Oda’s Conjecture

Cardedeu 28 Novembre 2024

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E, s) = L(f, s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E, s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E, s) = L(f, s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E, s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E,s) = L(f,s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E,s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that
1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E,s) = L(f,s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E,s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E,s) = L(f,s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E,s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E,s) = L(f,s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E,s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f,s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E,s) = L(f,s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E,s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

The Modularity Theorem

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ is modular.

- Modularity has two equivalent meanings:
  1. $\exists$ newform $f = \sum a_n q^n \in S_2(N)$ with $a_n \in \mathbb{Q}$ and $L(E,s) = L(f,s)$.
  2. E is a quotient of $\mathrm{Jac}(X_0(N))$.
- 1 gives analytic continuation of $L(E,s)$
- 2 provides Heegner points: $X_0(N) \to \mathrm{Jac}(X_0(N)) \to E$

Eichler-Shimura construction

Given $f \in S_2(N)$ eigenform with $a_n(f) \in \mathbb{Q}$, constructs $E_f/\mathbb{Q}$ such that

1. $L(E_f, s) = L(f, s)$
2. E_f is a quotient of $\mathrm{Jac}(X_0(N))$

- Modularity theorem: every $E/\mathbb{Q}$ arises from this construction

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C}\colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f\colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c \}$ acts on $\mathcal{H} = \{ z \in \mathbb{C} \colon \mathrm{Im}(z) > 0 \}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N) \backslash \mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N) \backslash \mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega \right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c \}$ acts on $\mathcal{H} = \{ z \in \mathbb{C} \colon \mathrm{Im}(z) > 0 \}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N) \backslash \mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N) \backslash \mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega \right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^{\vee} \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^{\vee}/H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C}\colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f\colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
- $$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

Modular forms, curves, and Jacobians (over $\mathbb{C}$)

- $\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \colon N \mid c\}$ acts on $\mathcal{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$
- The modular curve $X_0(N)/\mathbb{Q} \rightsquigarrow X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*$

Definition of modular form of weight 2

$f \colon \mathcal{H} \longrightarrow \mathbb{C}$ holomorphic s.t. $\omega_f = f(z)dz$ invariant by action of $\Gamma_0(N)$

- ω_f descends to a differential on $\Gamma_0(N)\backslash\mathcal{H}$
- If f is a cusp form ω_f descends to a differential on $X_0(N)$
- $S_2(\Gamma_0(N)) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1)$
- $H^1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{C} \simeq H^1_{\mathrm{dR}}(X_0(N)_{\mathbb{C}}) \simeq H^0(X_0(N)_{\mathbb{C}}, \Omega^1) \oplus \overline{H^0(X_0(N)_{\mathbb{C}}, \Omega^1)}$
-
$$\begin{array}{ccc} H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z}) & \hookrightarrow & H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee \\ \gamma & \longmapsto & \left(\omega \mapsto \int_\gamma \omega\right) \end{array}$$
- $\mathrm{Jac}(X_0(N)) = H^0(X_0(N)_{\mathbb{C}}, \Omega^1)^\vee / H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- To break it: use Hecke operators

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p \gamma = a_p(f)\gamma$ for all $p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p\gamma = a_p(f)\gamma \text{ for all } p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p \gamma = a_p(f)\gamma \text{ for all } p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p\gamma = a_p(f)\gamma$ for all $p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p\gamma = a_p(f)\gamma$ for all $p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^{+} + \mathbb{Z}\Omega^{-})$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^{+}}{\Omega^{-}}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p \gamma = a_p(f)\gamma \text{ for all } p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z}) \colon T_p\gamma = a_p(f)\gamma \text{ for all } p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

