Rational points on elliptic curves over almost totally complex quadratic extensions

Xevi Guitart[1] Víctor Rotger[2] Yu Zhao[3]

[1]Universitat Politècnica de Catalunya

[2]Universitat Politècnica de Catalunya

[3]McGill University

Comof 2011, Heidelberg

Outline

Outline

BSD over totally real fields

BSD over totally real fields

- F totally real field, E/F elliptic curve of conductor $\mathcal{N} \subseteq F$.

BSD over totally real fields

- F totally real field, E/F elliptic curve of conductor $\mathcal{N} \subseteq F$.
- Suppose that E/F is modular:
  - $L(E/F, s) = L(f, s)$ for some Hilbert modular form f.
  - Let $r_{an}(E/F) = \mathrm{ord}_{s=1} L(E/F, s)$.

BSD over totally real fields

- F totally real field, E/F elliptic curve of conductor $\mathcal{N} \subseteq F$.
- Suppose that E/F is modular:
  - $L(E/F, s) = L(f, s)$ for some Hilbert modular form f.
  - Let $r_{an}(E/F) = \mathrm{ord}_{s=1} L(E/F, s)$.
- Jacquet-Langlands hypothesis:
  (JL) either $[F\colon \mathbb{Q}]$ is odd or $v_{\mathfrak{p}}(\mathcal{N}) = 1$ for some $\mathfrak{p} \subseteq F$.

BSD over totally real fields

- F totally real field, E/F elliptic curve of conductor $\mathcal{N} \subseteq F$.
- Suppose that E/F is modular:
  - $L(E/F, s) = L(f, s)$ for some Hilbert modular form f.
  - Let $r_{an}(E/F) = \mathrm{ord}_{s=1} L(E/F, s)$.
- Jacquet-Langlands hypothesis:
  (JL) either $[F\colon \mathbb{Q}]$ is odd or $v_{\mathfrak{p}}(\mathcal{N}) = 1$ for some $\mathfrak{p} \subseteq F$.

Theorem (Gross-Zagier, Kolyvagin, Zhang)

If E satisfies (JL) and $r_{an}(E/F) \leq 1$ then

$$r_{an}(E/F) = r(E/F).$$

BSD over totally real fields

- F totally real field, E/F elliptic curve of conductor $\mathcal{N} \subseteq F$.
- Suppose that E/F is modular:
  - $L(E/F, s) = L(f, s)$ for some Hilbert modular form f.
  - Let $r_{an}(E/F) = \mathrm{ord}_{s=1} L(E/F, s)$.
- Jacquet-Langlands hypothesis:
  (JL) either $[F : \mathbb{Q}]$ is odd or $v_{\mathfrak{p}}(\mathcal{N}) = 1$ for some $\mathfrak{p} \subseteq F$.

Theorem (Gross-Zagier, Kolyvagin, Zhang)

If E satisfies (JL) and $r_{an}(E/F) \leq 1$ then

$$r_{an}(E/F) = r(E/F).$$

- Condition (JL) is needed to ensure geometric modularity:

$$\pi_E : Jac(X) \longrightarrow E, \quad X/F \text{ Shimura curve.}$$

BSD over totally real fields

- F totally real field, E/F elliptic curve of conductor $\mathcal{N} \subseteq F$.
- Suppose that E/F is modular:
  - $L(E/F, s) = L(f, s)$ for some Hilbert modular form f.
  - Let $r_{an}(E/F) = \mathrm{ord}_{s=1} L(E/F, s)$.
- Jacquet-Langlands hypothesis:
  (JL) either $[F : \mathbb{Q}]$ is odd or $v_{\mathfrak{p}}(\mathcal{N}) = 1$ for some $\mathfrak{p} \subseteq F$.

Theorem (Gross-Zagier, Kolyvagin, Zhang)

If E satisfies (JL) and $r_{an}(E/F) \leq 1$ then

$$r_{an}(E/F) = r(E/F).$$

- Condition (JL) is needed to ensure geometric modularity:

$$\pi_E : Jac(X) \longrightarrow E, \quad X/F \text{ Shimura curve.}$$

- Heegner points: for a quadratic CM extension K/F they belong to $Jac(X)(K^{ab})$ and can be projected to $E(K^{ab})$

BSD over totally real fields

- When $F = \mathbb{Q}$ they can be explicitly computed:

BSD over totally real fields

- When $F = \mathbb{Q}$ they can be explicitly computed:
  - Let f be the newform such that $L(E/\mathbb{Q}; s) = L(f; s)$.
  - Let $\omega_f = 2\pi i f(z)dz$, a differential on $X = X_0(N)$.
  - $\Lambda_f = \{\int_\gamma \omega_f \mid \gamma \in H_1(X, \mathbb{Z})\} \subseteq \mathbb{C}$
  - $\mathbb{C}/\Lambda_f \sim E$
  - $K = \mathbb{Q}(\tau)$ then the CM point is

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \sim E,$$

where $\Delta_\tau = \{\tau \to \infty\} \in C_1(X, \mathbb{Z})$.

