PARAMETRIZATION OF ABELIAN K-SURFACES WITH QUATERNIONIC MULTIPLICATION

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Abstract. We prove that the abelian K-surfaces whose endomorphism algebra is an indefinite rational quaternion algebra are parametrized, up to isogeny, by the K-rational points of the quotient of certain Shimura curves by the group of their Atkin-Lehner involutions.

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1. Introduction

Let K be a number field. An Abelian variety $A/\overline{K}$ is called an *Abelian K-variety* if for each $\sigma \in \mathrm{Gal}(\overline{K}/K)$ there exists an isogeny $\mu_\sigma : {}^\sigma A \to A$ such that $\psi \circ \mu_\sigma = \mu_\sigma \circ {}^\sigma\psi$ for all $\psi \in \mathrm{End}(A)$. In the case $K = \mathbf{Q}$, an interesting type of $\mathbf{Q}$-varieties are the *building blocks* (namely, those whose endomorphism algebra is a central division algebra over a totally real number field F, with Schur index $t = 1$ or $t = 2$ and $t[F : \mathbf{Q}] = \dim A$), since they are known to be the absolutely simple factors up to isogeny of the non-CM Abelian varieties of GL_2-type (see [4, Chapter 4]). After the validity of Serre's conjecture [6, 3.2.4$_?$] on representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, a theorem of Ribet ([5, Theorem 4.4]) implies that they are indeed the non-CM absolutely simple factors up to isogeny of the modular Jacobians $J_1(N)$.

Elkies proved in [3] that non-CM Abelian K-varieties of dimension one (also called elliptic K-curves) are parametrized up to isogeny by the non-cusp K-rational points without CM of the curves $X_0(N)/W(N)$ for square-free positive integers N, where $W(N)$ is the group of Atkin-Lehner involutions of $X_0(N)$. In this note we adapt Elkies's original argument to prove an analogous result for Abelian K-surfaces whose endomorphism algebra is an indefinite rational quaternion algebra; in this case, they are parametrized by the K-rational points of the quotient of a Shimura curve by its group of Atkin-Lehner involutions.

2. Shimura curves, Atkin-Lehner involutions and isogenies

The aim of this section is to recall the basic definitions and results concerning Shimura curves and their interpretation as moduli spaces of certain Abelian surfaces, and also to study the isogeny class of such Abelian surfaces in this context. The presentation of the background material is based mostly on the first chapter of [1].

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2.1. **QM-Abelian surfaces and Shimura curves.** Let B be an indefinite quaternion algebra over $\mathbf{Q}$ of discriminant D, and let $\mathcal{O}$ be an Eichler order in B of level N, which we suppose square-free and prime to D (we allow also the case $N=1$, in which $\mathcal{O}$ is actually a maximal order). An *Abelian surface with QM by* $\mathcal{O}$ is a pair $(A,\imath)$, where $A/\mathbf{C}$ is an Abelian surface and $\imath$ is an embedding $\mathcal{O}\hookrightarrow \mathrm{End}(A)$, satisfying that $H_1(A,\mathbf{Z})\simeq\mathcal{O}$ as left $\mathcal{O}$-modules (here the structure of $\mathcal{O}$-module in $H_1(A,\mathbf{Z})$ is given by $\imath$). If the order $\mathcal{O}$ is clear by the context, we will call them just *QM-Abelian surfaces*.

We will denote by $X(D,N)$ the Shimura curve defined over $\mathbf{Q}$ associated with the moduli problem of classifying isomorphism classes of Abelian surfaces $(A,\imath)$ with QM by $\mathcal{O}$. We remark that $X(D,N)$ can also be defined as the moduli space for quadruples $(A,\imath,P,Q_N)$, where $(A,\imath)$ is an Abelian surface with QM by a maximal order $\mathcal{O}_0$, P is a principal polarization on A satisfying certain compatibility conditions with $\imath$, and Q_N is a level N-structure (i.e. a subgroup of A isomorphic to $\mathbf{Z}/N\mathbf{Z}\times\mathbf{Z}/N\mathbf{Z}$ and cyclic as $\mathcal{O}_0$-module). Actually, both moduli problems coincide. Indeed, a result of Milne asserts that in this case there exists a unique compatible principal polarization, so we can remove it from the moduli problem. Moreover, considering the level N-structure is equivalent to considering embeddings of the Eichler order $\mathcal{O}\hookrightarrow\mathrm{End}(A)$, and in this way we obtain the moduli interpretation for $X(D,N)$ we started with, which is the one we will use. The reader can consult [2, §0.3.2] for more details on the several moduli interpretations for $X(D,N)$.

