543}{5943669315602} - \frac{591566427769149607}{10246297476835603402}\sqrt{293}\right)$
373	$2 \cdot 30 \cdot \left(\frac{298780258398}{62087183929}, -\frac{360867442327}{124174367858} - \frac{19368919551426449}{30940899762281434}\sqrt{373}\right)$

$p = 19$, $D = 6$, $\text{prec} = 19^{60}$

$$E_{114}: \quad y^2 + xy = x^3 - 8x$$

d_K	P
29	$1 \cdot 72 \cdot \left(-\frac{6}{25}\sqrt{29} - \frac{38}{25}, -\frac{18}{125}\sqrt{29} + \frac{86}{125}\right)$
53	$1 \cdot 72 \cdot \left(-\frac{1}{9}, \frac{7}{54}\sqrt{53} + \frac{1}{18}\right)$
173	$1 \cdot 72 \cdot \left(-\frac{3481}{13689}, \frac{347333}{3203226}\sqrt{173} + \frac{3481}{27378}\right)$
269	$1 \cdot 72 \cdot \left(\frac{1647149414400}{23887470525361}\sqrt{269} - \frac{43248475603556}{23887470525361}, \frac{2359447648611379200}{116749558330761905641}\sqrt{269} + \frac{2681774974170243075 64}{116749558330761905641}\right)$
293	$1 \cdot 72 \cdot \left(\frac{21289143620808}{4902225525409}, \frac{4567039561444642548}{10854002829131490673}\sqrt{293} - \frac{10644571810404}{4902225525409}\right)$
317	$1 \cdot 72 \cdot \left(-\frac{25}{9}, -\frac{5}{54}\sqrt{317} + \frac{25}{18}\right)$
341	$1 \cdot 72 \cdot \left(\frac{3449809443179}{499880896975}, \frac{3600393040902501011}{3935597293546963250}\sqrt{341} - \frac{3449809443179}{999761793950}\right)$
413	$1 \cdot 72 \cdot \left(\frac{59}{7}, \frac{113}{98}\sqrt{413} - \frac{59}{14}\right)$

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- Show an example of mixed signature archimedean Darmon point

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- In practice, one can compute ω_E!

- ω_E has a “Fourier-Bessel expansion”:

$$\omega_E(z,x,y) = \sum_{\substack{\alpha\in\mathcal{O}_F\\ \alpha_0>0}} \frac{a_{(\alpha)}}{N_{F/\mathbb{Q}}(\alpha)}\frac{\alpha_0}{\delta_0}\exp\left(-2\pi i\left(\frac{\alpha_0\bar{z}}{\delta_0}+\frac{\alpha_1 x}{\delta_1}+\frac{\alpha_2\bar{x}}{\delta_2}\right)\right)\mathbb{K}\left(\frac{\alpha_1 y}{\delta_1}\right)\cdot\begin{pmatrix}\frac{-dx}{y}\wedge d\bar{z}\\ \frac{dy}{y}\wedge d\bar{z}\\ \frac{d\bar{x}}{y}\wedge d\bar{z}\end{pmatrix}$$

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- We can compute the $a_{(\alpha)}$ by counting points on $E(\mathcal{O}_F/\mathfrak{p})$

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- The image of $J_\tau \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$ coincides (up to 32 digits of accuracy) with $10P$, where

$$P = \left(r - 1 : w - r^2 + 2r : 1\right) \in E(K)$$

is a point of infinite order!

Computation and some new instances of Darmon points

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