BSD over totally real fields

- When $F = \mathbb{Q}$ they can be explicitly computed:
  - Let f be the newform such that $L(E/\mathbb{Q}; s) = L(f; s)$.
  - Let $\omega_f = 2\pi i f(z)dz$, a differential on $X = X_0(N)$.
  - $\Lambda_f = \{\int_\gamma \omega_f \mid \gamma \in H_1(X, \mathbb{Z})\} \subseteq \mathbb{C}$
  - $\mathbb{C}/\Lambda_f \sim E$
  - $K = \mathbb{Q}(\tau)$ then the CM point is

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \sim E,$$

    where $\Delta_\tau = \{\tau \to \infty\} \in C_1(X, \mathbb{Z})$.
- When $F \neq \mathbb{Q}$, what if (JL) is not satisfied?

BSD over totally real fields

- When $F = \mathbb{Q}$ they can be explicitly computed:
  - Let f be the newform such that $L(E/\mathbb{Q}; s) = L(f; s)$.
  - Let $\omega_f = 2\pi i f(z)dz$, a differential on $X = X_0(N)$.
  - $\Lambda_f = \{\int_\gamma \omega_f \mid \gamma \in H_1(X, \mathbb{Z})\} \subseteq \mathbb{C}$
  - $\mathbb{C}/\Lambda_f \sim E$
  - $K = \mathbb{Q}(\tau)$ then the CM point is

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \sim E,$$

  where $\Delta_\tau = \{\tau \to \infty\} \in C_1(X, \mathbb{Z})$.
- When $F \neq \mathbb{Q}$, what if (JL) is not satisfied?
- The simplest case: F real quadratic field, E/F with $\mathcal{N} = 1$.

BSD over totally real fields

- When $F = \mathbb{Q}$ they can be explicitly computed:
  - Let f be the newform such that $L(E/\mathbb{Q}; s) = L(f; s)$.
  - Let $\omega_f = 2\pi i f(z)dz$, a differential on $X = X_0(N)$.
  - $\Lambda_f = \{\int_\gamma \omega_f \mid \gamma \in H_1(X, \mathbb{Z})\} \subseteq \mathbb{C}$
  - $\mathbb{C}/\Lambda_f \sim E$
  - $K = \mathbb{Q}(\tau)$ then the CM point is

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \sim E,$$

where $\Delta_\tau = \{\tau \to \infty\} \in C_1(X, \mathbb{Z})$.

- When $F \neq \mathbb{Q}$, what if (JL) is not satisfied?
- The simplest case: F real quadratic field, E/F with $\mathcal{N} = 1$.
- If M/F is a quadratic extension such that $\operatorname{sign} L(E/M, s) = -1$, is there a way of analytically constructing points on $E(M^{ab})$?

BSD over totally real fields

- When $F = \mathbb{Q}$ they can be explicitly computed:
  - Let f be the newform such that $L(E/\mathbb{Q}; s) = L(f; s)$.
  - Let $\omega_f = 2\pi i f(z)dz$, a differential on $X = X_0(N)$.
  - $\Lambda_f = \{\int_\gamma \omega_f \mid \gamma \in H_1(X, \mathbb{Z})\} \subseteq \mathbb{C}$
  - $\mathbb{C}/\Lambda_f \sim E$
  - $K = \mathbb{Q}(\tau)$ then the CM point is

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \sim E,$$

  where $\Delta_\tau = \{\tau \to \infty\} \in C_1(X, \mathbb{Z})$.
- When $F \neq \mathbb{Q}$, what if (JL) is not satisfied?
- The simplest case: F real quadratic field, E/F with $\mathcal{N} = 1$.
- If M/F is a quadratic extension such that $\operatorname{sign} L(E/M, s) = -1$, is there a way of analytically constructing points on $E(M^{ab})$?
- $\operatorname{sign} L(E/M, s) = -1$ if and only if M is Almost Totally Real (ATR) (i.e. M has exactly one complex place)

Outline

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.
- Let $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$ be the Hilbert modular form attached to E.

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.
- Let $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$ be the Hilbert modular form attached to E.
- Let $X = \mathcal{H} \times \mathcal{H}/\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular surface.