The period lattice of E_f

- Hecke operators T_p acting on $S_2(\Gamma_0(N))$ and $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})$
- f newform with rational coefficients: $T_p f = a_p(f) f$ with $a_p(f) \in \mathbb{Z}$
- $H_1(X_0(N), \mathbb{Z})_f = \{\gamma \in H_1(X_0(N), \mathbb{Z})\colon T_p\gamma = a_p(f)\gamma \text{ for all } p\}$
- $\Lambda_f = \{\int_\gamma \omega_f \colon \gamma \in H_1(X_0(N)_{\mathbb{C}}, \mathbb{Z})_f\}$ is a lattice in $\mathbb{C}$ and $\mathbb{C}/\Lambda_f \sim E_{f,\mathbb{C}}$
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f$ has rank 2, and we can decompose it further using the action of complex conjugation
- $H_1(X_0(N)_{\mathbb{C}}, \mathbb{Q})_f^{\pm} = \langle \gamma^{\pm} \rangle \rightsquigarrow \Omega^{\pm} = \int_{\gamma^{\pm}} \omega_f$
- $E_{f,\mathbb{C}} \sim \mathbb{C}/(\mathbb{Z}\Omega^+ + \mathbb{Z}\Omega^-)$
- The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega^+}{\Omega^-}$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F:\mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2 : F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) : \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2)dz_1dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F:\mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2\colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2)dz_1 dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F\colon \mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2\colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2)dz_1 dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F:\mathbb{Q}]=2$, so that there are two infinite places $v_1, v_2 \colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E,s)=L(E,s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H}\times\mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H}\times\mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ such that $f(z_1,z_2)dz_1dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H}\times\mathcal{H}$.
- f_E is associated to E means that $L(f_E,s)=L(E,s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F:\mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2 \colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c \} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2) dz_1 dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F\colon \mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2\colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2)dz_1dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F\colon \mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2\colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2)dz_1 dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F:\mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2 \colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F) \colon \mathfrak{N} \mid c \} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2) dz_1 dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N}) \backslash \mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Modularity over totally real fields

- E an elliptic curve over a totally real F
- For simplicity of exposition assume
  - $[F:\mathbb{Q}] = 2$, so that there are two infinite places $v_1, v_2\colon F \hookrightarrow \mathbb{R}$
  - F has narrow class number 1

Modularity Theorem

Every elliptic curve E/F is modular: there is a Hilbert modular form f_E such that $L(f_E, s) = L(E, s)$.

- $\mathfrak{N} \subset \mathcal{O}_F \rightsquigarrow \Gamma_0(\mathfrak{N}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}_F)\colon \mathfrak{N} \mid c\} \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
- $\Gamma_0(\mathfrak{N})$ acts on $\mathcal{H} \times \mathcal{H}$ diagonally via v_1 and v_2
- $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$ is an algebraic surface (the Hilbert modular surface)
  - Its compactification admits a model $X_0(\mathfrak{N})/\mathbb{Q}$
- A Hilbert modular form is a function $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $f(z_1, z_2)dz_1dz_2$ descends to a differential on $\Gamma_0(\mathfrak{N})\backslash\mathcal{H} \times \mathcal{H}$.
- f_E is associated to E means that $L(f_E, s) = L(E, s)$
- But now there is no morphism $X_0(\mathfrak{N}) \longrightarrow E$

Eichler–Shimura over totally real fields

Conjecture (Eichler-Shimura construction)

Given a Hilbert modular newform f of weight 2 and $a_{\mathfrak{p}}(f) \in \mathbb{Q}$, there exists an elliptic curve E_f/F such that $L(E_f, s) = L(f, s)$

- It is implied by the Hodge conjecture (Blasius)
- Known if $[F : \mathbb{Q}]$ is odd or there is a prime $\mathfrak{q} \mid\mid \mathfrak{N}$
  - In this case the system $\{a_{\mathfrak{p}}(f)\}$ of $T_{\mathfrak{p}}$-eigenvalues of f appears in the cohomology of a Shimura curve, and we have geometry.

Eichler–Shimura over totally real fields

Conjecture (Eichler-Shimura construction)

Given a Hilbert modular newform f of weight 2 and $a_{\mathfrak{p}}(f) \in \mathbb{Q}$, there exists an elliptic curve E_f/F such that $L(E_f, s) = L(f, s)$

- It is implied by the Hodge conjecture (Blasius)
- Known if $[F : \mathbb{Q}]$ is odd or there is a prime $\mathfrak{q} \mid\mid \mathfrak{N}$
  - In this case the system $\{a_{\mathfrak{p}}(f)\}$ of $T_{\mathfrak{p}}$-eigenvalues of f appears in the cohomology of a Shimura curve, and we have geometry.

Eichler–Shimura over totally real fields

Conjecture (Eichler-Shimura construction)

Given a Hilbert modular newform f of weight 2 and $a_{\mathfrak{p}}(f) \in \mathbb{Q}$, there exists an elliptic curve E_f/F such that $L(E_f, s) = L(f, s)$

- It is implied by the Hodge conjecture (Blasius)
- Known if $[F : \mathbb{Q}]$ is odd or there is a prime $\mathfrak{q} \mid\mid \mathfrak{N}$
  - In this case the system $\{a_{\mathfrak{p}}(f)\}$ of $T_{\mathfrak{p}}$-eigenvalues of f appears in the cohomology of a Shimura curve, and we have geometry.

