A quaternionic construction of p-adic singular moduli

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RSME-SEMA-SCM-PTM Mathematical Meeting

Outline

1. The classical theory: singular moduli
2. A recent theory: p-adic singular moduli
3. Our proposal: a quaternionic version of the p-adic theory

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- Explicit CFT: can we write down a collection of generators of K^{ab}?
  - This is known in some cases: $K = \mathbb{Q}$, imaginary quadratic.

Theorem (Kronecker–Webber)

- Every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field.
- So $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\{e^{2\pi i m}\}_{m\in\mathbb{Q}})$.

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The j-function and explicit CFT of IQF

- It is a function $j\colon \mathcal{H} = \{\tau \in \mathbb{C}\colon \mathrm{Im}(\tau) > 0\} \longrightarrow \mathbb{C}$.
  - $\tau \in \mathcal{H} \rightsquigarrow E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ elliptic curve and $j(\tau) := j(E_\tau)$.
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- If K = imaginary quadratic and $\tau \in K \setminus \mathbb{Q} \rightsquigarrow j(\tau)$ is algebraic.
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- $K(\{j(\tau)\}_{\tau \in K\setminus\mathbb{Q}}) \subset K^{\mathrm{ab}}$.
- To get K^{ab} need to adjoint coordinates of torsion points of E_τ.

Question (Hilbert 12th problem)

Is there something similar for more general K?

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Darmon–Vonk's p-adic singular moduli

- K a real quadratic field and $\tau \in K \setminus \mathbb{Q}$ (call τ an RM point)
- Problem: $\tau \notin \mathcal{H}$ so $j(\tau)$ doesn't make sense
  - replace $\mathcal{H} = (\mathbb{C} \setminus \mathbb{R})^+$ by $\mathcal{H}_p = \mathbb{C}_p \setminus \mathbb{Q}_p$ the p-adic upper half plane
  - If p does not split in K, then $\tau \in \mathcal{H}_p$.
  - Consider the action of $\Gamma := \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ on $\mathcal{H}_p$ and on

$$\mathcal{M}^\times = \{\text{non-zero rigid meromorphic functions on } \mathcal{H}_p\}.$$

  - The analog of j now would be a function on $H^0(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]), \mathcal{M}^\times)$.
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Rigid meromorphic cycles

Definition (Rigid meromorphic cocycles)

Classes in $H^1(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]), \mathcal{M}^\times)$ whose restriction to Γ_∞ is constant.

- They are maps $J \colon \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) \longrightarrow \mathcal{M}^\times$ (+ cocycle condition)
- Can be evaluated at RM points $\tau \in K \setminus \mathbb{Q}$:
  - $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])}(\tau) = \langle \pm\gamma_\tau \rangle \rightsquigarrow J(\gamma_\tau) \in \mathcal{M}^\times$
  - $J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p$

Conjecture (Darmon–Vonk)

- If τ is RM and J a rigid meromorphic cocycle, $J[\tau]$ is algebraic.
- Field can be made more precise (compositum of ring class fields)

Can we compute RMC?

- Darmon–Vonk: given any RM point $\theta \rightsquigarrow$ explicit RMC J_θ
- So for an RM point τ we expect $J_\theta[\tau] \in K^{ab}$

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- Darmmon–Vonk give a method for the effective computation of J_θ, and many numerical evidence that $J_\theta[\tau]$ is algebraic for RM θ, τ
- Computing $J_\theta(\gamma) = \prod_{w \in \Gamma \cdot \theta} (z - w)^{\delta_\gamma(w)}$ via the definition takes exponential time (in the p-adic precision)
- Overconvergent method: iteration of a U_p-operator.
- $J_{\frac{1+\sqrt{5}}{2}}[2\sqrt{6}] = (3 + 8\sqrt{2} + 12\sqrt{-1} + 2\sqrt{-2})/17 \pmod{7^{100}}$ in $\mathbb{C}_7$

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To give a similar construction that allows for more general extensions $K/\mathbb{Q}$ and more general p-arithmetic groups.

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Outline

1. The classical theory: singular moduli
2. A recent theory: p-adic singular moduli
3. Our proposal: a quaternionic version of the p-adic theory

Quaternionic p-adic singular moduli: the setting

- F a totally real number field of narrow class number 1.
- B/F an almost totally definite quaternion algebra: $B \otimes_{v_\infty} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ for a single infinite place v_∞ of F.
- $R \subset B$ a maximal order and $\Gamma_0 = R_1^\times$ (plays the role of $\mathrm{SL}_2(\mathbb{Z})$).
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- B acts on $\mathcal{H}$ and on $\mathcal{H}_p$ via the splittings at v_∞ and $\iota_\mathfrak{p}$
- Let K/F be a quadratic extension admitting $K \hookrightarrow B$ such that v_∞ splits in K, and all other infinite places ramify (ATC extension)

Structure of the construction

- For an ATC point $\theta \in K_1 \setminus F \rightsquigarrow J_\theta$ a cocycle
- For an ATC point $\tau \in K_2 \setminus F \rightsquigarrow J_\theta[\tau] \in \mathbb{C}_p$ should be "algebraic"

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- Let K/F be a quadratic extension admitting $K \hookrightarrow B$ such that v_∞ splits in K, and all other infinite places ramify (ATC extension)

Structure of the construction

- For an ATC point $\theta \in K_1 \setminus F \rightsquigarrow J_\theta$ a cocycle
- For an ATC point $\tau \in K_2 \setminus F \rightsquigarrow J_\theta[\tau] \in \mathbb{C}_p$ should be "algebraic"

