Modular forms over cubic fields and algebraic points on elliptic curves

Xevi Guitart[1] Marc Masdeu[2] Haluk Sengun[2]

[1]Institute for Experimental Mathematics, Essen

[2]University of Warwick

Bilbo

Outline

Outline

1. Heegner points
2. Darmon points (archimedean)
3. A construction over a cubic field of mixed signature
4. Numerical evidence for the conjecture

Heegner points

- E elliptic curve with rational coefficients

$$E\colon\ y^2 + b_1xy + b_3y = x^3 + b_2x^2 + b_4x + b_6, \quad b_i \in \mathbb{Z}$$

Heegner points

- E elliptic curve with rational coefficients

$$E\colon\ y^2 + b_1xy + b_3y = x^3 + b_2x^2 + b_4x + b_6, \quad b_i \in \mathbb{Z}$$

- Heegner points on E are a canonical collection of algebraic points defined over (abelian extensions of) quadratic imaginary fields.

Heegner points

- E elliptic curve with rational coefficients

$$E\colon\ y^2 + b_1xy + b_3y = x^3 + b_2x^2 + b_4x + b_6, \quad b_i \in \mathbb{Z}$$

- Heegner points on E are a canonical collection of algebraic points defined over (abelian extensions of) quadratic imaginary fields.
- They are a key tool in the study of Mordell–Weil groups of E (e.g., partial results on Birch-Swinnerton–Dyer Conjecture).

Heegner points

- E elliptic curve with rational coefficients

$$E\colon\ y^2 + b_1xy + b_3y = x^3 + b_2x^2 + b_4x + b_6, \quad b_i \in \mathbb{Z}$$

- Heegner points on E are a canonical collection of algebraic points defined over (abelian extensions of) quadratic imaginary fields.
- They are a key tool in the study of Mordell–Weil groups of E (e.g., partial results on Birch-Swinnerton–Dyer Conjecture).
- Can be computed explicitly $\rightsquigarrow$ efficient algorithms

Heegner points

- E elliptic curve with rational coefficients

$$E\colon\ y^2 + b_1xy + b_3y = x^3 + b_2x^2 + b_4x + b_6, \quad b_i \in \mathbb{Z}$$

- Heegner points on E are a canonical collection of algebraic points defined over (abelian extensions of) quadratic imaginary fields.
- They are a key tool in the study of Mordell–Weil groups of E (e.g., partial results on Birch-Swinnerton–Dyer Conjecture).
- Can be computed explicitly $\rightsquigarrow$ efficient algorithms
- Key fact for the construction of Heegner points: E is modular.

Modular forms and elliptic curves

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az + b}{cz + d}$$

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$$

- $\Gamma_0(N)\backslash\mathcal{H}$ (suitably compactified) is a Riemann surface.

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az + b}{cz + d}$$

- $\Gamma_0(N)\backslash\mathcal{H}$ (suitably compactified) is a Riemann surface.
- There is the modular curve $X_0(N)/\mathbb{Q}$, and $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N)\backslash\mathcal{H}$

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az + b}{cz + d}$$

- $\Gamma_0(N)\backslash\mathcal{H}$ (suitably compactified) is a Riemann surface.
- There is the modular curve $X_0(N)/\mathbb{Q}$, and $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N)\backslash\mathcal{H}$
- A modular form f is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that $f(z)dz$ descends to a differential on $X_0(N)$

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az + b}{cz + d}$$

- $\Gamma_0(N)\backslash\mathcal{H}$ (suitably compactified) is a Riemann surface.
- There is the modular curve $X_0(N)/\mathbb{Q}$, and $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N)\backslash\mathcal{H}$
- A modular form f is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that $f(z)dz$ descends to a differential on $X_0(N)$
- They have a Fourier expansion: $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$.

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$$

- $\Gamma_0(N)\backslash\mathcal{H}$ (suitably compactified) is a Riemann surface.
- There is the modular curve $X_0(N)/\mathbb{Q}$, and $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N)\backslash\mathcal{H}$
- A modular form f is a holomorphic function $f : \mathcal{H}\to\mathbb{C}$ such that $f(z)dz$ descends to a differential on $X_0(N)$
- They have a Fourier expansion: $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$.