Eichler–Shimura over totally real fields

Conjecture (Eichler-Shimura construction)

Given a Hilbert modular newform f of weight 2 and $a_{\mathfrak{p}}(f) \in \mathbb{Q}$, there exists an elliptic curve E_f/F such that $L(E_f, s) = L(f, s)$

- It is implied by the Hodge conjecture (Blasius)
- Known if $[F : \mathbb{Q}]$ is odd or there is a prime $\mathfrak{q} \mid\mid \mathfrak{N}$
  - In this case the system $\{a_{\mathfrak{p}}(f)\}$ of $T_{\mathfrak{p}}$-eigenvalues of f appears in the cohomology of a Shimura curve, and we have geometry.

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^{\pm} = \int_{\gamma^{\pm}} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F:\mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M}) \backslash \mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^{\pm} = \int_{\gamma^{\pm}} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F:\mathbb{Q}]=2$, $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\mathfrak{q}\mathfrak{M}$ with $\mathfrak{q}\nmid\mathfrak{M}$
- B/F quaternion algebra: $B=F+iF+jF+ijF$, $i^2,j^2\in F$, $ij=-ji$
  - splits at a single real place: $B\otimes_{F,v_1}\mathbb{R}\simeq \mathrm{M}_2(\mathbb{R})$ and $B\otimes_{F,v_2}\mathbb{R}\simeq\mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B\otimes_F F_{\mathfrak{l}}\not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l}=\mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M})\subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M})\subset (B\otimes_{F,v_1}\mathbb{R})^\times\subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}\rightsquigarrow X_0^B(\mathfrak{M})=\Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon\mathcal{H}\to\mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B\in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}}f_B=a_{\mathfrak{p}}(f)f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B}=\{\int_\gamma\omega_{f_B}:\gamma\in H_1(X_0^B(\mathfrak{M}),\mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}}\sim\mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm=\int_{\gamma^\pm}\omega_f$
  - $E_{f,\mathbb{C}}\sim\mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_{\mathfrak{l}} \not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}} f_B = a_{\mathfrak{p}}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \simeq \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \simeq \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F:\mathbb{Q}]=2$, $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\mathfrak{q}\mathfrak{M}$ with $\mathfrak{q}\nmid\mathfrak{M}$
- B/F quaternion algebra: $B=F+iF+jF+ijF$, $i^2,j^2\in F$, $ij=-ji$
  - splits at a single real place: $B\otimes_{F,v_1}\mathbb{R}\simeq \mathrm{M}_2(\mathbb{R})$ and $B\otimes_{F,v_2}\mathbb{R}\simeq\mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B\otimes_F F_{\mathfrak{l}}\not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l}=\mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M})\subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M})\subset(B\otimes_{F,v_1}\mathbb{R})^\times\subset\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}\rightsquigarrow X_0^B(\mathfrak{M})=\Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon\mathcal{H}\to\mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B\in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}}f_B=a_{\mathfrak{p}}(f)f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B}=\{\int_\gamma\omega_{f_B}\colon\gamma\in H_1(X_0^B(\mathfrak{M}),\mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}}\sim\mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^{\pm}=\int_{\gamma^\pm}\omega_f$
  - $E_{f,\mathbb{C}}\sim\mathbb{C}/\mathbb{Z}\Omega_B^+ +\mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F:\mathbb{Q}]=2$, $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\mathfrak{q}\mathfrak{M}$ with $\mathfrak{q}\nmid\mathfrak{M}$
- B/F quaternion algebra: $B=F+iF+jF+ijF$, $i^2,j^2\in F$, $ij=-ji$
  - splits at a single real place: $B\otimes_{F,v_1}\mathbb{R}\simeq \mathrm{M}_2(\mathbb{R})$ and $B\otimes_{F,v_2}\mathbb{R}\simeq\mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B\otimes_F F_{\mathfrak{l}}\not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l}=\mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M})\subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M})\subset(B\otimes_{F,v_1}\mathbb{R})^\times\subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}\rightsquigarrow X_0^B(\mathfrak{M})=\Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon\mathcal{H}\to\mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B\in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}}f_B=a_{\mathfrak{p}}(f)f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B}=\{\int_\gamma\omega_{f_B}\colon\gamma\in H_1(X_0^B(\mathfrak{M}),\mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}}\sim\mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm=\int_{\gamma^\pm}\omega_f$
  - $E_{f,\mathbb{C}}\sim\mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_\mathfrak{l} \not\simeq \mathrm{M}_2(F_\mathfrak{l}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_\mathfrak{p} f_B = a_\mathfrak{p}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} : \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F\colon \mathbb{Q}] = 2$, $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \mathfrak{q}\mathfrak{M}$ with $\mathfrak{q} \nmid \mathfrak{M}$
- B/F quaternion algebra: $B = F + iF + jF + ijF$, $i^2, j^2 \in F$, $ij = -ji$
  - splits at a single real place: $B \otimes_{F,v_1} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})$ and $B \otimes_{F,v_2} \mathbb{R} \simeq \mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B \otimes_F F_\mathfrak{l} \not\simeq \mathrm{M}_2(F_\mathfrak{l}) \iff \mathfrak{l} = \mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M}) \subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M}) \subset (B \otimes_{F,v_1} \mathbb{R})^\times \subset \mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H} \rightsquigarrow X_0^B(\mathfrak{M}) = \Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon \mathcal{H} \to \mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B \in S_2^B(\mathfrak{M})$ such that $T_\mathfrak{p} f_B = a_\mathfrak{p}(f) f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B} = \{\int_\gamma \omega_{f_B} \colon \gamma \in H_1(X_0^B(\mathfrak{M}), \mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}} \sim \mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm = \int_{\gamma^\pm} \omega_f$
  - $E_{f,\mathbb{C}} \sim \mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Shimura curves

