Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

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  - $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) = \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms

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Theorem (Josep González, 2011)

$\mathcal{A}^{\mathrm{split}}_{2,1}$ contains 28 algebras coming from modular $A/\mathbb{Q}$ (+ gives explicit list)

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3. Here $A_{\overline{\mathbb{Q}}} \sim E^2$ with E with CM by M: here is where the work is

Squares of CM elliptic curves

Central question

If $A/\mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^2$ and E has CM by M, what are the possible M's?

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Squares of CM elliptic curves

Central question

If $A/\mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^2$ and E has CM by M, what are the possible M's?

Theorem (Fité–G., 2015)

Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_2$, or $\mathrm{C}_2 \times \mathrm{C}_2$

- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves
  - Let $K/\mathbb{Q}$ be the minimal such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathrm{End}(A_K)$
  - Known that $\mathrm{Gal}(K/M) \simeq \mathrm{C}_1, \mathrm{C}_r, \mathrm{D}_r$ with $r \in \{2, 3, 4, 6\}$
  - E is an M-curve: $\forall \sigma \in \mathrm{Gal}(K/M) \rightsquigarrow \mu_\sigma : {}^\sigma E \to E$ compatible with M
  - $c_E(\sigma, \tau) = \mu_\sigma \circ {}^\sigma\mu_\tau \circ \mu_{\sigma\tau}^{-1} \in M^\times$ and $c_E \in H^2(\mathrm{Gal}(K/M), M^\times)$
  - One shows that c_E is 2-torsion
  - Every 2 torsion class is trivialized when restricted to $\mathrm{Gal}(K/N)$ with $\mathrm{Gal}(N/M)$ of exponent 2 (so $\mathrm{Gal}(N/M) \simeq \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_2 \times \mathrm{C}_2$)
  - Weil Descent Theorem: E can be defined over N up to isogeny
  - CM theory $H \subset N$ so $\mathrm{Gal}(H/M)$ is a quotient of $\mathrm{Gal}(N/M)$
- Real question: of these possible M's, which ones do really occur?
  - Give a construction of A's for some M's and rule out the other M's

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  - Can assume $\mathrm{Gal}(K/M)$ has an element β of order $r = 4$ or $r = 6$.

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  - $\rho_\sigma \rho_\tau = c_{E^*}(\sigma, \tau)\rho_{\sigma\tau}$ projective representation (c_{E^*}-representation)

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- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too

Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

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