Modular forms over fields of mixed signature and algebraic points in elliptic curves

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  - (Automorphic) There is a modular form $f_E(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$ such that $a_p = p + 1 - \#E(\mathbb{Z}/p\mathbb{Z})$ for all p.

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  - Idea: to find an analog of the analytic formula

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1. Heegner points
2. Darmon points (archimedean)
3. A construction over fields of mixed signature
4. Numerical evidence for the conjecture

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- K/F a quadratic ATR extension: $K \otimes_{v_0} \mathbb{R} = \mathbb{C}$, $K \otimes_{v_1} \mathbb{R} = \mathbb{R} \oplus \mathbb{R}$

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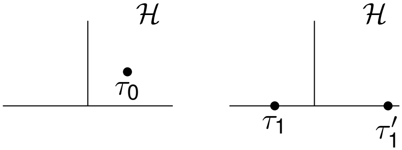

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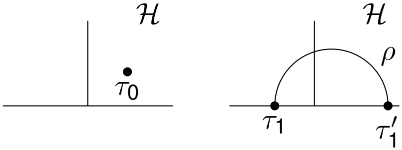

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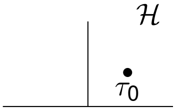

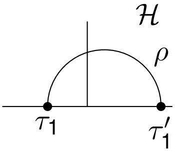

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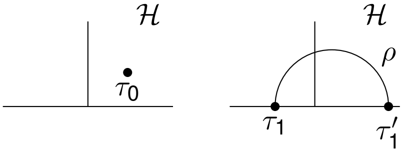

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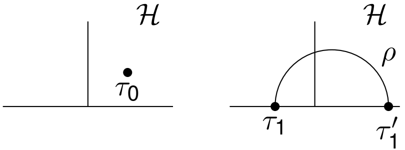

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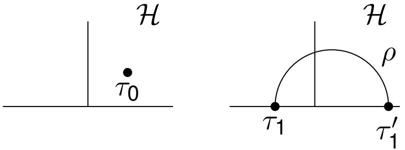

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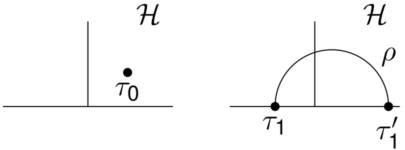

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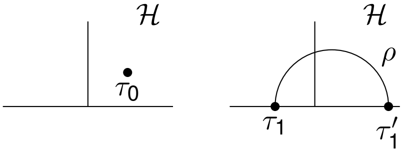

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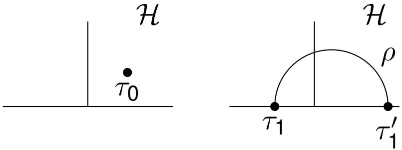

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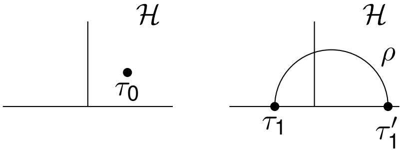

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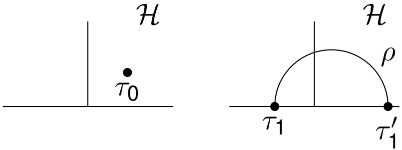

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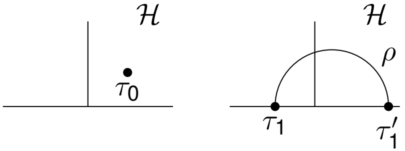

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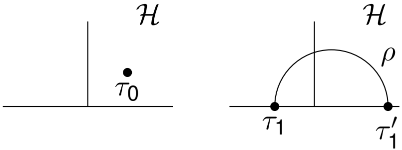

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- Our aim: propose a similar construction if F is not totally real

Outline

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- As before, ω_E is determined by its Fourier–Bessel expansion.

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- We can compute the $a_{(\alpha)}$ by counting points on $E(\mathcal{O}_F/\mathfrak{p})$

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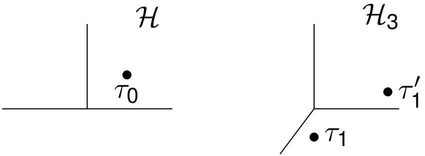

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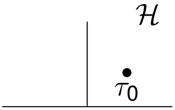

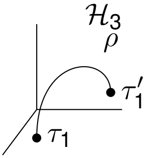

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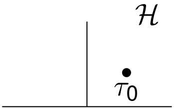

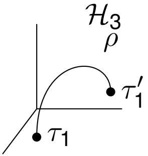

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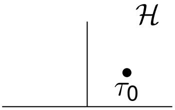

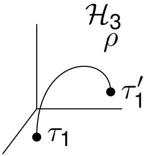

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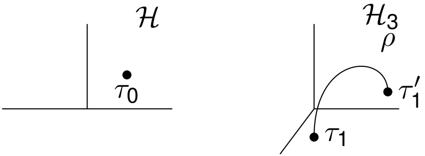

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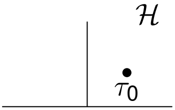

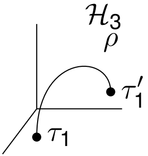

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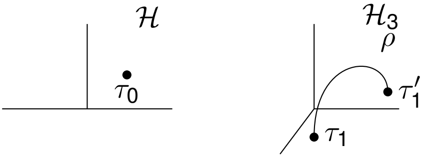

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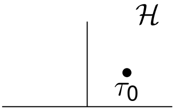

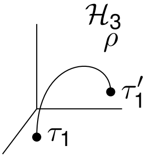

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Conjecture

$P_\Psi \in E_0(H)$, with H the Hilbert class field of K.

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Construction of the points

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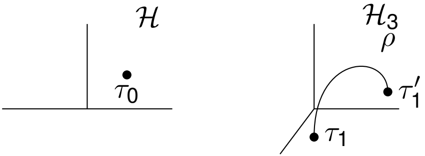

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- We found some numerical evidence for the conjecture.

Outline

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1 = 0$

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- The image of $P_\Psi \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$ coincides (up to 32 digits of accuracy) with $10P$, where

$$P = \left(r - 1 : w - r^2 + 2r : 1\right) \in E(K)$$

Modular forms over fields of mixed signature and algebraic points in elliptic curves

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