Let $\hat{\mathcal{O}}=\mathcal{O}\otimes\hat{\mathbf{Z}}$, $\hat{B}=B\otimes\hat{\mathbf{Z}}$ and $\mathbf{P}=\mathbf{C}\smallsetminus\mathbf{R}$. By fixing an isomorphism $B\otimes\mathbf{R}\simeq\mathrm{M}_2(\mathbf{R})$, we can make $B^\times$ act on $\mathbf{P}$ by fractional linear transformations, and there is a canonical identification $X(D,N)(\mathbf{C})\simeq\mathbf{P}/\mathcal{O}^\times$. It associates to $\tau\in\mathbf{P}$ the pair $(\mathbf{C}^2/\Lambda_\tau,\imath_\tau)$, where Λ_τ is the lattice $\mathcal{O}\binom{1}{\tau}$ and $\imath_\tau$ is the natural inclusion given by the action of $\mathcal{O}$ on $\mathbf{C}^2$. Starting from an Abelian surface $(A,\imath)$ with QM by $\mathcal{O}$, we can write $A\simeq\mathbf{C}^2/\Lambda$ for some lattice Λ. Since $\Lambda\simeq\mathcal{O}$, and after scaling the lattice if necessary, we have that $\Lambda\simeq\mathcal{O}\binom{1}{\tau}$ for some $\tau\in\mathbf{P}$, and this gives the reverse map.

Since B is indefinite we have that $\#(\hat{\mathcal{O}}^\times\setminus\hat{B}^\times/B^\times)=1$, and this implies that

$$\mathbf{P}/\mathcal{O}^\times\simeq(\hat{\mathcal{O}}^\times\setminus\hat{B}^\times\times\mathbf{P})/B^\times\simeq(\hat{\mathcal{O}}^\times\setminus\hat{B}^\times/\mathbf{Q}^\times\times\mathbf{P})/B^\times.$$

Note that the double coset $\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times$ represents the set of left fractional ideals in B, modulo the relation given by the multiplication of the ideals by rational numbers. Each fractional ideal J can be multiplied by a rational number to obtain an integral ideal I which is contained in no proper ideal of the form $k\mathcal{O}$, with $k\in\mathbf{Z}$. In this way, we can also identify $\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times$ with the set of left integral ideals that are not contained in any proper ideal of the form $k\mathcal{O}$ with $k\in\mathbf{Z}$. Therefore, any point in $X(D,N)(\mathbf{C})$ can be represented by a pair of the form (I,τ), where I is a left ideal of $\mathcal{O}$ and τ belongs to $\mathbf{C}\smallsetminus\mathbf{R}$; it is easy to see that the QM-Abelian surface corresponding to this point in the moduli interpretation is $\left(\mathbf{C}^2/I\binom{1}{\tau},\imath_\tau\right)$, were $\imath_\tau$ is the natural inclusion given by the action of $\mathcal{O}$ on $\mathbf{C}^2$ (note that this gives a well defined action on $\mathbf{C}^2/I\binom{1}{\tau}$ because I is a left ideal).

Finally, since the class number of $\mathbf{Q}$ is 1 we have that $\mathbf{Q}^\times\hat{\mathbf{Z}}^\times=\hat{\mathbf{Q}}^\times$, and therefore we also have the identification $X(D,N)\simeq(\hat{\mathcal{O}}^\times\setminus\hat{B}^\times/\hat{\mathbf{Q}}^\times\times\mathbf{P})/B^\times$.