- $[F:\mathbb{Q}]=2$, $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\mathfrak{q}\mathfrak{M}$ with $\mathfrak{q}\nmid\mathfrak{M}$
- B/F quaternion algebra: $B=F+iF+jF+ijF$, $i^2,j^2\in F$, $ij=-ji$
  - splits at a single real place: $B\otimes_{F,v_1}\mathbb{R}\simeq \mathrm{M}_2(\mathbb{R})$ and $B\otimes_{F,v_2}\mathbb{R}\simeq\mathbb{H}$
  - has discriminant $\mathfrak{q}$: $B\otimes_F F_{\mathfrak{l}}\not\simeq \mathrm{M}_2(F_{\mathfrak{l}}) \iff \mathfrak{l}=\mathfrak{q}$
- $\Gamma_0^B(\mathfrak{M})\subset B^\times$ arithmetic group
  - $\Gamma_0^B(\mathfrak{M})\subset(B\otimes_{F,v_1}\mathbb{R})^\times\subset\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{H}\rightsquigarrow X_0^B(\mathfrak{M})=\Gamma_0^B(\mathfrak{M})\backslash\mathcal{H}$
  - $X_0^B(\mathfrak{M})$ is a curve, and has a model over F
- Modular form: $g\colon\mathcal{H}\to\mathbb{C}$ s.t. $g(z)dz$ descends to $X_0^B(\mathfrak{M})$

Jacquet-Langlands correspondence

There is $f_B\in S_2^B(\mathfrak{M})$ such that $T_{\mathfrak{p}}f_B=a_{\mathfrak{p}}(f)f_B$ for almost all $\mathfrak{p}$

- E_f is a factor of $\mathrm{Jac}(X_0^B(\mathfrak{M}))$
  - $\Lambda_{f_B}=\{\int_\gamma\omega_{f_B}\colon\gamma\in H_1(X_0^B(\mathfrak{M}),\mathbb{Z})_{f_B}\}$ is a lattice in $\mathbb{C}$ and $E_{f,\mathbb{C}}\sim\mathbb{C}/\Lambda_{f_B}$
  - $\Omega_B^\pm=\int_{\gamma^\pm}\omega_f$
  - $E_{f,\mathbb{C}}\sim\mathbb{C}/\mathbb{Z}\Omega_B^+ + \mathbb{Z}\Omega_B^-$
  - The period of $E_{f,\mathbb{C}}$ (up to isogeny) is $\frac{\Omega_B^+}{\Omega_B^-}$

