L-series of building blocks

Xavier Guitart, Jordi Quer

Universitat Politècnica de Catalunya

Québec-Vermont Number Theory Seminar, 2010

Outline

1. Building blocks and statement of the problem (modularity)
2. Characterization of strongly modular abelian varieties
3. Examples of strongly modular abelian surfaces

The Shimura-Taniyama conjecture

Modularity Theorem (Wiles, Taylor, Breuil, Conrad, Diamond)

Every elliptic curve C defined over $\mathbb{Q}$ is modular

The Shimura-Taniyama conjecture

Modularity Theorem (Wiles, Taylor, Breuil, Conrad, Diamond)

Every elliptic curve C defined over $\mathbb{Q}$ is modular

C is modular if either of these two equivalent conditions is satisfied:

1. There exists a non-constant homomorphism $J_0(N) \to C$.

2. There is a newform $f \in S_2(\Gamma_0(N))$ such that $L(f;s) = L(C/\mathbb{Q};s)$

The Shimura-Taniyama conjecture

Modularity Theorem (Wiles, Taylor, Breuil, Conrad, Diamond)

Every elliptic curve C defined over $\mathbb{Q}$ is modular

C is modular if either of these two equivalent conditions is satisfied:

1. There exists a non-constant homomorphism $J_0(N) \to C$.

2. There is a newform $f \in S_2(\Gamma_0(N))$ such that $L(f;s) = L(C/\mathbb{Q};s)$

Equivalence: Eichler-Shimura construction, congruence of Eichler-Shimura and Faltings's isogeny theorem.

The Shimura-Taniyama conjecture

Modularity Theorem (Wiles, Taylor, Breuil, Conrad, Diamond)

Every elliptic curve C defined over $\mathbb{Q}$ is modular

C is modular if either of these two equivalent conditions is satisfied:

1. There exists a non-constant homomorphism $J_0(N) \to C$.
   - Key step in the proof of the Birch and Swinnerton-Dyer conjecture in the case of analytic rank at most 1 (Gross-Zagier, Kolyvagin). Heegner points in $J_0(N)$ are projected via this uniformisation to produce rational points in C.
2. There is a newform $f \in S_2(\Gamma_0(N))$ such that $L(f; s) = L(C/\mathbb{Q}; s)$

Equivalence: Eichler-Shimura construction, congruence of Eichler-Shimura and Faltings's isogeny theorem.

The Shimura-Taniyama conjecture

Modularity Theorem (Wiles, Taylor, Breuil, Conrad, Diamond)

Every elliptic curve C defined over $\mathbb{Q}$ is modular

C is modular if either of these two equivalent conditions is satisfied:

1. There exists a non-constant homomorphism $J_0(N) \to C$.
   - Key step in the proof of the Birch and Swinnerton-Dyer conjecture in the case of analytic rank at most 1 (Gross-Zagier, Kolyvagin). Heegner points in $J_0(N)$ are projected via this uniformisation to produce rational points in C.
2. There is a newform $f \in S_2(\Gamma_0(N))$ such that $L(f; s) = L(C/\mathbb{Q}; s)$
   - It implies the Hasse conjecture for C: $L(C/\mathbb{Q}; s)$ is entire and it satisfies a functional equation.
   - Modularity of Frey curves can be used to solve diophantine equations of Fermat–type.

Equivalence: Eichler-Shimura construction, congruence of Eichler-Shimura and Faltings's isogeny theorem.

Characterization of modular abelian varieties over $\mathbb{Q}$

Characterization of modular abelian varieties over $\mathbb{Q}$

Definition

$A/\mathbb{Q}$ is said to be of GL_2-type if $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{\mathbb{Q}}(A)$ is a field E with $[E : \mathbb{Q}] = \dim A$.

Characterization of modular abelian varieties over $\mathbb{Q}$

Definition

$A/\mathbb{Q}$ is said to be of GL_2-type if $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{\mathbb{Q}}(A)$ is a field E with $[E : \mathbb{Q}] = \dim A$.

Theorem (consequence of Serre's conjecture)

A simple abelian variety $A/\mathbb{Q}$ is modular if and only if it is of GL_2-type.

Characterization of modular abelian varieties over $\mathbb{Q}$

Definition

$A/\mathbb{Q}$ is said to be of GL_2-type if $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{\mathbb{Q}}(A)$ is a field E with $[E : \mathbb{Q}] = \dim A$.

Theorem (consequence of Serre's conjecture)

A simple abelian variety $A/\mathbb{Q}$ is modular if and only if it is of GL_2-type.

Here modular means either of these two equivalent conditions:

1. There exists a non-constant homomorphism $J_1(N) \to A$.
2. $L(A/\mathbb{Q}; s) = \prod_{\sigma : E_f \hookrightarrow \mathbb{C}} L({}^{\sigma}f; s)$ for some newform $f \in S_2(\Gamma_1(N))$.

Characterization of modular abelian varieties over $\mathbb{Q}$

Definition

$A/\mathbb{Q}$ is said to be of GL_2-type if $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{\mathbb{Q}}(A)$ is a field E with $[E : \mathbb{Q}] = \dim A$.

Theorem (consequence of Serre's conjecture)

A simple abelian variety $A/\mathbb{Q}$ is modular if and only if it is of GL_2-type.

Here modular means either of these two equivalent conditions:

1. There exists a non-constant homomorphism $J_1(N) \to A$.
2. $L(A/\mathbb{Q}; s) = \prod_{\sigma : E_f \hookrightarrow \mathbb{C}} L({}^{\sigma}f; s)$ for some newform $f \in S_2(\Gamma_1(N))$. A is $\mathbb{Q}$-isogenous to A_f (a.v. attached to f by Eichler-Shimura)

Characterization of modular abelian varieties over $\mathbb{Q}$

Definition

$A/\mathbb{Q}$ is said to be of GL_2-type if $\mathrm{End}^0_{\mathbb{Q}}(A) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}_{\mathbb{Q}}(A)$ is a field E with $[E:\mathbb{Q}] = \dim A$.

Theorem (consequence of Serre's conjecture)

A simple abelian variety $A/\mathbb{Q}$ is modular if and only if it is of GL_2-type.