2.2. **Trees and Atkin-Lehner involutions.** We have a decomposition

$$\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\hat{\mathbf{Q}}^\times \simeq \prod_\ell{}' \mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times,$$

where $\mathcal{O}_\ell = \mathcal{O}\otimes\mathbf{Z}_\ell$, $B_\ell = B\otimes\mathbf{Z}_\ell$ and $\prod'_\ell$ denotes the restricted product over all primes. When $\ell\nmid DN$ then $\mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times \simeq \mathrm{PGL}_2(\mathbf{Z}_\ell)\backslash\mathrm{PGL}_2(\mathbf{Q}_\ell)$, which is identified with the set of vertices of the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbf{Q}_\ell)$, a regular tree of degree $\ell+1$. If $\ell\mid N$ then, since we are assuming N to be square-free, we have $\mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times \simeq \Gamma_0(\ell)\backslash\mathrm{PGL}_2(\mathbf{Q}_\ell)$, which is identified with the set of oriented edges of the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbf{Q}_\ell)$. If $\ell\mid D$ then $\mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$ has only two elements, and we identify them with an oriented edge (that is, each element corresponds to one orientation of the edge). Hence, for $\ell\mid ND$ there is a natural involution on $\mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$; namely, the one that reverses the orientation of the edges. This involution extends to an *Atkin-Lehner involution* W_ℓ on $X(D,N)$, and we denote by $W(D,N)=\langle W_\ell : \ell\mid ND\rangle$. As usual, if $n\mid ND$ then W_n stands for the composition of all the W_ℓ with $\ell\mid n$.

A maximal order $\mathcal{O}_0$ such that $\mathcal{O}\subseteq\mathcal{O}_0$ gives rise to a natural morphism $\phi : X(D,N)\to X(D,1)$, which at the level of complex points is the natural map $(\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\hat{\mathbf{Q}}^\times\times\mathbf{P})/B^\times\to(\hat{\mathcal{O}}_0^\times\backslash\hat{B}^\times/\hat{\mathbf{Q}}^\times\times\mathbf{P})/B^\times$. We can use ϕ to define another morphism $\varphi : X(D,N)\to X(D,1)\times X(D,1)$ by means of $\varphi(P)=(\phi(P),\phi(W_N(P)))$. If $(b_\ell)_\ell$ belongs to $\prod'_\ell \mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$ and τ is a non-real complex number, then $((b_\ell)_\ell,\tau)$ represents a point in $X(D,N)$. It is sent by ϕ to the point represented by $((b'_\ell)_\ell,\tau)$, where $b'_\ell=b_\ell$ for $\ell\nmid N$, and b'_ℓ is the origin of the edge b_ℓ for $\ell\mid N$. This interpretation of ϕ makes it clear the fact that φ is injective.

2.3. **The isogeny class of a QM-Abelian surface.** An *isogeny* between two QM-Abelian surfaces $(A,\imath)$ and $(A',\imath')$ is an isogeny $\mu : A\to A'$ that respects the action of $\mathcal{O}$, i.e. such that $\imath'(\psi)\circ\mu=\mu\circ\imath(\psi)$ for all ψ in $\mathcal{O}$. We will denote by $[(A,\imath)]$, or for ease of notation just by $[A,\imath]$ in some cases, the isogeny class of $(A,\imath)$; that is, the set of all QM-Abelian surfaces isogenous to $(A,\imath)$ up to isomorphism. In the following lemma we characterize $[A,\imath]$ as a subset of $X(D,N)(\mathbf{C})$:

Lemma 1. *Let $I\subseteq\mathcal{O}$ be a left ideal, τ a non-real complex number and let $(A,\imath)$ be the point in $X(D,N)(\mathbf{C})$ represented by (I,τ). Then $[A,\imath]=(\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\})/B^\times\subseteq X(D,N)(\mathbf{C})$. Moreover, if $(A,\imath)$ does not have CM then we can identify $[A,\imath]$ with $\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\}$.*