Modularity Theorem (Wiles et. al.)

Let E be an elliptic curve over $\mathbb{Q}$. Then for some $N \in \mathbb{Z}_{\geq 1}$:

1. (geometric version): There exists a morphism $X_0(N)\to E$ over $\mathbb{Q}$.

Modular forms and elliptic curves

- Poincaré upper half plane: $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\} \subset \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az + b}{cz + d}$$

- $\Gamma_0(N)\backslash\mathcal{H}$ (suitably compactified) is a Riemann surface.
- There is the modular curve $X_0(N)/\mathbb{Q}$, and $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N)\backslash\mathcal{H}$
- A modular form f is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that $f(z)dz$ descends to a differential on $X_0(N)$
- They have a Fourier expansion: $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$.

Modularity Theorem (Wiles et. al.)

Let E be an elliptic curve over $\mathbb{Q}$. Then for some $N \in \mathbb{Z}_{\geq 1}$:

1. (geometric version): There exists a morphism $X_0(N) \to E$ over $\mathbb{Q}$.
2. (automorphic version): There exists a modular form $f_E(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ such that $a_p = p + 1 - \#E(\mathbb{Z}/p\mathbb{Z})$ for all primes p.

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H} \to E$

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H} \to E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H}\rightarrow E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

- An explicit formula:

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H} \to E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

- An explicit formula:
  - Let $P_\tau = 2\pi i \int_\tau^{i\infty} f_E(z)dz$

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H}\to E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

- An explicit formula:
  - Let $P_\tau = 2\pi i \int_\tau^{i\infty} f_E(z)dz \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H} \to E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

- An explicit formula:
  - Let $P_\tau = 2\pi i \int_\tau^{i\infty} f_E(z)dz \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$
- Heegner points generalize to the following setting:
  - E is defined over a totally real field F.
  - K is a quadratic totally imaginary extension of F.
  - Heegner points on E, defined over abelian extensions of K

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H} \to E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

- An explicit formula:
  - Let $P_\tau = 2\pi i \int_\tau^{i\infty} f_E(z)dz \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$
- Heegner points generalize to the following setting:
  - E is defined over a totally real field F.
  - K is a quadratic totally imaginary extension of F.
  - Heegner points on E, defined over abelian extensions of K
- What if F is totally real, but K is not totally imaginary?

Definition of Heegner points

- $K = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ a quadratic imaginary field and $\tau \in K \cap \mathcal{H}$
- A Heegner point P_τ is the image of τ under $\Gamma_0(N)\backslash\mathcal{H} \to E$

Theorem (consequence of the theory of Complex multiplication)

P_τ has coordinates in a finite abelian extension of K.

- An explicit formula:
  - Let $P_\tau = 2\pi i \int_\tau^{i\infty} f_E(z)dz \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$
- Heegner points generalize to the following setting:
  - E is defined over a totally real field F.
  - K is a quadratic totally imaginary extension of F.
  - Heegner points on E, defined over abelian extensions of K
- What if F is totally real, but K is not totally imaginary?
  - There are some conjectural constructions, proposed by H. Darmon.

Outline

Elliptic curves over totally real fields

- E an elliptic curve over F.
- F a real quadratic field ($h_F^+ = 1$), so that $v_0, v_1 : F \hookrightarrow \mathbb{R}$

Elliptic curves over totally real fields

- E an elliptic curve over F.
- F a real quadratic field ($h_F^+ = 1$), so that $v_0, v_1 : F \hookrightarrow \mathbb{R}$

Modularity Theorem

There is a Hilbert modular form f_E associated to E.

Elliptic curves over totally real fields

- E an elliptic curve over F.
- F a real quadratic field ($h_F^+ = 1$), so that $v_0, v_1 \colon F \hookrightarrow \mathbb{R}$

Modularity Theorem

There is a Hilbert modular form f_E associated to E.

- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c \} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$

Elliptic curves over totally real fields

- E an elliptic curve over F.
- F a real quadratic field ($h_F^+ = 1$), so that $v_0, v_1 : F \hookrightarrow \mathbb{R}$

Modularity Theorem

There is a Hilbert modular form f_E associated to E.

- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$

Elliptic curves over totally real fields

- E an elliptic curve over F.
- F a real quadratic field ($h_F^+ = 1$), so that $v_0, v_1 : F \hookrightarrow \mathbb{R}$

Modularity Theorem

There is a Hilbert modular form f_E associated to E.

- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)

Elliptic curves over totally real fields

- E an elliptic curve over F.
- F a real quadratic field ($h_F^+ = 1$), so that $v_0, v_1 \colon F \hookrightarrow \mathbb{R}$

Modularity Theorem

There is a Hilbert modular form f_E associated to E.

- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c \} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$
- $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
- A Hilbert modular form is a function $f \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2) dz_1 dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}$.

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)

Darmon's ATR points

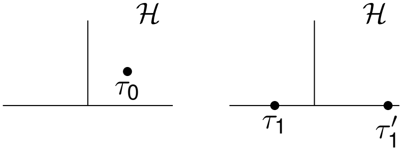
- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:

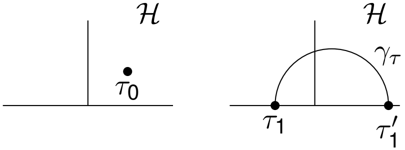
Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



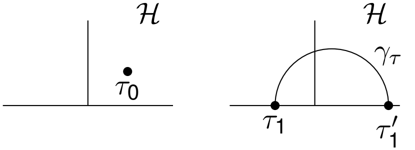
Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



Darmon's ATR points

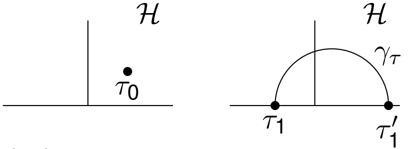
- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:

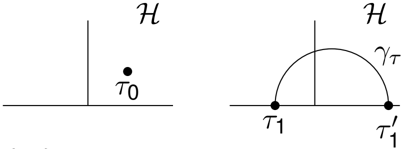


$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:

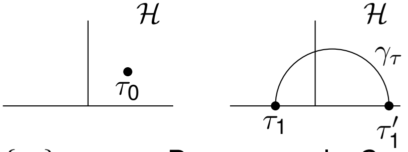


$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$
- There is a 2-dim chain Δ_τ with $\partial\Delta_\tau = C_\tau$

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:

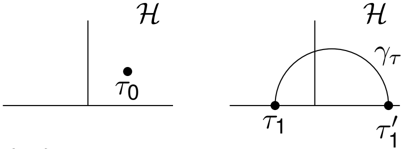


$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$
- There is a 2-dim chain Δ_τ with $\partial\Delta_\tau = C_\tau$
- Darmon point: $P_\tau = \int\int_{\Delta_\tau} \tilde{f}_E(z_1, z_2) dz_1 dz_2 \in \mathbb{C}/\Lambda_E$

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

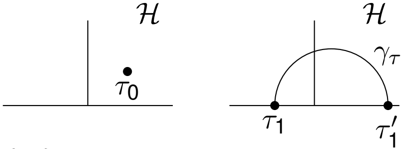
- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$
- There is a 2-dim chain Δ_τ with $\partial\Delta_\tau = C_\tau$
- Darmon point: $P_\tau = \int\int_{\Delta_\tau} \tilde{f}_E(z_1, z_2) dz_1 dz_2 \in \mathbb{C}/\Lambda_E$

Conjecture (Darmon)

$P_\tau \in E(H)$ with H a finite abelian extension of K.

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$
- There is a 2-dim chain Δ_τ with $\partial\Delta_\tau = C_\tau$
- Darmon point: $P_\tau = \int\int_{\Delta_\tau} \tilde{f}_E(z_1, z_2) dz_1 dz_2 \in \mathbb{C}/\Lambda_E$

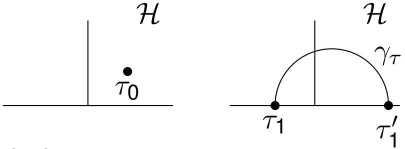
Conjecture (Darmon)

$P_\tau \in E(H)$ with H a finite abelian extension of K.