Here modular means either of these two equivalent conditions:

1. There exists a non-constant homomorphism $J_1(N) \to A$.
2. $L(A/\mathbb{Q}; s) = \prod_{\sigma: E_f \hookrightarrow \mathbb{C}} L({}^{\sigma}f; s)$ for some newform $f \in S_2(\Gamma_1(N))$. A is $\mathbb{Q}$-isogenous to A_f (a.v. attached to f by Eichler-Shimura)

- GL_2-type varieties are not necessarily simple up to $\bar{\mathbb{Q}}$-isogeny
- Ribet studied their $\bar{\mathbb{Q}}$-decomposition and characterized their $\bar{\mathbb{Q}}$-simple factors: they are a kind of varieties called building blocks.

$\bar{\mathbb{Q}}$-factors of GL_2-type varieties

Definition

A building block is an abelian variety $B/\bar{\mathbb{Q}}$ such that

- $\forall\, \sigma \in G_{\mathbb{Q}}$ there exists an isogeny $\mu_\sigma : {}^\sigma B \to B$ compatible with $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$:

$$\begin{array}{ccc} {}^\sigma B & \xrightarrow{\mu_\sigma} & B \\ {\scriptstyle {}^\sigma\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ {}^\sigma B & \xrightarrow{\mu_\sigma} & B. \end{array}$$

- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ is:
  - A totally real number field F with $[F : \mathbb{Q}] = \dim B$
  - A quaternion division algebra over F with $2[F : \mathbb{Q}] = \dim B$

$\bar{\mathbb{Q}}$-factors of GL_2-type varieties

Definition

A building block is an abelian variety $B/\bar{\mathbb{Q}}$ such that

- $\forall\, \sigma \in G_{\mathbb{Q}}$ there exists an isogeny $\mu_\sigma : {}^\sigma B \to B$ compatible with $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$:

$$\begin{array}{ccc} {}^\sigma B & \xrightarrow{\mu_\sigma} & B \\ {\scriptstyle {}^\sigma\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ {}^\sigma B & \xrightarrow{\mu_\sigma} & B. \end{array}$$

- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ is:
  - A totally real number field F with $[F : \mathbb{Q}] = \dim B$
  - A quaternion division algebra over F with $2[F : \mathbb{Q}] = \dim B$

Theorem (Ribet-Pyle)

Let $A/\mathbb{Q}$ be a GL_2-type variety (such that $A_{\bar{\mathbb{Q}}}$ is non-CM).

$\bar{\mathbb{Q}}$-factors of GL_2-type varieties

Definition

A building block is an abelian variety $B/\bar{\mathbb{Q}}$ such that

- $\forall\, \sigma \in G_{\mathbb{Q}}$ there exists an isogeny $\mu_\sigma : {}^\sigma B \to B$ compatible with $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$:

$$\begin{array}{ccc} {}^\sigma B & \xrightarrow{\mu_\sigma} & B \\ {}^\sigma\varphi \downarrow & & \downarrow \varphi \\ {}^\sigma B & \xrightarrow{\mu_\sigma} & B. \end{array}$$

- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ is:
  - A totally real number field F with $[F:\mathbb{Q}] = \dim B$
  - A quaternion division algebra over F with $2[F:\mathbb{Q}] = \dim B$

Theorem (Ribet-Pyle)

Let $A/\mathbb{Q}$ be a GL_2-type variety (such that $A_{\bar{\mathbb{Q}}}$ is non-CM).

- Then $A \sim_{\bar{\mathbb{Q}}} B^n$, where $B/\bar{\mathbb{Q}}$ is a building block.

$\bar{\mathbb{Q}}$-factors of GL_2-type varieties

Definition

A building block is an abelian variety $B/\bar{\mathbb{Q}}$ such that

- $\forall\, \sigma \in G_{\mathbb{Q}}$ there exists an isogeny $\mu_\sigma : {}^\sigma B \to B$ compatible with $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$:

$$\begin{array}{ccc} {}^\sigma B & \xrightarrow{\mu_\sigma} & B \\ {\scriptstyle {}^\sigma\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ {}^\sigma B & \xrightarrow{\mu_\sigma} & B. \end{array}$$

- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ is:
  - A totally real number field F with $[F:\mathbb{Q}] = \dim B$
  - A quaternion division algebra over F with $2[F:\mathbb{Q}] = \dim B$

Theorem (Ribet-Pyle)

Let $A/\mathbb{Q}$ be a GL_2-type variety (such that $A_{\bar{\mathbb{Q}}}$ is non-CM).

- Then $A \sim_{\bar{\mathbb{Q}}} B^n$, where $B/\bar{\mathbb{Q}}$ is a building block.
- Conversely, given a building block $B/\bar{\mathbb{Q}}$ there exists $A/\mathbb{Q}$ of GL_2-type as $A \sim_{\bar{\mathbb{Q}}} B^n$ for some n.

Geometric modular varieties over number $\bar{\mathbb{Q}}$

Combining this result with the fact that every GL_2-type variety is a quotient of some $J_1(N)$ we obtain that:

- building blocks are "modular" over $\bar{\mathbb{Q}}$, i.e. there exists

$$J_1(N)_{\bar{\mathbb{Q}}} \longrightarrow B.$$

Geometric modular varieties over number $\bar{\mathbb{Q}}$

Combining this result with the fact that every GL_2-type variety is a quotient of some $J_1(N)$ we obtain that:

- building blocks are "modular" over $\bar{\mathbb{Q}}$, i.e. there exists

$$J_1(N)_{\bar{\mathbb{Q}}} \longrightarrow B.$$

- This uniformisation can be used in a similar way as in the case of varieties over $\mathbb{Q}$.

Geometric modular varieties over number $\bar{\mathbb{Q}}$

Combining this result with the fact that every GL_2-type variety is a quotient of some $J_1(N)$ we obtain that:

- building blocks are "modular" over $\bar{\mathbb{Q}}$, i.e. there exists

$$J_1(N)_{\bar{\mathbb{Q}}} \longrightarrow B.$$

- This uniformisation can be used in a similar way as in the case of varieties over $\mathbb{Q}$.
- Building blocks of dimension 1: elliptic curves $C/\bar{\mathbb{Q}}$ isogenous to all of their Galois conjugates (also known as $\mathbb{Q}-$ *curves*).

Geometric modular varieties over number $\bar{\mathbb{Q}}$

Combining this result with the fact that every GL_2-type variety is a quotient of some $J_1(N)$ we obtain that:

- building blocks are "modular" over $\bar{\mathbb{Q}}$, i.e. there exists

$$J_1(N)_{\bar{\mathbb{Q}}} \longrightarrow B.$$

- This uniformisation can be used in a similar way as in the case of varieties over $\mathbb{Q}$.
- Building blocks of dimension 1: elliptic curves $C/\bar{\mathbb{Q}}$ isogenous to all of their Galois conjugates (also known as $\mathbb{Q}-$ *curves*).
- If $C/\bar{\mathbb{Q}}$ is a $\mathbb{Q}$-curve then

$$J_1(N)_{\bar{\mathbb{Q}}} \longrightarrow C$$

  and Heegner points on $J_1(N)$ can be used to produce results in the direction of BSD for C.