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F : \mathbb{Q}] = 2$ and $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \square$?
- $H^2(X_0(\mathfrak{N}), \mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{++} \oplus H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{+-} \oplus H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{-+} \oplus H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{--}$
  - $\Omega^{++} = \int_{\gamma^{++}} \omega_f$, $\Omega^{+-} = \int_{\gamma^{+-}} \omega_f$, $\Omega^{-+} = \int_{\gamma^{-+}} \omega_f$, $\Omega^{--} = \int_{\gamma^{--}} \omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f \otimes_{v_1} \mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}} \doteq \frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F\colon \mathbb{Q}] = 2$ and $f \in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N} = \square$?
- $H^2(X_0(\mathfrak{N}), \mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{++} \oplus H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{+-} \oplus H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{-+} \oplus H^2(X_0(\mathfrak{N}), \mathbb{Q})_f^{--}$
  - $\Omega^{++} = \int_{\gamma^{++}} \omega_f,\ \Omega^{+-} = \int_{\gamma^{+-}} \omega_f,\ \Omega^{-+} = \int_{\gamma^{-+}} \omega_f,\ \Omega^{--} = \int_{\gamma^{--}} \omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f \otimes_{v_1} \mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}} \doteq \frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Oda Conjecture

- What happens if $[F:\mathbb{Q}]=2$ and $f\in S_2(\Gamma_0(\mathfrak{N}))$ and $\mathfrak{N}=\square$?
- $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f$ now has rank 4
  - Now we have two involutions T_{v_1} and T_{v_2}
  - $H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{++}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{+-}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{-+}\oplus H^2(X_0(\mathfrak{N}),\mathbb{Q})_f^{--}$
  - $\Omega^{++}=\int_{\gamma^{++}}\omega_f,\ \Omega^{+-}=\int_{\gamma^{+-}}\omega_f,\ \Omega^{-+}=\int_{\gamma^{-+}}\omega_f,\ \Omega^{--}=\int_{\gamma^{--}}\omega_f$

Oda Conjecture

$\frac{\Omega^{++}}{\Omega^{-+}}$ is the period of $E_f\otimes_{v_1}\mathbb{R}$

- Oda: it is true if f is a base change from a form $f_0/\mathbb{Q}$

Theorem (G-Molina, in progress)

Oda's conjecture is true if f admits a JL lift f_B to a Shimura curve

- Strategy: show that $\frac{\Omega^{++}}{\Omega^{-+}}\doteq\frac{\Omega_B^{+}}{\Omega_B^{-}}$ using the relation of periods to special values of L-functions.

Relation to L-functions

- f Hilbert modular form $\rightsquigarrow \Omega^{++}, \Omega^{-+}$
- f_B quaternionic form over $B \rightsquigarrow \Omega_B^{+}, \Omega_B^{-}$

Theorem (Shimura)

Let $\chi_K : \mathrm{Gal}(K/F) \to \{\pm 1\}$ and $\varepsilon_i \in \{\pm\}$, $\varepsilon_i = +$ if v_i splits

$$L(1, f \otimes \chi_K) \doteq \Omega^{\varepsilon_0 \varepsilon_1}$$

Theorem (Molina)

M/F quadratic, split at v_1 and ramified at v_2
$\xi : \mathrm{Gal}(N/M) \to \{\pm 1\}$ anticyclotomic, $\varepsilon = +$ iff the places above v_1 split

$$L(1, f_M \otimes \xi) \doteq (\Omega_B^{\varepsilon})^2$$

- If $N = M \cdot K$ then $L(1, f_M \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_K \chi_M)$
- If $\mathfrak{N} \neq \square$ can find τ of any infinity type such that $L(1, f \otimes \tau) \neq 0$

Relation to L-functions

- f Hilbert modular form $\rightsquigarrow \Omega^{++}, \Omega^{-+}$
- f_B quaternionic form over $B \rightsquigarrow \Omega_B^{+}, \Omega_B^{-}$

Theorem (Shimura)

Let $\chi_K : \mathrm{Gal}(K/F) \to \{\pm 1\}$ and $\varepsilon_i \in \{\pm\}$, $\varepsilon_i = +$ if v_i splits

$$L(1, f \otimes \chi_K) \doteq \Omega^{\varepsilon_0 \varepsilon_1}$$

Theorem (Molina)