- There is some numerical evidence in support of the conjecture

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$
- There is a 2-dim chain Δ_τ with $\partial\Delta_\tau = C_\tau$
- Darmon point: $P_\tau = \int\int_{\Delta_\tau} \tilde{f}_E(z_1, z_2) dz_1 dz_2 \in \mathbb{C}/\Lambda_E$

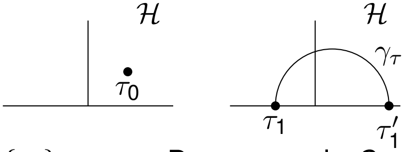
Conjecture (Darmon)

$P_\tau \in E(H)$ with H a finite abelian extension of K.

- There is some numerical evidence in support of the conjecture
- This can be generalized to arbitrary totally real F

Darmon's ATR points

- K quadratic almost totally real extension of F (one complex place and two real places)
- Assume: all primes dividing $\mathfrak{N}_E$ split in K
- Let $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H}$ and $\tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H}$:



$\{\tau_0\} \times \gamma_\tau$ in the quotient is $\simeq S^1$

- $\{\tau_0\} \times \gamma_\tau \rightsquigarrow$ Darmon cycle $C_\tau \in H^1(\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}, \mathbb{Z})$
- There is a 2-dim chain Δ_τ with $\partial\Delta_\tau = C_\tau$
- Darmon point: $P_\tau = \int\int_{\Delta_\tau} \tilde{f}_E(z_1, z_2) dz_1 dz_2 \in \mathbb{C}/\Lambda_E$

Conjecture (Darmon)

$P_\tau \in E(H)$ with H a finite abelian extension of K.

- There is some numerical evidence in support of the conjecture
- This can be generalized to arbitrary totally real F
- Our aim: propose a similar construction if F is not totally real

Outline

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$
- $\mathrm{SL}_2(\mathbb{C})$ acts on the upper half space

$$\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$$

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$
- $\mathrm{SL}_2(\mathbb{C})$ acts on the upper half space

$$\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0} \subset \mathbb{H} = \mathbb{R} \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$$

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$
- $\mathrm{SL}_2(\mathbb{C})$ acts on the upper half space

$$\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0} \subset \mathbb{H} = \mathbb{R} \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = (az + b)(cz + d)^{-1}$$

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$
- $\mathrm{SL}_2(\mathbb{C})$ acts on the upper half space

$$\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0} \subset \mathbb{H} = \mathbb{R} \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = (az + b)(cz + d)^{-1}$$

- Consider $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}_3$. It is not an algebraic variety (has real dimension 5), but it is a real differential manifold anyway.

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F\hookrightarrow\mathbb{R}$, $v_1,\bar{v}_1\colon F\hookrightarrow\mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N}\subset\mathcal{O}_F$: $\Gamma_0(\mathfrak{N})=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathcal{O}_F)\colon\mathfrak{N}\mid c\}\subset\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$
- $\mathrm{SL}_2(\mathbb{C})$ acts on the upper half space

$$\mathcal{H}_3=\mathbb{C}\times\mathbb{R}_{>0}\subset\mathbb{H}=\mathbb{R}\oplus\mathbb{R}\cdot i\oplus\mathbb{R}\cdot j\oplus\mathbb{R}\cdot k$$

$$\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\cdot z=(az+b)(cz+d)^{-1}$$

- Consider $\Gamma_0(\mathfrak{N})\backslash\mathcal{H}\times\mathcal{H}_3$. It is not an algebraic variety (has real dimension 5), but it is a real differential manifold anyway.

Generalized Modularity Conjecture

There is a harmonic differential 2-form ω_E on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H}\times\mathcal{H}_3$ associated to E.