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K;s)$ is a product of L-series of modular forms.

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K; s)$ is a product of L-series of modular forms.
- This property is important in some applications of modularity:

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K; s)$ is a product of L-series of modular forms.
- This property is important in some applications of modularity:
  - it gives the analytic continuation of $L(B/K; s)$

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K; s)$ is a product of L-series of modular forms.
- This property is important in some applications of modularity:
  - it gives the analytic continuation of $L(B/K; s)$
  - the use of $\mathbb{Q}$-curves to solve certain diophantine equations (e.g. L. Dieulefait and J. Jiménez)

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K; s)$ is a product of L-series of modular forms.
- This property is important in some applications of modularity:
  - it gives the analytic continuation of $L(B/K; s)$
  - the use of $\mathbb{Q}$-curves to solve certain diophantine equations (e.g. L. Dieulefait and J. Jiménez)

Definition

B/K es strongly modular if $L(B/K; s) \sim \prod_f L(f; s)$, for some newforms $f \in S_2(\Gamma_1(N_f))$.

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K;s)$ is a product of L-series of modular forms.
- This property is important in some applications of modularity:
  - it gives the analytic continuation of $L(B/K;s)$
  - the use of $\mathbb{Q}$-curves to solve certain diophantine equations (e.g. L. Dieulefait and J. Jiménez)

Definition

B/K es strongly modular if $L(B/K;s) \sim \prod_f L(f;s)$, for some newforms $f \in S_2(\Gamma_1(N_f))$.

Aim

- Give a characterization of (non-CM) strongly modular abelian varieties in terms of the geometry of B.

- However, if B is a building block and we consider a model of B over a number field K, in general it is not true that $L(B/K;s)$ is a product of L-series of modular forms.
- This property is important in some applications of modularity:
  - it gives the analytic continuation of $L(B/K;s)$
  - the use of $\mathbb{Q}$-curves to solve certain diophantine equations (e.g. L. Dieulefait and J. Jiménez)

Definition

B/K es strongly modular if $L(B/K;s) \sim \prod_f L(f;s)$, for some newforms $f \in S_2(\Gamma_1(N_f))$.

Aim

- Give a characterization of (non-CM) strongly modular abelian varieties in terms of the geometry of B.
- Give some examples of strongly modular abelian varieties constructed without any use of modularity, and use this characterization to decide their strong modularity.

Outline

Outline

- B/K a non-CM abelian variety
- Suppose that B is $\bar{\mathbb{Q}}$-simple and that $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$.

- B/K a non-CM abelian variety
- Suppose that B is $\bar{\mathbb{Q}}$-simple and that $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$.

Observation 1

B is strongly modular over K $\Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B \sim_{\mathbb{Q}} \prod_f A_f$.

- B/K a non-CM abelian variety
- Suppose that B is $\bar{\mathbb{Q}}$-simple and that $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$.

Observation 1

B is strongly modular over $K \Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B \sim_{\mathbb{Q}} \prod_f A_f$.

- $L(B/K; s) = L((\mathrm{Res}_{K/\mathbb{Q}} B)/\mathbb{Q}; s)$
- B/K strongly modular $\Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B/\mathbb{Q}$ strongly modular
- A variety $A/\mathbb{Q}$ is strongly modular over $\mathbb{Q} \Leftrightarrow A \sim_{\mathbb{Q}} \prod_f A_f$ (consequence of Faltings's isogeny theorem)

- B/K a non-CM abelian variety
- Suppose that B is $\bar{\mathbb{Q}}$-simple and that $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$.

Observation 1

B is strongly modular over $K \Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B \sim_{\mathbb{Q}} \prod_f A_f$.

- $L(B/K; s) = L((\mathrm{Res}_{K/\mathbb{Q}} B)/\mathbb{Q}; s)$
- B/K strongly modular $\Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B/\mathbb{Q}$ strongly modular
- A variety $A/\mathbb{Q}$ is strongly modular over $\mathbb{Q} \Leftrightarrow A \sim_{\mathbb{Q}} \prod_f A_f$ (consequence of Faltings's isogeny theorem)

Observation 2

If B is strongly modular over K, then $K/\mathbb{Q}$ is abelian.

- B/K a non-CM abelian variety
- Suppose that B is $\bar{\mathbb{Q}}$-simple and that $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$.

Observation 1

B is strongly modular over $K \Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B \sim_{\mathbb{Q}} \prod_f A_f$.

- $L(B/K; s) = L((\mathrm{Res}_{K/\mathbb{Q}} B)/\mathbb{Q}; s)$
- B/K strongly modular $\Leftrightarrow \mathrm{Res}_{K/\mathbb{Q}} B/\mathbb{Q}$ strongly modular
- A variety $A/\mathbb{Q}$ is strongly modular over $\mathbb{Q} \Leftrightarrow A \sim_{\mathbb{Q}} \prod_f A_f$ (consequence of Faltings's isogeny theorem)

Observation 2

If B is strongly modular over K, then $K/\mathbb{Q}$ is abelian.

- $\mathrm{Res}_{K/\mathbb{Q}} B \sim_{\mathbb{Q}} \prod A_f$
- The endomorphisms of $\prod A_f$ are defined over an abelian number field L.
- The endomorphisms of $\mathrm{Res}_{K/\mathbb{Q}} B$ are defined over L.
- $K \subseteq L$.

Proposition

If B is strongly modular over K then B is a building block completely defined over K.

Proposition

If B is strongly modular over K then B is a building block completely defined over K.

Definition

A building block B/K is completely defined over K if for each $s \in \mathrm{Gal}(K/\mathbb{Q})$ there exists a compatible isogeny $\mu_s \colon {}^sB \to B$ defined over K.

Proposition

If B is strongly modular over K then B is a building block completely defined over K.

- $\prod A_f \sim_{\mathbb{Q}} \mathrm{Res}_{K/\mathbb{Q}} B \simeq_K \prod_{s \in \mathrm{Gal}(K/\mathbb{Q})} {}^s B$
- $A_f \sim_K B^n$

Definition

A building block B/K is completely defined over K if for each $s \in \mathrm{Gal}(K/\mathbb{Q})$ there exists a compatible isogeny $\mu_s \colon {}^s B \to B$ defined over K.