- Some differences with the construction of Darmon–Vonk:
  - Cocycles obtained by iterating $U_\mathfrak{p}$ (only use arithmetic groups)
  - $\mathcal{M}^\times$ is replaced by $\Lambda = \{$ functions on the unit ball$\}$
  - Additive: obtained quantities should be logs of algebraic numbers

Quaternionic p-adic singular moduli: the setting

- F a totally real number field of narrow class number 1.
- B/F an almost totally definite quaternion algebra: $B \otimes_{v_\infty} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ for a single infinite place v_∞ of F.
- $R \subset B$ a maximal order and $\Gamma_0 = R_1^\times$ (plays the role of $\mathrm{SL}_2(\mathbb{Z})$).
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Quaternionic p-adic singular moduli: construction

- Let $\mathfrak{p} = (\varpi)$ and $\Lambda = \overline{\mathbb{Z}}_\mathfrak{p}[[\varpi z]]$
- Integration pairing

$$\begin{array}{ccc} \Lambda \times \mathrm{Div}^0\mathcal{H}_p & \longrightarrow & \mathbb{C}_p \\ (f, Q-P) & \longmapsto & \int_P^Q f(x)dx, \end{array}$$

- $\langle\cdot,\cdot\rangle : H^1(\Gamma_0(\mathfrak{p}),\Lambda) \times H_1(\Gamma_0(\mathfrak{p}),\mathrm{Div}^0\mathcal{H}_p) \longrightarrow \mathbb{C}_p.$

Main construction

- $\theta \in K_1 \setminus F \rightsquigarrow J_\theta \in H^1(\Gamma_0(\mathfrak{p}),\Lambda)$
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- Define $J_\theta[\tau] := \langle J_\theta, c_\tau\rangle \in \mathbb{C}_p$

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$J_\theta[\tau]$ is the logarithm of an algebraic number lying in a compositum of ring class fields of K_1 and K_2.

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Quaternionic singular moduli: examples

- We have computed $J_\theta[\tau]$ in many particular cases to high precision and in many cases we have been able to recognize these quantities as close to logarithms of algebraic numbers in the expected ring class fields.

Example: $F = \mathbb{Q}(\omega)$, $\omega = \frac{1+\sqrt{5}}{2}$, $p = 11$, $B = (-w, -2)_F$, $D_B = (2)$

- $K_1 = F(\theta)$, $\theta = \sqrt{1 - 2\omega}$ and $K_2 = F(\tau)$, $\tau = \sqrt{9 - 14\omega}$
- $J_\theta[\tau] = 265083386108501156984620884744997022962466460875569079195483 8 + O(11^{59})$
- Satisfies: $25420x^4 - 227820x^3 + 2200011x^2 - 27566220x + 372174220$ which generates an unramified extension of $K_1 \cdot K_2$.

Darmon–Gehrmann–Lipnowski '23

Rigid meromorphic cocycles in a very general setting (orthogonal groups) + conjecture of rationality + numerical evidence

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A quaternionic construction of p-adic singular moduli

Xevi Guitart (UB) Marc Masdeu (UAB) Xavier Xarles (UAB)

RSME-SEMA-SCM-PTM Mathematical Meeting

Quaternionic singular moduli: examples

J_ψ^+

	8	12	53	77	92	93
8		-	-	3, 5	2, 3	5
12	-		5	??	2	-
53	-	5		?	3, 23, 31	2, 5, 41
77	3, 5	??	?		??	??
92	2, 3	2	3, 23, 31	??		??
93	5	-	2, 5, 41	??	??	

J_ψ^-

	8	12	53	77	92	93
8		1	-	3, 5	2, 3	2, 5
12	1		2, 5	??	1	1
53	-	2, 5		3, 5	2, 3, 23, 31	2, 5, 41
77	3, 5	??	3, 5		??	??
92	2, 3	1	2, 3, 23, 31	??		??
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Table: Tables for $D = 6$, $p = 5$, plus-minus classes.

Moving p and B

- $K_1 = \mathbb{Q}(\sqrt{53})$ and $K_2 = \mathbb{Q}(\sqrt{23})$ can be embedded in B_6 and B_{10}.
  - Compute using B_{10} and $p = 3$:

$$J_{\theta,3}^{\text{even}}(\tau) = 671432593119615754\ldots + 854036156664899807\ldots\frac{1+\sqrt{53}}{2} + O(3^{195}).$$

  - Compute using B_6 and $p = 5$:

$$J_{\theta,5}^{\text{even}}(\tau) = 223515896705660593\ldots + 188812945396004677\ldots\frac{1+\sqrt{53}}{2} + O(5^{197}).$$

- M number field generated by a root of $x^8 - 4x^7 + 84x^6 - 238x^5 + 1869x^4 - 3346x^3 + 7260x^2 - 5626x + 3497$
  - $\iota_3 : M \hookrightarrow \mathbb{C}_3$ and $\iota_5 : M \hookrightarrow \mathbb{C}_5$
  - There exists $\alpha \in M$ and units u_1, u_2 in $\mathbb{Q}(\sqrt{23}, \sqrt{53})$ such that

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Question

Are the quantities $J_{\theta,p}^{\text{even}}$ the local manifestation of a global object?

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A quaternionic construction of p-adic singular moduli

Xevi Guitart (UB) Marc Masdeu (UAB) Xavier Xarles (UAB)

RSME-SEMA-SCM-PTM Mathematical Meeting