Modular forms and modularity

- $F/\mathbb{Q}$ cubic field of signature $(1,1)$: $v_0\colon F \hookrightarrow \mathbb{R}$, $v_1, \bar{v}_1\colon F \hookrightarrow \mathbb{C}$.
- Let E be an elliptic curve over F.
- $\mathfrak{N} \subset \mathcal{O}_F$: $\Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{C})$.
- $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half plane $\mathcal{H}$
- $\mathrm{SL}_2(\mathbb{C})$ acts on the upper half space

$$\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0} \subset \mathbb{H} = \mathbb{R} \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = (az+b)(cz+d)^{-1}$$

- Consider $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}_3$. It is not an algebraic variety (has real dimension 5), but it is a real differential manifold anyway.

Generalized Modularity Conjecture

There is a harmonic differential 2-form ω_E on $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}_3$ associated to E.

- As before, ω_E is determined by its Fourier–Bessel expansion.

- ω_E has a "Fourier-Bessel expansion":

$$\omega_E(z,x,y)=\sum_{\substack{\alpha\in\mathcal{O}_F\\ \alpha_0>0}}\frac{a_{(\alpha)}}{N_{F/\mathbb{Q}}(\alpha)}\frac{\alpha_0}{\delta_0}\exp\left(-2\pi i\left(\frac{\alpha_0\bar{z}}{\delta_0}+\frac{\alpha_1 x}{\delta_1}+\frac{\alpha_2\bar{x}}{\delta_2}\right)\right)\mathbb{K}\left(\frac{\alpha_1 y}{\delta_1}\right)\cdot\begin{pmatrix}\frac{-dx}{y}\wedge d\bar{z}\\ \frac{dy}{y}\wedge d\bar{z}\\ \frac{d\bar{x}}{y}\wedge d\bar{z}\end{pmatrix}$$

- ω_E has a "Fourier-Bessel expansion":

$$\omega_E(z,x,y) = \sum_{\substack{\alpha\in\mathcal{O}_F\\ \alpha_0>0}} \frac{a_{(\alpha)}}{N_{F/\mathbb{Q}}(\alpha)} \frac{\alpha_0}{\delta_0} \exp\left(-2\pi i\left(\frac{\alpha_0\bar{z}}{\delta_0}+\frac{\alpha_1 x}{\delta_1}+\frac{\alpha_2\bar{x}}{\delta_2}\right)\right)\mathbb{K}\left(\frac{\alpha_1 y}{\delta_1}\right)\cdot\begin{pmatrix}\frac{-dx}{y}\wedge d\bar{z}\\ \frac{dy}{y}\wedge d\bar{z}\\ \frac{d\bar{x}}{y}\wedge d\bar{z}\end{pmatrix}$$

- It is completely determined by its Fourier coefficients $a_{(\alpha)}$

- ω_E has a "Fourier-Bessel expansion":

$$\omega_E(z,x,y) = \sum_{\substack{\alpha\in\mathcal{O}_F\\ \alpha_0>0}} \frac{a_{(\alpha)}}{N_{F/\mathbb{Q}}(\alpha)} \frac{\alpha_0}{\delta_0} \exp\left(-2\pi i\left(\frac{\alpha_0\bar{z}}{\delta_0} + \frac{\alpha_1 x}{\delta_1} + \frac{\alpha_2\bar{x}}{\delta_2}\right)\right) \mathbb{K}\left(\frac{\alpha_1 y}{\delta_1}\right)\cdot\begin{pmatrix}\frac{-dx}{y}\wedge d\bar{z}\\ \frac{dy}{y}\wedge d\bar{z}\\ \frac{d\bar{x}}{y}\wedge d\bar{z}\end{pmatrix}$$

- It is completely determined by its Fourier coefficients $a_{(\alpha)}$
- We can compute the $a_{(\alpha)}$ by counting points on $E(\mathcal{O}_F/\mathfrak{p})$

Construction of the points

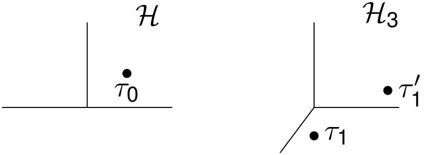
- K a totally imaginary quadratic extension of F