Proposition

If B is strongly modular over K then B is a building block completely defined over K.

- $\prod A_f \sim_{\mathbb{Q}} \mathrm{Res}_{K/\mathbb{Q}} B \simeq_K \prod_{s \in \mathrm{Gal}(K/\mathbb{Q})} {}^s B$
- $A_f \sim_K B^n$

Definition

A building block B/K is completely defined over K if for each $s \in \mathrm{Gal}(K/\mathbb{Q})$ there exists a compatible isogeny $\mu_s \colon {}^s B \to B$ defined over K.

- Not every building block B/K completely defined over K is strongly modular.

Proposition

If B is strongly modular over K then B is a building block completely defined over K.

- $\prod A_f \sim_{\mathbb{Q}} \mathrm{Res}_{K/\mathbb{Q}} B \simeq_K \prod_{s\in \mathrm{Gal}(K/\mathbb{Q})} {}^sB$
- $A_f \sim_K B^n$

Definition

A building block B/K is completely defined over K if for each $s \in \mathrm{Gal}(K/\mathbb{Q})$ there exists a compatible isogeny $\mu_s \colon {}^sB \to B$ defined over K.

- Not every building block B/K completely defined over K is strongly modular.
- The characterization is given in terms of a 2-cohomology class attached to B/K.

Building blocks and Galois Cohomology

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

$$B \xrightarrow{\mu_{st}^{-1}} {}^{st}B \xrightarrow{{}^s\mu_t} {}^sB \xrightarrow{\mu_s} B$$

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

$$B \xrightarrow{\mu_{st}^{-1}} {}^{st}B \xrightarrow{{}^s\mu_t} {}^sB \xrightarrow{\mu_s} B$$

- In fact, $c_{B/K}(s,t) \in F^*$, and it is a 2-cocycle.
- $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)$ (considering the trivial action).

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

$$B \xrightarrow{\mu_{st}^{-1}} {}^{st}B \xrightarrow{{}^s\mu_t} {}^sB \xrightarrow{\mu_s} B$$

- In fact, $c_{B/K}(s,t) \in F^*$, and it is a 2-cocycle.
- $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)$ (considering the trivial action).
- $[c_{B/K}]$ does not depend on the choice of the μ_s and is an invariant of the K-isogeny class of B.

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

$$B \xrightarrow{\mu_{st}^{-1}} {}^{st}B \xrightarrow{{}^s\mu_t} {}^sB \xrightarrow{\mu_s} B$$

- In fact, $c_{B/K}(s,t) \in F^*$, and it is a 2-cocycle.
- $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)$ (considering the trivial action).
- $[c_{B/K}]$ does not depend on the choice of the μ_s and is an invariant of the K-isogeny class of B.
- $[c_{B/K}]$ is 2-torsion.

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

$$B \xrightarrow{\mu_{st}^{-1}} {}^{st}B \xrightarrow{{}^s\mu_t} {}^sB \xrightarrow{\mu_s} B$$

- In fact, $c_{B/K}(s,t) \in F^*$, and it is a 2-cocycle.
- $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)$ (considering the trivial action).
- $[c_{B/K}]$ does not depend on the choice of the μ_s and is an invariant of the K-isogeny class of B.
- $[c_{B/K}]$ is 2-torsion.
- $[c_B] = \mathrm{Inf}[c_{B/K}] \in H^2(G_{\mathbb{Q}}, F^*)$ (trivial action)
- $[c_B]$ is an invariant of the $\bar{\mathbb{Q}}$-isogeny class of B.

Building blocks and Galois Cohomology

B/K building block completely defined over K.

- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ let $\mu_s : {}^sB \to B$ be a compatible isogeny.
- Let $F = Z(\mathrm{End}^0(B))$.
- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1}$

$$B \xrightarrow{\mu_{st}^{-1}} {}^{st}B \xrightarrow{{}^s\mu_t} {}^sB \xrightarrow{\mu_s} B$$

- In fact, $c_{B/K}(s,t) \in F^*$, and it is a 2-cocycle.
- $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)$ (considering the trivial action).
- $[c_{B/K}]$ does not depend on the choice of the μ_s and is an invariant of the K-isogeny class of B.
- $[c_{B/K}]$ is 2-torsion.
- $[c_B] = \mathrm{Inf}[c_{B/K}] \in H^2(G_{\mathbb{Q}}, F^*)$ (trivial action)
- $[c_B]$ is an invariant of the $\bar{\mathbb{Q}}$-isogeny class of B.
- Ribet used $[c_B]$ in his study of building blocks.

Building blocks and Galois Cohomology

- $[c_B]$ contains a lot of arithmetic information about B (in fact about its $\bar{\mathbb{Q}}$-isogeny class).

Building blocks and Galois Cohomology

- $[c_B]$ contains a lot of arithmetic information about B (in fact about its $\overline{\mathbb{Q}}$-isogeny class).

Theorem (Chi)

The Brauer class of $\mathrm{End}^0(B)$ is the image of $[c_B]$ under

$$H^2(G_{\mathbb{Q}}, F^*) \xrightarrow{Res} H^2(G_F, F^*) \longrightarrow H^2(G_F, \overline{F}^*) \simeq \mathrm{Br}(F)$$

Building blocks and Galois Cohomology

- $[c_B]$ contains a lot of arithmetic information about B (in fact about its $\bar{\mathbb{Q}}$-isogeny class).

Theorem (Chi)

The Brauer class of $\mathrm{End}^0(B)$ is the image of $[c_B]$ under

$$H^2(G_{\mathbb{Q}}, F^*) \xrightarrow{Res} H^2(G_F, F^*) \longrightarrow H^2(G_F, \overline{F}^*) \simeq \mathrm{Br}(F)$$

Theorem (Ribet)

There exists a model of B defined over L (up to $\bar{\mathbb{Q}}$-isogeny) if and only if $[c_B]$ lies in the kernel of

$$\mathrm{Res}\colon H^2(G_{\mathbb{Q}}, F^*) \longrightarrow H^2(G_L, F^*).$$

Building blocks and Galois Cohomology

- $[c_B]$ contains a lot of arithmetic information about B (in fact about its $\bar{\mathbb{Q}}$-isogeny class).