M/F quadratic, split at v_1 and ramified at v_2
$\xi : \mathrm{Gal}(N/M) \to \{\pm 1\}$ anticyclotomic, $\varepsilon = +$ iff the places above v_1 split

$$L(1, f_M \otimes \xi) \doteq (\Omega_B^{\varepsilon})^2$$

- If $N = M \cdot K$ then $L(1, f_M \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_K \chi_M)$
- If $\mathfrak{N} \neq \square$ can find τ of any infinity type such that $L(1, f \otimes \tau) \neq 0$

Relation to L-functions

- f Hilbert modular form $\rightsquigarrow \Omega^{++}, \Omega^{-+}$
- f_B quaternionic form over $B \rightsquigarrow \Omega_B^{+}, \Omega_B^{-}$

Theorem (Shimura)

Let $\chi_K : \mathrm{Gal}(K/F) \to \{\pm 1\}$ and $\varepsilon_i \in \{\pm\}$, $\varepsilon_i = +$ if v_i splits

$$L(1, f \otimes \chi_K) \doteq \Omega^{\varepsilon_0 \varepsilon_1}$$

Theorem (Molina)

M/F quadratic, split at v_1 and ramified at v_2
$\xi : \mathrm{Gal}(N/M) \to \{\pm 1\}$ anticyclotomic, $\varepsilon = +$ iff the places above v_1 split

$$L(1, f_M \otimes \xi) \doteq (\Omega_B^{\varepsilon})^2$$

- If $N = M \cdot K$ then $L(1, f_M \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_K \chi_M)$
- If $\mathfrak{N} \neq \square$ can find τ of any infinity type such that $L(1, f \otimes \tau) \neq 0$

Relation to L-functions

- f Hilbert modular form $\rightsquigarrow \Omega^{++}, \Omega^{-+}$
- f_B quaternionic form over $B \rightsquigarrow \Omega_B^+, \Omega_B^-$

Theorem (Shimura)

Let $\chi_K \colon \mathrm{Gal}(K/F) \to \{\pm 1\}$ and $\varepsilon_i \in \{\pm\}$, $\varepsilon_i = +$ if v_i splits

$$L(1, f \otimes \chi_K) \doteq \Omega^{\varepsilon_0 \varepsilon_1}$$

Theorem (Molina)

M/F quadratic, split at v_1 and ramified at v_2
$\xi \colon \mathrm{Gal}(N/M) \to \{\pm 1\}$ anticyclotomic, $\varepsilon = +$ iff the places above v_1 split

$$L(1, f_M \otimes \xi) \doteq (\Omega_B^{\varepsilon})^2$$

- If $N = M \cdot K$ then $L(1, f_M \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_K \chi_M)$
- If $\mathfrak{N} \neq \square$ can find τ of any infinity type such that $L(1, f \otimes \tau) \neq 0$

Relation to L-functions

- f Hilbert modular form $\rightsquigarrow \Omega^{++}, \Omega^{-+}$
- f_B quaternionic form over $B \rightsquigarrow \Omega_B^{+}, \Omega_B^{-}$

Theorem (Shimura)

Let $\chi_K \colon \mathrm{Gal}(K/F) \to \{\pm 1\}$ and $\varepsilon_i \in \{\pm\}$, $\varepsilon_i = +$ if v_i splits

$$L(1, f \otimes \chi_K) \doteq \Omega^{\varepsilon_0 \varepsilon_1}$$

Theorem (Molina)

M/F quadratic, split at v_1 and ramified at v_2
$\xi \colon \mathrm{Gal}(N/M) \to \{\pm 1\}$ anticyclotomic, $\varepsilon = +$ iff the places above v_1 split

$$L(1, f_M \otimes \xi) \doteq (\Omega_B^{\varepsilon})^2$$

- If $N = M \cdot K$ then $L(1, f_M \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_K \chi_M)$
- If $\mathfrak{N} \neq \square$ can find τ of any infinity type such that $L(1, f \otimes \tau) \neq 0$

Relation to L-functions

- f Hilbert modular form $\rightsquigarrow \Omega^{++}, \Omega^{-+}$
- f_B quaternionic form over $B \rightsquigarrow \Omega_B^{+}, \Omega_B^{-}$

Theorem (Shimura)