Construction of the points

- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$

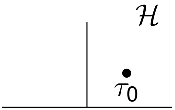
Construction of the points

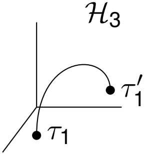
- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$



Construction of the points

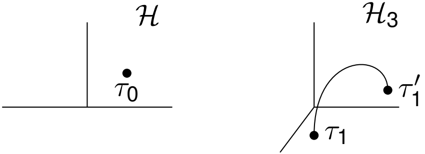
- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$





Construction of the points

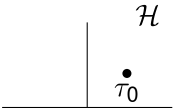
- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$

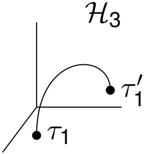


- This gives a 1-cycle C_τ on $\Gamma_0(\mathfrak{N}) \setminus \mathcal{H} \times \mathcal{H}_3$

Construction of the points

- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$

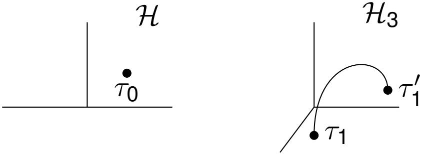




- This gives a 1-cycle C_τ on $\Gamma_0(\mathfrak{N}) \setminus \mathcal{H} \times \mathcal{H}_3$
- As before, there is a 2-dimensional chain Δ_τ such that $\partial\Delta_\tau = C_\tau$

Construction of the points

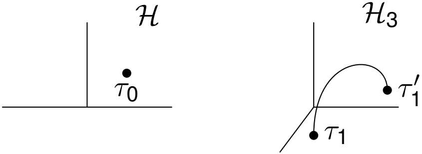
- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$



- This gives a 1-cycle C_τ on $\Gamma_0(\mathfrak{N}) \setminus \mathcal{H} \times \mathcal{H}_3$
- As before, there is a 2-dimensional chain Δ_τ such that $\partial\Delta_\tau = C_\tau$
- Define: $P_\tau = \int\int_{\Delta_\tau} \omega_E \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$

Construction of the points

- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$



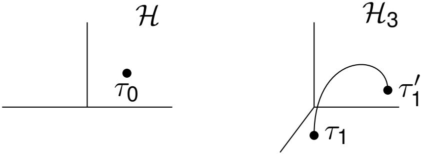
- This gives a 1-cycle C_τ on $\Gamma_0(\mathfrak{N}) \setminus \mathcal{H} \times \mathcal{H}_3$
- As before, there is a 2-dimensional chain Δ_τ such that $\partial\Delta_\tau = C_\tau$
- Define: $P_\tau = \int\int_{\Delta_\tau} \omega_E \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$

Conjecture

$P_\tau \in E(H)$ with H a finite abelian extension of K.

Construction of the points

- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$



- This gives a 1-cycle C_τ on $\Gamma_0(\mathfrak{N}) \setminus \mathcal{H} \times \mathcal{H}_3$
- As before, there is a 2-dimensional chain Δ_τ such that $\partial\Delta_\tau = C_\tau$
- Define: $P_\tau = \int\int_{\Delta_\tau} \omega_E \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$

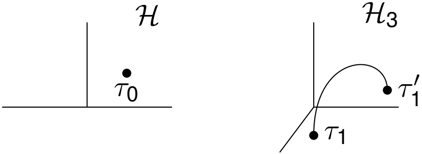
Conjecture

$P_\tau \in E(H)$ with H a finite abelian extension of K.

- C_τ is not algebraic, but also this time $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}_3$ is neither!

Construction of the points

- K a totally imaginary quadratic extension of F
- $\tau \in K \setminus F \rightsquigarrow \tau_0 \in \mathcal{H},\ \tau_1, \tau_1' \in \mathbb{C} = \partial\mathcal{H}_3$



- This gives a 1-cycle C_τ on $\Gamma_0(\mathfrak{N}) \setminus \mathcal{H} \times \mathcal{H}_3$
- As before, there is a 2-dimensional chain Δ_τ such that $\partial\Delta_\tau = C_\tau$
- Define: $P_\tau = \int\int_{\Delta_\tau} \omega_E \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$

Conjecture

$P_\tau \in E(H)$ with H a finite abelian extension of K.