Theorem (Chi)

The Brauer class of $\mathrm{End}^0(B)$ is the image of $[c_B]$ under

$$H^2(G_{\mathbb{Q}}, F^*) \xrightarrow{Res} H^2(G_F, F^*) \longrightarrow H^2(G_F, \overline{F}^*) \simeq \mathrm{Br}(F)$$

Theorem (Ribet)

There exists a model of B defined over L (up to $\bar{\mathbb{Q}}$-isogeny) if and only if $[c_B]$ lies in the kernel of

$$\mathrm{Res}\colon H^2(G_{\mathbb{Q}}, F^*) \longrightarrow H^2(G_L, F^*).$$

Theorem

There exists a model of B completely defined over L (up to $\bar{\mathbb{Q}}$-isogeny) if and only if $[c_B]$ lies in the image of the inflation map

$$\mathrm{Inf}\colon H^2(L/\mathbb{Q}, F^*) \longrightarrow H^2(G_{\mathbb{Q}}, F^*)$$

Endomorphism algebra of the restriction of scalars

- $[c_{B/K}]$ contains information about the K-isogeny class of B.

Endomorphism algebra of the restriction of scalars

- $[c_{B/K}]$ contains information about the K-isogeny class of B.

Proposition

$\mathrm{End}^0_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}} B) \simeq \mathrm{End}^0(B) \otimes F^{c_{B/K}}[\mathrm{Gal}(K/\mathbb{Q})]$

Endomorphism algebra of the restriction of scalars

- $[c_{B/K}]$ contains information about the K-isogeny class of B.

Proposition

$\mathrm{End}^0_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}} B) \simeq \mathrm{End}^0(B) \otimes F^{c_{B/K}}[\mathrm{Gal}(K/\mathbb{Q})]$

- $A = \mathrm{Res}_{K/\mathbb{Q}} B$
- $A_K \simeq \prod_{s \in \mathrm{Gal}(K/\mathbb{Q})} {}^sB$
- $\mathrm{End}^0(B) \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A) \colon \varphi \mapsto ({}^s\varphi)_{s \in \mathrm{Gal}(K/\mathbb{Q})}$

Endomorphism algebra of the restriction of scalars

- $[c_{B/K}]$ contains information about the K-isogeny class of B.

Proposition

$\mathrm{End}^0_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}} B) \simeq \mathrm{End}^0(B) \otimes F^{c_{B/K}}[\mathrm{Gal}(K/\mathbb{Q})]$

- $A = \mathrm{Res}_{K/\mathbb{Q}} B$
- $A_K \simeq \prod_{s \in \mathrm{Gal}(K/\mathbb{Q})} {}^sB$
- $\mathrm{End}^0(B) \hookrightarrow \mathrm{End}^0_{\mathbb{Q}}(A)\colon \varphi \mapsto ({}^s\varphi)_{s \in \mathrm{Gal}(K/\mathbb{Q})}$
- Let $\lambda_s\colon A_K \to A_K$ such that ${}^{ts}B \xrightarrow{{}^t\mu_s} {}^tB$.
- In fact $\lambda_s \in \mathrm{End}^0_{\mathbb{Q}}(A)$ and $\lambda_s \lambda_t = c_{B/K}(s,t)\lambda_{st}$.

Characterization of strongly modular abelian varieties

Theorem

Let B/K be a non-CM building block with $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$. Then B is strongly modular over K if and only if

- $K/\mathbb{Q}$ is Galois abelian.
- B is completely defined over K.
- $[c_{B/K}]$ is symmetric: $c_{B/K}(s,t) = c_{B/K}(t,s)$.

Characterization of strongly modular abelian varieties

Theorem

Let B/K be a non-CM building block with $\mathrm{End}_{\bar{\mathbb{Q}}}(B) = \mathrm{End}_K(B)$. Then B is strongly modular over K if and only if

- $K/\mathbb{Q}$ is Galois abelian.
- B is completely defined over K.
- $[c_{B/K}]$ is symmetric: $c_{B/K}(s,t) = c_{B/K}(t,s)$.

- $\mathrm{End}^0_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}} B) \simeq \mathrm{End}_K(B) \otimes F^{c_{B/K}}[\mathrm{Gal}(K/\mathbb{Q})]$
- If $c_{B/K}$ is symmetric then $F^{c_{B/K}}[\mathrm{Gal}(K/\mathbb{Q})]$ is abelian and it is a product of number fields.
- $\mathrm{End}^0_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}} B) \simeq \mathrm{End}^0_{\bar{\mathbb{Q}}}(B) \otimes \prod E_i \simeq \prod \mathrm{M}_t(E_i)$
- $\mathrm{Res}_{K/\mathbb{Q}} B \sim_{\mathbb{Q}} \prod A_i^t$ with $\mathrm{End}^0_{\mathbb{Q}}(A_i) \simeq E_i$ and $[E_i : \mathbb{Q}] = \dim A_i$.

Strongly modular twists

- Let $K/\mathbb{Q}$ be an abelian extension
- B/K a building block completely defined over K
- B/K is strongly modular if and only if $c_{B/K}$ is symmetric.

Strongly modular twists

- Let $K/\mathbb{Q}$ be an abelian extension
- B/K a building block completely defined over K
- B/K is strongly modular if and only if $c_{B/K}$ is symmetric.
- Suppose that B/K is not strongly modular.
- Does there exist some twist of B strongly modular over K?

Strongly modular twists

- Let $K/\mathbb{Q}$ be an abelian extension
- B/K a building block completely defined over K
- B/K is strongly modular if and only if $c_{B/K}$ is symmetric.
- Suppose that B/K is not strongly modular.
- Does there exist some twist of B strongly modular over K?

Proposition

There exists a twist of B (a variety B_0/K isogenous to B over $\bar{\mathbb{Q}}$) that is strongly modular over K if and only if K contains a splitting field for $[c_B]$

Strongly modular twists

- Let $K/\mathbb{Q}$ be an abelian extension
- B/K a building block completely defined over K
- B/K is strongly modular if and only if $c_{B/K}$ is symmetric.
- Suppose that B/K is not strongly modular.
- Does there exist some twist of B strongly modular over K?

Proposition

There exists a twist of B (a variety B_0/K isogenous to B over $\bar{\mathbb{Q}}$) that is strongly modular over K if and only if K contains a splitting field for $[c_B]$

- Tate's theorem: $H^2(G_{\mathbb{Q}}, \bar{F}^*) = \{1\}$ (trivial $G_{\mathbb{Q}}$-action)
- The image of $[c_B]$ in $H^2(G_{\mathbb{Q}}, \bar{F}^*)$ is trivial: there exist maps $\beta\colon G_{\mathbb{Q}} \to \bar{F}^*$ such that

$$c_B(\sigma, \tau) = \beta(\sigma)\beta(\tau)\beta(\sigma\tau)^{-1}.$$

- The map $\beta \mod F^*$ is a homomorphism.
- The field $\bar{\mathbb{Q}}^{\ker(\beta \mod F^*)}$ is a splitting field for $[c_B]$.

