Modular forms over fields of mixed signature and algebraic points in elliptic curves

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Outline

1. Heegner points
2. Darmon points (archimedean)
3. A construction over a cubic field of mixed signature
4. Numerical evidence for the conjecture

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- Key fact for the construction of Heegner points: E is modular.

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  - (Certain) E defined over a totally real field F.
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  - There are some conjectural constructions, proposed by H. Darmon.

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- (But in this case there is no algebraic map $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H} \to E$)

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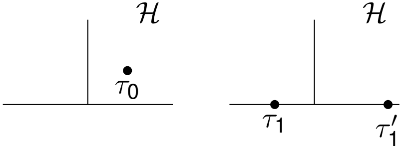

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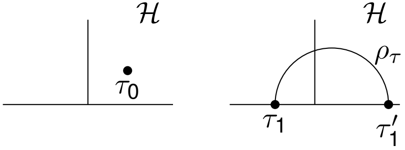

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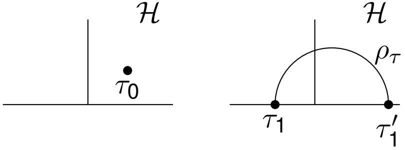

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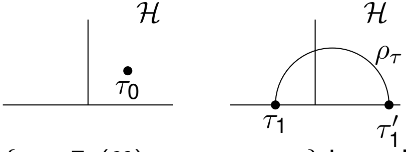

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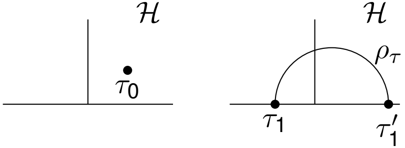

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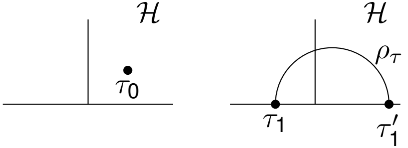

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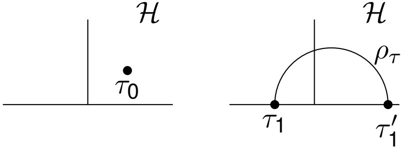

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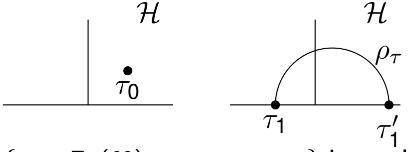

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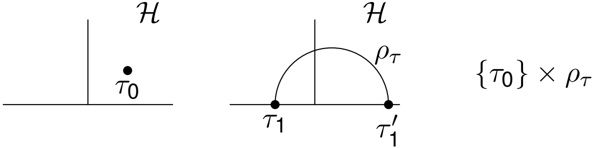

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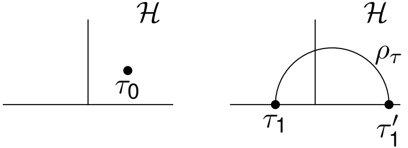

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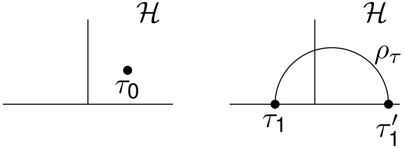

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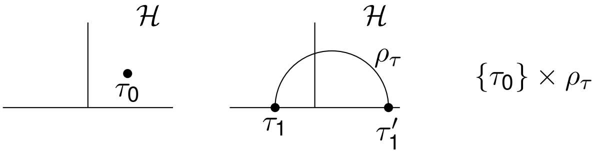

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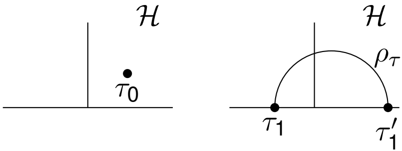

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- Our aim: propose a similar construction if F is not totally real

Outline

1. Heegner points

2. Darmon points (archimedean)

3. A construction over a cubic field of mixed signature

4. Numerical evidence for the conjecture

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.