Proof. First of all, we claim that there is a one-to-one correspondence between the isogenies $(A',\imath')\to(A,\imath)$ of degree n^2 and the left ideals of $\mathcal{O}$ of norm n. Indeed, if we write $A\simeq\mathbf{C}^2/H_1(A,\mathbf{Z})$ and $A'\simeq\mathbf{C}^2/H_1(A',\mathbf{Z})$, giving an isogeny $A'\to A$ is equivalent to giving an inclusion $H_1(A',\mathbf{Z})\subseteq H_1(A,\mathbf{Z})\simeq\mathcal{O}$, and the condition on the isogeny to be compatible with the action of $\mathcal{O}$ translates into the condition on $H_1(A',\mathbf{Z})$ to be a left ideal of $\mathcal{O}$. In addition, if the degree of the isogeny is n^2, then $\#\mathcal{O}/H_1(A',\mathbf{Z})=n^2$, and therefore the norm of the ideal $H_1(A',\mathbf{Z})$ is equal to n. This proves the claim. Now, we observe that ideals of the form $k\mathcal{O}$ for some $k\in\mathbf{Z}$ give rise to isogenies $(A',\imath')\to(A,\imath)$ with $(A',\imath')\simeq(A,\imath)$, because they correspond to the isogenies 'multiplication by k' in $(A,\imath)$. Since $\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times$ is the set of all left ideals not contained in any proper ideal of the form $k\mathcal{O}$ with $k\in\mathbf{Z}$, we see that the orbit $(\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\})/B^\times\subseteq X(D,N)(\mathbf{C})$ contains a representative for each $(A',\imath')$ isogenous to $(A,\imath)$. If $(A,\imath)$ does not have CM this orbit can be identified with $\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\}$, because then any element $b\in B^\times$ such that $b\tau=\tau$ necessarily belongs to $\mathbf{Q}^\times$. □

Corollary 2. *If $(A,\imath)$ does not have CM, then $\phi([A,\imath])\subseteq[\phi(A,\imath)]$ and $W_n([A,\imath])=[A,\imath]$ for all $n\mid ND$.*

Proof. By the lemma and by the description of ϕ at the level of complex points we have that $\phi([A,\imath])=\phi(\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\})\subseteq\hat{\mathcal{O}}_0^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\}=[\phi(A,\imath)]$. By the definition of W_n we see that $W_n([A,\imath])=W_n(\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\})=\hat{\mathcal{O}}^\times\backslash\hat{B}^\times/\mathbf{Q}^\times\times\{\tau\}=[A,\imath]$, and this gives the second statement. □

3. K-surfaces with QM

Let $\mathcal{O}$ be an Eichler order of square-free level N in an indefinite rational quaternion algebra of discriminant D, and let $\mathcal{O}_0$ be a maximal order with $\mathcal{O}\subseteq\mathcal{O}_0$. Let P be a non-CM K-rational point in $X(D,N)/W(D,N)$. A preimage $Q\in X(D,N)$ of P under the quotient map corresponds to an Abelian surface $(A,\imath)/\overline{K}$ with QM by $\mathcal{O}$, and $\phi(Q)$ corresponds to an Abelian surface $(A_0,\imath_0)/\overline{K}$ with QM by $\mathcal{O}_0$. For each $\sigma\in\mathrm{Gal}(\overline{K}/K)$ there exists an integer $n\mid ND$ such that ${}^\sigma Q=W_n(Q)$, and therefore ${}^\sigma\phi(Q)=\phi({}^\sigma Q)=\phi(W_n(Q))$. But ${}^\sigma\phi(Q)$ corresponds to $({}^\sigma A_0,{}^\sigma\imath_0)$, and by corollary 2 we have that $\phi(W_n(Q))$ belongs to $[A_0,\imath_0]$. This means that $(A_0,\imath_0)$ and $({}^\sigma A_0,{}^\sigma\imath_0)$ are isogenous for all $\sigma\in\mathrm{Gal}(\overline{K}/K)$; we say that $(A_0,\imath_0)$ is an *Abelian K-surface with QM by $\mathcal{O}_0$*. We also have the following converse to this construction:

Theorem 3. *Let $(A_0,\imath_0)$ be a non-CM Abelian K-surface with QM by the maximal order $\mathcal{O}_0$. Then there exists a square-free N, depending only on the isogeny class of $(A_0,\imath_0)$, such that $(A_0,\imath_0)$ is isogenous to the QM-Abelian surface obtained by the above procedure applied to some K-rational point in $X(D,N)/W(D,N)$. Moreover, if a K-rational point of $X(D,N')/W(D,N')$ parametrizes an Abelian K-surface with QM isogenous to $(A_0,\imath_0)$, then $N\mid N'$.*

Proof. The pair $(A_0,\imath_0)$ gives a point in $X(D,1)$, that we can represent by $((b_\ell)_\ell,\tau)$ for some $(b_\ell)_\ell\in\prod'_\ell\mathcal{O}_{0\ell}^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$ and some complex number τ. Recall that $[A_0,\imath_0]=\prod'_\ell\mathcal{O}_{0\ell}^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times\times\{\tau\}$, and that for $\ell\nmid D$ we identify $\mathcal{O}_{0\ell}^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$ with an homogenous tree of degree $\ell+1$: its vertexes are the pairs $(A_0',\imath_0')$ isogenous to $(A_0,\imath_0)$ with an isogeny of degree a power of ℓ, and two vertexes are connected if there exists an isogeny of degree ℓ^2 between them. Also for $\ell\mid D$ we identify $\mathcal{O}_{0\ell}^\times\setminus B_\ell^\times/\mathbf{Q}_\ell^\times$ with an oriented edge. Denote by π_ℓ the projection $[A_0,\imath_0]\to\mathcal{O}_{0\ell}^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$, and by $\langle A_0,\imath_0\rangle$ the finite set of Abelian surfaces up to isomorphism with QM by $\mathcal{O}_0$ that are $\mathrm{Gal}(\overline{K}/K)$-conjugated to $(A_0,\imath_0)$. Note that $\langle A_0,\imath_0\rangle\subseteq[A_0,\imath_0]$, because $(A_0,\imath_0)$ is an Abelian K-surface with QM. We consider the action of $\mathrm{Gal}(\overline{K}/K)$ on $\pi_\ell\langle A_0,\imath_0\rangle$ defined by ${}^\sigma(\pi_\ell(B,\jmath))=\pi_\ell({}^\sigma B,{}^\sigma\jmath)$. Note that $\pi_\ell\langle A_0,\imath_0\rangle$ will contain a single vertex for all but finitely many primes ℓ. Following Elkies, for each ℓ we construct an edge or a vertex of $\pi_\ell\langle A_0,\imath_0\rangle$ fixed by $\mathrm{Gal}(\overline{K}/K)$: it is the central vertex or edge of any path of maximum length joining two vertexes in $\pi_\ell\langle A_0,\imath_0\rangle$, and we will call it the *center* of $\pi_\ell\langle A_0,\imath_0\rangle$. It is a well known property of trees that this vertex or edge does not depend on the path chosen, and since $\mathrm{Gal}(\overline{K}/K)$ takes one path of maximum length to another, it is clear that the center is fixed by $\mathrm{Gal}(\overline{K}/K)$. Define