Outline

1. Building blocks and statement of the problem (modularity)
2. Characterization of strongly modular abelian varieties
3. Examples of strongly modular abelian surfaces

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

- Equations of strongly modular varieties over $\mathbb{Q}$

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

- Equations of strongly modular varieties over $\mathbb{Q}$
  - Curves: $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$.

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

- Equations of strongly modular varieties over $\mathbb{Q}$
  - Curves: $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$.
  - Surfaces: $A = \mathrm{Jac}(C)$ with $C/\mathbb{Q}$ a genus 2 curve $\mathrm{End}^0_{\mathbb{Q}}(A)$ a quadratic number field.
    Examples by Cardona-González-Rio-Lario, González-Guardia-Rotger,...

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

- Equations of strongly modular varieties over $\mathbb{Q}$
  - Curves: $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$.
  - Surfaces: $A = \mathrm{Jac}(C)$ with $C/\mathbb{Q}$ a genus 2 curve $\mathrm{End}^0_{\mathbb{Q}}(A)$ a quadratic number field.
    Examples by Cardona-González-Rio-Lario, González-Guardia-Rotger,...
- Equations of strongly modular varieties over number fields

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

- Equations of strongly modular varieties over $\mathbb{Q}$
  - Curves: $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$.
  - Surfaces: $A = \mathrm{Jac}(C)$ with $C/\mathbb{Q}$ a genus 2 curve $\mathrm{End}^0_{\mathbb{Q}}(A)$ a quadratic number field.
    Examples by Cardona-González-Rio-Lario, González-Guardia-Rotger,...

- Equations of strongly modular varieties over number fields
  - Curves: $\mathbb{Q}$-curves E/K with $[c_{E/K}]$ symmetric.
    Examples by Quer.

Examples of strongly modular varieties

Aim

Give explicit equations of strongly modular varieties over number fields, constructed without any use of modular forms.

- Equations of strongly modular varieties over $\mathbb{Q}$
  - Curves: $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$.
  - Surfaces: $A = \mathrm{Jac}(C)$ with $C/\mathbb{Q}$ a genus 2 curve $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A)$ a quadratic number field.
    Examples by Cardona-González-Rio-Lario, González-Guardia-Rotger,...
- Equations of strongly modular varieties over number fields
  - Curves: $\mathbb{Q}$-curves E/K with $[c_{E/K}]$ symmetric.
    Examples by Quer.
  - Surfaces: $B = \mathrm{Jac}(C)$ with C/K a genus 2 curve.
    B is a $\mathbb{Q}$-variety and $\mathrm{End}^0_{\overline{\mathbb{Q}}}(B)$ is a quaternion algebra.
    We compute $[c_B]$ and $[c_{B/K}]$ to guarantee strong modularity.

Baba-Granath family of genus 2 curves

$$
\begin{aligned}
C_j:\quad Y^2 = {} & \left(-4+3\sqrt{-6j}\right)X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28+9\sqrt{-6j}\right)X^4 \\
& + 16(27j+16)^2X^3 + 12(27j+16)2\left(28-9\sqrt{-6j}\right)X^2 \\
& - 48(27j+16)^3X + 8(27j+16)3\left(4+3\sqrt{-6j}\right)
\end{aligned}
$$

- Let $B_j = \mathrm{Jac}(C_j)$ for $j \in \mathbb{Q}$.

Baba-Granath family of genus 2 curves

$$
\begin{aligned}
C_j:\quad Y^2 = &\left(-4+3\sqrt{-6j}\right)X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28+9\sqrt{-6j}\right)X^4\\
&+16(27j+16)^2X^3 + 12(27j+16)2\left(28-9\sqrt{-6j}\right)X^2\\
&-48(27j+16)^3X + 8(27j+16)3\left(4+3\sqrt{-6j}\right)
\end{aligned}
$$

- Let $B_j = \mathrm{Jac}(C_j)$ for $j \in \mathbb{Q}$.
- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B_j) \simeq (2,3)_{\mathbb{Q}}$. (Hashimoto-Murabayashi)

Baba-Granath family of genus 2 curves

$$C_j: \quad Y^2 = \left(-4+3\sqrt{-6j}\right)X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28+9\sqrt{-6j}\right)X^4$$
$$+ 16(27j+16)^2X^3 + 12(27j+16)2\left(28-9\sqrt{-6j}\right)X^2$$
$$- 48(27j+16)^3X + 8(27j+16)3\left(4+3\sqrt{-6j}\right)$$

- Let $B_j = \mathrm{Jac}(C_j)$ for $j \in \mathbb{Q}$.
- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B_j) \simeq (2,3)_{\mathbb{Q}}$. (Hashimoto-Murabayashi)
- ${}^{\sigma}C_j \simeq C_j$ for each $\sigma \in G_{\mathbb{Q}} \Rightarrow \phi_\sigma : {}^{\sigma}B_j \to B_j$, but not necessarily compatible

Baba-Granath family of genus 2 curves

$$
\begin{aligned}
C_j: \quad Y^2 = & \left(-4+3\sqrt{-6j}\right)X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28+9\sqrt{-6j}\right)X^4 \\
& + 16(27j+16)^2X^3 + 12(27j+16)2\left(28-9\sqrt{-6j}\right)X^2 \\
& - 48(27j+16)^3X + 8(27j+16)3\left(4+3\sqrt{-6j}\right)
\end{aligned}
$$

- Let $B_j = \mathrm{Jac}(C_j)$ for $j \in \mathbb{Q}$.
- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B_j) \simeq (2,3)_{\mathbb{Q}}$. (Hashimoto-Murabayashi)
- ${}^{\sigma}C_j \simeq C_j$ for each $\sigma \in G_{\mathbb{Q}} \Rightarrow \phi_\sigma : {}^{\sigma}B_j \to B_j$, but not necessarily compatible
- Skolem-Noether: $\exists\ \psi_\sigma \in \mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ with $\psi_\sigma \circ \phi_\sigma$ compatible.