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Generalized Modularity Conjecture

There is a harmonic differential 2-form ω_E on $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}_3$ associated to E (the eigenvalues of the Hecke operators match the $a_{\mathfrak{p}}$'s of E).

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Generalized Modularity Conjecture

There is a harmonic differential 2-form ω_E on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}_3$ associated to E (the eigenvalues of the Hecke operators match the $a_{\mathfrak{p}}$'s of E).

- As before, ω_E is determined by its Fourier–Bessel expansion.

- ω_E has a "Fourier-Bessel expansion":

$$\omega_E(z,x,y) = \sum_{\substack{\alpha\in\mathcal{O}_F \\ \alpha_0>0}} \frac{a_{(\alpha)}}{N_{F/\mathbb{Q}}(\alpha)} \frac{\alpha_0}{\delta_0} \exp\left(-2\pi i \left(\frac{\alpha_0 \bar{z}}{\delta_0} + \frac{\alpha_1 x}{\delta_1} + \frac{\alpha_2 \bar{x}}{\delta_2}\right)\right) \mathbb{K}\left(\frac{\alpha_1 y}{\delta_1}\right) \cdot \begin{pmatrix} \frac{-dx}{y} \wedge d\bar{z} \\ \frac{dy}{y} \wedge d\bar{z} \\ \frac{d\bar{x}}{y} \wedge d\bar{z} \end{pmatrix}$$

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- We can compute the $a_{(\alpha)}$ by counting points on $E(\mathcal{O}_F/\mathfrak{p})$

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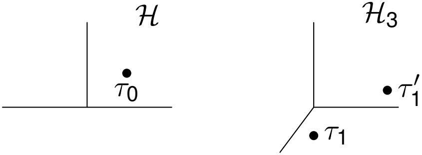

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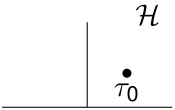

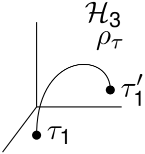

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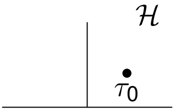

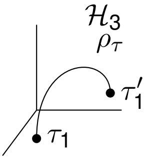

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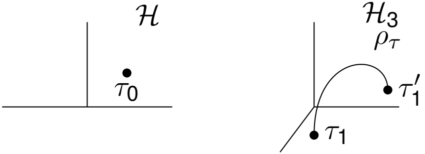

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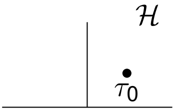

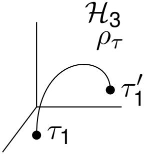

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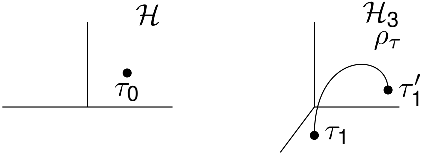

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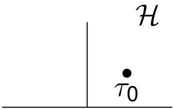

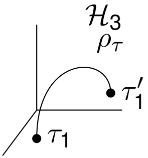

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$P_\tau \in E(H)$ with H a finite abelian extension of K.

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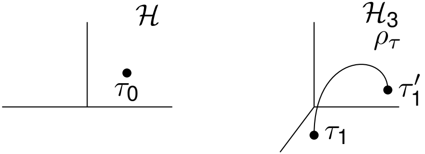

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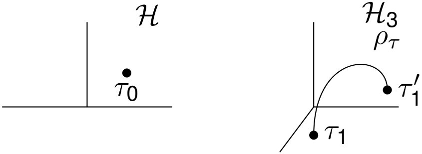

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- We found some numerical evidence for the conjecture.

Outline

1 Heegner points

2 Darmon points (archimedean)

3 A construction over a cubic field of mixed signature

4 Numerical evidence for the conjecture

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$

$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$

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- The image of $J_\tau \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$ coincides (up to 32 digits of accuracy) with $10P$, where

$$P = \left(r - 1 : w - r^2 + 2r : 1 \right) \in E(K)$$

Modular forms over fields of mixed signature and algebraic points in elliptic curves

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