N to be the product of all the primes $\ell \nmid D$ such that the center of $\pi_\ell\langle A_0, \imath_0\rangle$ is an edge, and let $\mathcal{O}$ be an Eichler order of level N. The fact that $\pi_\ell[A_0, \imath_0]$ is a tree implies that if the center of $\pi_\ell\langle A_0, \imath_0\rangle$ is an edge, then it is necessarily the only edge or vertex in $\pi_\ell[A_0, \imath_0]$ fixed by $\mathrm{Gal}(\overline{K}/K)$ (otherwise there would exist a cycle; this can be seen by considering the action on the tree of a $\sigma \in \mathrm{Gal}(\overline{K}/K)$ that swaps the vertices of the central edge). Thus any pair in $[A_0, \imath_0]$ produces the same N. For each $\ell \mid N$, choose an orientation of the center and call b'_ℓ this oriented edge in the graph $\mathrm{PGL}_2(\mathbf{Z}_\ell)\backslash \mathrm{PGL}_2(\mathbf{Q}_\ell)$; recall that we can identify b'_ℓ with an element of $\mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$. For each $\ell \nmid N$ let $b'_\ell = b_\ell$, but viewed as an element in $\mathcal{O}_\ell^\times\backslash B_\ell^\times/\mathbf{Q}_\ell^\times$. Now the pair $((b'_\ell)_\ell, \tau)$ defines a point $Q = (A, \imath) \in X(D, N)(\overline{K})$, with the property that $\phi(Q) \in [A_0, \imath_0]$. If we represent $\phi(Q)$ by $((c_\ell)_\ell, \tau)$ and ${}^\sigma\phi(Q)$ by $((c'_\ell)_\ell, \tau)$, then $c_\ell = c'_\ell$ for all $\ell \nmid ND$. But for some $\ell \mid N$, c'_ℓ can be the vertex of the center of $\pi_\ell\langle A_0, \imath_0\rangle$ which is different from c_ℓ, and for some $\ell \mid D$, c'_ℓ and c_ℓ can have opposite orientation. If n is the product of the primes ℓ where c_ℓ and c'_ℓ differ, we have that ${}^\sigma\phi(Q) = \phi(W_n(Q))$, which implies that $\phi({}^\sigma Q) = \phi(W_n(Q))$. A similar argument shows that $\phi(W_N({}^\sigma Q)) = \phi(W_N W_n(Q))$. Therefore, by the injectivity of the map φ defined in the Section 2 we have that ${}^\sigma Q = W_n Q$, and the image of Q by the quotient map $X(D, N) \to X(D, N)/W(D, N)$ is a K-rational point P. Since $\phi(Q) \in [A_0, \imath_0]$, it is clear that applying to P the process for obtaining an Abelian K-surface with QM described at the beginning of the section, we obtain a pair isogenous to $(A_0, \imath_0)$. Finally, to see the last statement in the theorem, note that if $\ell \mid N$ and $(A'_0, \imath'_0)$ comes from a K-rational point in $X(D, N')/W(D, N')$ for some N' not divisible by ℓ, then $\pi_\ell\langle A'_0, \imath'_0\rangle$ would be a vertex fixed by $\mathrm{Gal}(\overline{K}/K)$, which is not possible because $\pi_\ell[A_0, \imath_0]$ contains an edge fixed by $\mathrm{Gal}(\overline{K}/K)$. □

Remark 4. For all but finitely many values of D and N the curve $X(D, N)/W(D, N)$ has genus at least 2 (see [2, Corollary 50]), and therefore it has a finite number of K-rational points.

Remark 5. So far in this section we have seen that a K-rational point in the quotient $X(D, N)/W(D, N)$ produces an Abelian K-surface $(A_0, \imath_0)$ with QM by a maximal order $\mathcal{O}_0$, and that any such pair arises from a K-rational point in $X(D, N)/W(D, N)$ for some square-free N. This result in fact gives slightly more information than the strictly needed in the setting we described in the introduction; hence, if we are interested only in Abelian K-surfaces with QM up to isogeny, and we do not care about the precise embedding $\imath$, we can just forget this information. Indeed, a K-rational point in $X(D, N)/W(D, N)$ gives rise to an Abelian K-surface A whose endomorphism algebra is isomorphic to the quaternion algebra over $\mathbf{Q}$ of discriminant D. Conversely, given an Abelian K-surface with endomorphism algebra isomorphic to the quaternion algebra over $\mathbf{Q}$ of discriminant D, we can find in its isogeny class a variety A_0 such that there exists an embedding $\imath_0 : \mathcal{O}_0 \hookrightarrow \mathrm{End}(A_0)$ for some maximal order $\mathcal{O}_0$. Then this variety is in turn isogenous to one arising from a K-rational point in $X(D, N)/W(D, N)$ for some N.

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