Baba-Granath family of genus 2 curves

$$C_j: \quad Y^2 = \left(-4+3\sqrt{-6j}\right)X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28+9\sqrt{-6j}\right)X^4 + 16(27j+16)^2X^3 + 12(27j+16)2\left(28-9\sqrt{-6j}\right)X^2 - 48(27j+16)^3X + 8(27j+16)3\left(4+3\sqrt{-6j}\right)$$

- Let $B_j = \mathrm{Jac}(C_j)$ for $j \in \mathbb{Q}$.
- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B_j) \simeq (2,3)_{\mathbb{Q}}$. (Hashimoto-Murabayashi)
- ${}^{\sigma}C_j \simeq C_j$ for each $\sigma \in G_{\mathbb{Q}} \Rightarrow \phi_\sigma : {}^{\sigma}B_j \to B_j$, but not necessarily compatible
- Skolem-Noether: $\exists\ \psi_\sigma \in \mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ with $\psi_\sigma \circ \phi_\sigma$ compatible.
- B_j is a building block: there exist newforms f such that
$$A_f \sim_{\bar{\mathbb{Q}}} B_j^n.$$
- B_j is completely defined over $K = \mathbb{Q}(\sqrt{-6j}, \sqrt{j}, \sqrt{-(27j+16)}, \sqrt{-2(27j+16)})$

The cohomology class $[c_{B_j}]$

We can compute $[c_{B_j}]$, and to give an explicit expression we use the isomorphism:

$$\begin{array}{cccccc} H^2(G_{\mathbb{Q}},\mathbb{Q}^*)[2] & \simeq & \mathrm{Hom}(G_{\mathbb{Q}},\mathbb{Q}^*/\{\pm 1\}\mathbb{Q}^{*2}) & \times & H^2(G_{\mathbb{Q}},\{\pm 1\}) \\ c_{B_j} & \leftrightarrow & (\,\overline{[c_{B_j}]} & , & [c_{B_j}]_{\pm}\,) \end{array}$$

- $c_{B_j}^2(\sigma,\tau)=\delta(\sigma)\delta(\tau)\delta(\sigma\tau)^{-1} \Rightarrow \overline{c_{B_j}}(\sigma)=\delta(\sigma)\mathrm{mod}\{\pm 1\}\mathbb{Q}^{*2}$.
- $c_{B_j\pm}(\sigma,\tau)=\mathrm{sign}(c_{B_j}(\sigma,\tau))$.

The cohomology class $[c_{B_j}]$

We can compute $[c_{B_j}]$, and to give an explicit expression we use the isomorphism:

$$\begin{array}{ccccc} H^2(G_{\mathbb{Q}},\mathbb{Q}^*)[2] & \simeq & \mathrm{Hom}(G_{\mathbb{Q}},\mathbb{Q}^*/\{\pm1\}\mathbb{Q}^{*2}) & \times & H^2(G_{\mathbb{Q}},\{\pm1\}) \\ c_{B_j} & \leftrightarrow & (\,\overline{[c_{B_j}]} & , & [c_{B_j}]_{\pm}\,) \end{array}$$

- $c_{B_j}^2(\sigma,\tau)=\delta(\sigma)\delta(\tau)\delta(\sigma\tau)^{-1} \Rightarrow \overline{c_{B_j}}(\sigma)=\delta(\sigma)\mathrm{mod}\{\pm1\}\mathbb{Q}^{*2}$.
- $c_{B_j\pm}(\sigma,\tau)=\mathrm{sign}(c_{B_j}(\sigma,\tau))$.

Propositon

Let $\mathrm{Gal}(\mathbb{Q}(\sqrt{-(27j+16)},\sqrt{-j(27j+16)})/\mathbb{Q})=\langle\sigma,\tau\rangle$.

- $\overline{[c_{B_j}]}:\sigma\mapsto 3\ \ \tau\mapsto 2$
- $[c_{B_j}]_{\pm}=(-(27j+16),3)_{\mathbb{Q}}\cdot(-j(27j+16),2)_{\mathbb{Q}}\cdot(2,3)_{\mathbb{Q}}$

The cohomology class $[c_{B_j}]$

We can compute $[c_{B_j}]$, and to give an explicit expression we use the isomorphism:

$$\begin{array}{ccccc} H^2(G_{\mathbb{Q}},\mathbb{Q}^*)[2] & \simeq & \mathrm{Hom}(G_{\mathbb{Q}},\mathbb{Q}^*/\{\pm1\}\mathbb{Q}^{*2}) & \times & H^2(G_{\mathbb{Q}},\{\pm1\}) \\ c_{B_j} & \leftrightarrow & (\overline{[c_{B_j}]} & , & [c_{B_j}]_\pm\,) \end{array}$$

- $c_{B_j}^2(\sigma,\tau)=\delta(\sigma)\delta(\tau)\delta(\sigma\tau)^{-1}\Rightarrow \overline{c_{B_j}}(\sigma)=\delta(\sigma)\mathrm{mod}\{\pm1\}\mathbb{Q}^{*2}$.
- $c_{B_j\pm}(\sigma,\tau)=\mathrm{sign}(c_{B_j}(\sigma,\tau))$.

Propositon

Let $\mathrm{Gal}(\mathbb{Q}(\sqrt{-(27j+16)},\sqrt{-j(27j+16)})/\mathbb{Q})=\langle\sigma,\tau\rangle$.

- $\overline{[c_{B_j}]}:\sigma\mapsto 3\quad \tau\mapsto 2$
- $[c_{B_j}]_\pm=(-(27j+16),3)_{\mathbb{Q}}\cdot(-j(27j+16),2)_{\mathbb{Q}}\cdot(2,3)_{\mathbb{Q}}$

- $c_{B_j}(\sigma,\tau)^2=c_{B_j}(\sigma,\tau)c_{B_j}(\sigma,\tau)'=(\psi_\sigma\circ\psi'_\sigma)(\psi_\tau\circ\psi'_\tau)(\psi_{\sigma\tau}\circ\psi'_{\sigma\tau})$
- A formula of Quer giving $\mathrm{End}^0_{\mathbb{Q}}(B_j)$ in terms of $\overline{[c_{B_j}]}$ and $[c_{B_j}]_\pm$.