- C_τ is not algebraic, but also this time $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}_3$ is neither!
- We found some numerical evidence for the conjecture.

Outline

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$

$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$
$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$
- $K = F(w)$, where w satisfies $w^2 + (r+1)w + 2r^2 - 3r + 3$.

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$
$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$
- $K = F(w)$, where w satisfies $w^2 + (r+1)w + 2r^2 - 3r + 3$.
- Take τ to be equal to w

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$

$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$

- $K = F(w)$, where w satisfies $w^2 + (r+1)w + 2r^2 - 3r + 3$.
- Take τ to be equal to w
- $\mathrm{Stab}_{\tau_0}(\Gamma_0(\mathfrak{N})) = \langle \gamma_\tau \rangle$ with $\gamma_\tau = \begin{pmatrix} -4r-3 & -r^2+2r+3 \\ -2r^2-4r-3 & -r^2+4r+2 \end{pmatrix}$

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$

$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$

- $K = F(w)$, where w satisfies $w^2 + (r+1)w + 2r^2 - 3r + 3$.
- Take τ to be equal to w
- $\mathrm{Stab}_{\tau_0}(\Gamma_0(\mathfrak{N})) = \langle \gamma_\tau \rangle$ with $\gamma_\tau = \begin{pmatrix} -4r-3 & -r^2+2r+3 \\ -2r^2-4r-3 & -r^2+4r+2 \end{pmatrix}$
- Finding Δ_τ such that $\partial \Delta_\tau = C_\tau$ can be reduced to decompose γ_τ into elementary matrices (effective congruence subgroup problem).

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$

$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$

- $K = F(w)$, where w satisfies $w^2 + (r+1)w + 2r^2 - 3r + 3$.
- Take τ to be equal to w
- $\mathrm{Stab}_{\tau_0}(\Gamma_0(\mathfrak{N})) = \langle \gamma_\tau \rangle$ with $\gamma_\tau = \begin{pmatrix} -4r-3 & -r^2+2r+3 \\ -2r^2-4r-3 & -r^2+4r+2 \end{pmatrix}$
- Finding Δ_τ such that $\partial \Delta_\tau = C_\tau$ can be reduced to decompose γ_τ into elementary matrices (effective congruence subgroup problem).
- $P_\tau = \sum_i \int_{\tau_i^1}^{\tau_i^2} \int_O^{\gamma_i \cdot O} \omega_E \simeq 0.141967077 - 0.055099463\sqrt{-1}$

A concrete calculation

- $F = \mathbb{Q}(r)$ with $r^3 - r^2 + 1$

$$E : y^2 + (r-1)xy + (r^2 - r)y = x^3 + (-r^2 - 1)x^2 + r^2 x.$$

- $K = F(w)$, where w satisfies $w^2 + (r+1)w + 2r^2 - 3r + 3$.
- Take τ to be equal to w
- $\mathrm{Stab}_{\tau_0}(\Gamma_0(\mathfrak{N})) = \langle \gamma_\tau \rangle$ with $\gamma_\tau = \begin{pmatrix} -4r-3 & -r^2+2r+3 \\ -2r^2-4r-3 & -r^2+4r+2 \end{pmatrix}$
- Finding Δ_τ such that $\partial \Delta_\tau = C_\tau$ can be reduced to decompose γ_τ into elementary matrices (effective congruence subgroup problem).
- $P_\tau = \sum_i \int_{\tau_i^1}^{\tau_i^2} \int_O^{\gamma_i \cdot O} \omega_E \simeq 0.141967077 - 0.055099463\sqrt{-1}$
- The image of $J_\tau \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$ coincides (up to 32 digits of accuracy) with $10P$, where

$$P = \left(r - 1 : w - r^2 + 2r : 1 \right) \in E(K)$$

Modular forms over cubic fields and algebraic points on elliptic curves

Xevi Guitart[1] Marc Masdeu[2] Haluk Sengun[2]

[1]Institute for Experimental Mathematics, Essen

[2]University of Warwick

Bilbo