Let $\chi_K\colon \mathrm{Gal}(K/F) \to \{\pm 1\}$ and $\varepsilon_i \in \{\pm\}$, $\varepsilon_i = +$ if v_i splits

$$L(1, f \otimes \chi_K) \doteq \Omega^{\varepsilon_0 \varepsilon_1}$$

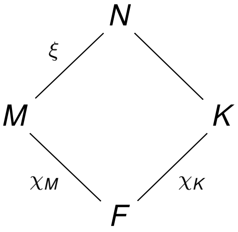
Theorem (Molina)

M/F quadratic, split at v_1 and ramified at v_2
$\xi\colon \mathrm{Gal}(N/M) \to \{\pm 1\}$ anticyclotomic, $\varepsilon = +$ iff the places above v_1 split

$$L(1, f_M \otimes \xi) \doteq (\Omega_B^{\varepsilon})^2$$

- If $N = M \cdot K$ then $L(1, f_M \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_K \chi_M)$
- If $\mathfrak{N} \neq \square$ can find τ of any infinity type such that $L(1, f \otimes \tau) \neq 0$

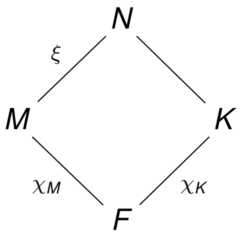
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

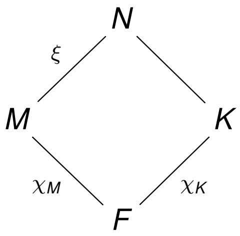
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

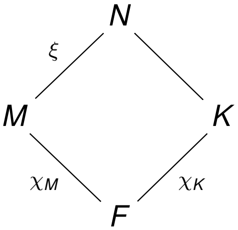
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

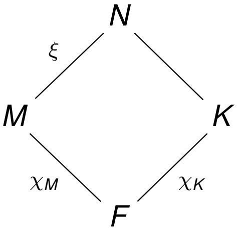
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

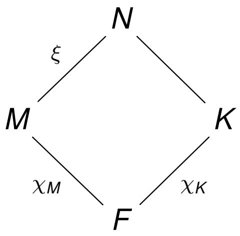
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

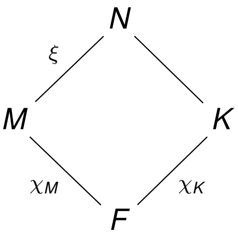
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

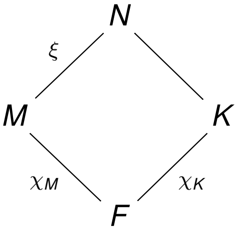
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} \doteq \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

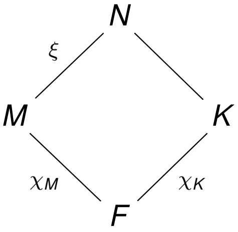
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

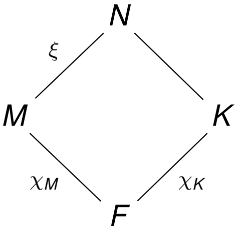
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

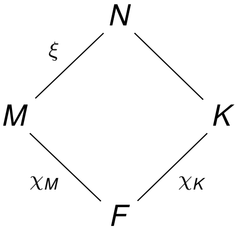
Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

Sketch of the proof



- $\chi_{M,\infty} = +-$
- If $\chi_{K,\infty} = ++$ then $\xi_\infty = +$
- If $\chi_{K,\infty} = --$ then $\xi_\infty = -$

- $L(1, f_K \otimes \xi) = L(1, f \otimes \chi_K) \cdot L(1, f \otimes \chi_M \chi_K)$
  - If $\chi_{K,\infty} = ++ \rightsquigarrow (\Omega_B^+)^2 \doteq \Omega^{++}\Omega^{+-}$
  - If $\chi_{K,\infty} = -- \rightsquigarrow (\Omega_B^-)^2 \doteq \Omega^{--}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \frac{\Omega^{++}\Omega^{+-}}{\Omega^{--}\Omega^{-+}}$
- Period relations of Riemann–Hodge (Oda): $\Omega^{++}\Omega^{--} = \Omega^{+-}\Omega^{-+}$
- $\left(\frac{\Omega_B^+}{\Omega_B^-}\right)^2 \doteq \left(\frac{\Omega^{++}}{\Omega^{-+}}\right)^2 \rightsquigarrow \frac{\Omega^{++}}{\Omega^{-+}} \in \sqrt{\mathbb{Q}}\frac{\Omega_B^+}{\Omega_B^-}$

Oda's Conjecture

Cardedeu 28 Novembre 2024