A concrete example: $B = B_{-4/27}$

A concrete example: $B = B_{-4/27}$

- $K = \mathbb{Q}(\sqrt{-6}, \sqrt{-3})$
- We do not know $[c_{B/K}]$, but $\mathrm{Inf}([c_{B/K}]) = [c_B]$.
- $[c_B]$ is not the inflation of any symmetric element, so $[c_{B/K}]$ is not symmetric
- K does not contain any splitting field for $[c_B]$

A concrete example: $B = B_{-4/27}$

- $K = \mathbb{Q}(\sqrt{-6}, \sqrt{-3})$
- We do not know $[c_{B/K}]$, but $\mathrm{Inf}([c_{B/K}]) = [c_B]$.
- $[c_B]$ is not the inflation of any symmetric element, so $[c_{B/K}]$ is not symmetric
- K does not contain any splitting field for $[c_B]$
- $L = K(\sqrt{-1})$, then some twist of B is strongly modular over L.
- $[c_{B/L}]$ is not symmetric $\to$ B/L is not strongly modular.

A concrete example: $B = B_{-4/27}$

- $K = \mathbb{Q}(\sqrt{-6}, \sqrt{-3})$
- We do not know $[c_{B/K}]$, but $\mathrm{Inf}([c_{B/K}]) = [c_B]$.
- $[c_B]$ is not the inflation of any symmetric element, so $[c_{B/K}]$ is not symmetric
- K does not contain any splitting field for $[c_B]$
- $L = K(\sqrt{-1})$, then some twist of B is strongly modular over L.
- $[c_{B/L}]$ is not symmetric $\rightarrow$ B/L is not strongly modular.
- We consider the twist of C by $\gamma = \sqrt{6} + \sqrt{18}$:

$$C_\gamma : \ \gamma Y^2 = f(X)$$

- $[c_{B_\gamma/L}]$ it is symmetric $\rightarrow$ B_γ/L is strongly modular.

A concrete example: $B = B_{-4/27}$

- $K = \mathbb{Q}(\sqrt{-6}, \sqrt{-3})$
- We do not know $[c_{B/K}]$, but $\mathrm{Inf}([c_{B/K}]) = [c_B]$.
- $[c_B]$ is not the inflation of any symmetric element, so $[c_{B/K}]$ is not symmetric
- K does not contain any splitting field for $[c_B]$
- $L = K(\sqrt{-1})$, then some twist of B is strongly modular over L.
- $[c_{B/L}]$ is not symmetric $\rightarrow$ B/L is not strongly modular.
- We consider the twist of C by $\gamma = \sqrt{6} + \sqrt{18}$:

$$C_\gamma : \ \gamma Y^2 = f(X)$$

- $[c_{B_\gamma/L}]$ it is symmetric $\rightarrow$ B_γ/L is strongly modular.
- γ is the solution of an embedding problem in Galois theory (associated to the non-symmetric part of $[c_{B_j/L}]_\pm \in H^2(L/\mathbb{Q}, \{\pm 1\})$)

$$1 \rightarrow \mathrm{Gal}(L(\sqrt{\gamma})/L) \simeq \{\pm 1\} \rightarrow \mathrm{Gal}(L(\sqrt{\gamma})/\mathbb{Q}) \rightarrow \mathrm{Gal}(L/\mathbb{Q}) \rightarrow 1$$

A concrete example: $j = -4/27$

We find $f \in S_2(\Gamma_0(2^4 \cdot 3^4), \chi)$:

$$\begin{aligned} f &= q - \sqrt{3}\, q^5 + 3i\, q^7 - 3\sqrt{3}\, q^{11} + q^{13} - 2i\sqrt{3}\, q^{17} - 6i\, q^{19} \\ &+ 3\sqrt{3}\, q^{23} + 2\, q^{25} - 5\sqrt{3}i\, q^{29} - 3i\, q^{31} + \cdots \end{aligned}$$

and $g \in S_2(\Gamma_0(2^6 \cdot 3^4), \varepsilon)$:

$$\begin{aligned} g &= q - \sqrt{3}\, q^5 + 3i\, q^7 - 3\sqrt{3}\, q^{11} - q^{13} + 2i\sqrt{3}\, q^{17} + 6i\, q^{19} \\ &- 3\sqrt{3}\, q^{23} + 2\, q^{25} - 5\sqrt{3}i\, q^{29} - 3i\, q^{31} + \cdots \end{aligned}$$

such that

$$L(B_\gamma/L; T) = (L(f; s)L({}^\sigma f; s)L(g; s)L({}^\tau g; s))^2$$

$$\mathrm{Res}_{L/\mathbb{Q}} B_\gamma \sim A_f^2 \times A_g^2$$

- In this example it is enough to go to a quadratic extension L/K to obtain a twist which is strongly modular over L.

- In this example it is enough to go to a quadratic extension L/K to obtain a twist which is strongly modular over L.

Proposition

There exist surfaces B_j in the Baba-Granath family such any L with B_j strongly modular over L is arbitrary large.

- In this example it is enough to go to a quadratic extension L/K to obtain a twist which is strongly modular over L.

Proposition

There exist surfaces B_j in the Baba-Granath family such any L with B_j strongly modular over L is arbitrary large.

Proposition

There exist surfaces B_j in the Baba-Granath family such that any A_f with $A_f \sim_{\bar{\mathbb{Q}}} B_j^n$ has dimension arbitrary large.

- In this example it is enough to go to a quadratic extension L/K to obtain a twist which is strongly modular over L.

Proposition

There exist surfaces B_j in the Baba-Granath family such any L with B_j strongly modular over L is arbitrary large.

Proposition

There exist surfaces B_j in the Baba-Granath family such that any A_f with $A_f \sim_{\bar{\mathbb{Q}}} B_j^n$ has dimension arbitrary large.

- Let f be a modular form such that

$$A_f \sim_{\bar{\mathbb{Q}}} B_j^n,$$

  and let ε be the Nebentypus of f.
- $[c_{B_j}]_\pm$ is ramified at $p \Leftrightarrow \varepsilon_p(-1) = -1$.

- In this example it is enough to go to a quadratic extension L/K to obtain a twist which is strongly modular over L.

Proposition

There exist surfaces B_j in the Baba-Granath family such any L with B_j strongly modular over L is arbitrary large.

Proposition

There exist surfaces B_j in the Baba-Granath family such that any A_f with $A_f \sim_{\bar{\mathbb{Q}}} B_j^n$ has dimension arbitrary large.

- Let f be a modular form such that

$$A_f \sim_{\bar{\mathbb{Q}}} B_j^n,$$

  and let ε be the Nebentypus of f.
- $[c_{B_j}]_\pm$ is ramified at $p \Leftrightarrow \varepsilon_p(-1) = -1$.
- Using the formulas for $[c_{B_j}]_\pm$ we can force the order of any such ε to be high.

L-series of building blocks

Xavier Guitart, Jordi Quer

Universitat Politècnica de Catalunya

Québec-Vermont Number Theory Seminar, 2010