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.
- Let $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$ be the Hilbert modular form attached to E.
- Let $X = \mathcal{H} \times \mathcal{H}/\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular surface.
- Let ω_f be the differential 2-form on X

$$\omega_f = (2\pi i)^2 f(z_0, z_1) dz_0 dz_1$$

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.
- Let $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$ be the Hilbert modular form attached to E.
- Let $X = \mathcal{H} \times \mathcal{H}/\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular surface.
- Let ω_f be the differential 2-form on X

$$\omega_f = (2\pi i)^2 f(z_0, z_1) dz_0 dz_1 - (2\pi i)^2 f(u_0 z_0, u_1 \bar{z}_1) d(u_0 z_0) d(u_1 \bar{z}_1)$$

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.
- Let $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$ be the Hilbert modular form attached to E.
- Let $X = \mathcal{H} \times \mathcal{H}/\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular surface.
- Let ω_f be the differential 2-form on X

$$\omega_f = (2\pi i)^2 f(z_0, z_1) dz_0 dz_1 - (2\pi i)^2 f(u_0 z_0, u_1 \bar{z}_1) d(u_0 z_0) d(u_1 \bar{z}_1)$$

and let $\Lambda_f = \left\{ \int_\gamma \omega_f, \;\; \gamma \in H_2(X(\mathbb{C}), \mathbb{Z}) \right\} \subseteq \mathbb{C}$.

Definition of the ATR points

- F a real quadratic field, E/F an elliptic curve of conductor 1.
- M/F quadratic ATR extension, think $M \subseteq \mathbb{C}$ via the complex place.
- Let $f \in S_2(\mathrm{SL}_2(\mathcal{O}_F))$ be the Hilbert modular form attached to E.
- Let $X = \mathcal{H} \times \mathcal{H}/\mathrm{SL}_2(\mathcal{O}_F)$ be the Hilbert modular surface.
- Let ω_f be the differential 2-form on X

$$\omega_f = (2\pi i)^2 f(z_0, z_1) dz_0 dz_1 - (2\pi i)^2 f(u_0 z_0, u_1 \bar{z}_1) d(u_0 z_0) d(u_1 \bar{z}_1)$$

and let $\Lambda_f = \left\{ \int_\gamma \omega_f, \ \ \gamma \in H_2(X(\mathbb{C}), \mathbb{Z}) \right\} \subseteq \mathbb{C}$.

Conjecture (Oda)

$\mathbb{C}/\Lambda_f$ is isogenous to E.

Definition of the ATR points

Definition of the ATR points

- Let $M = F(\tau)$.

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f$$

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f$$

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$

- Analogous to Heegner points, and it is explicitly computable.

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$

- Analogous to Heegner points, and it is explicitly computable.

Conjecture (Darmon)

The isogeny ι can be chosen such that $\iota(J_\tau)$ belongs to $E(M^{\mathrm{ab}})$.

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$

- Analogous to Heegner points, and it is explicitly computable.

Conjecture (Darmon)

The isogeny ι can be chosen such that $\iota(J_\tau)$ belongs to $E(M^{\mathrm{ab}})$.

- It does not assume (JL): it also applies to elliptic curves which are not expected to be geometrically modular in general.

Definition of the ATR points

- Let $M = F(\tau)$.
- Darmon defines 2-dimensional chain $\Delta_\tau \in C_2(X, \mathbb{Z})$ so that the ATR point is defined as

$$J_\tau = \int_{\Delta_\tau} \omega_f \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$

- Analogous to Heegner points, and it is explicitly computable.

Conjecture (Darmon)

The isogeny ι can be chosen such that $\iota(J_\tau)$ belongs to $E(M^{\mathrm{ab}})$.

- It does not assume (JL): it also applies to elliptic curves which are not expected to be geometrically modular in general.
- There is a special type of elliptic curves called $\mathbb{Q}$-curves. Even if they do not satisfy (JL), they are known to be geometrically modular. Maybe a construction using Heegner points is available.

Outline

Heegner points on $\mathbb{Q}$-curves

- N a squarefree odd integer, and let $F = \mathbb{Q}(\sqrt{N})$
- ε be a quadratic character of conductor N.
- Let $f_{\mathbb{Q}} \in S_2(N, \varepsilon)$ be a modular form over $\mathbb{Q}$ such that $\dim A_{f_{\mathbb{Q}}} = 2$.

Heegner points on $\mathbb{Q}$-curves

- N a squarefree odd integer, and let $F = \mathbb{Q}(\sqrt{N})$
- ε be a quadratic character of conductor N.
- Let $f_{\mathbb{Q}} \in S_2(N, \varepsilon)$ be a modular form over $\mathbb{Q}$ such that $\dim A_{f_{\mathbb{Q}}} = 2$.
- Then $A_{f_{\mathbb{Q}}} \sim_F E^2$, where E/F is a $\mathbb{Q}$-curve of conductor 1.

Heegner points on $\mathbb{Q}$-curves

- N a squarefree odd integer, and let $F = \mathbb{Q}(\sqrt{N})$
- ε be a quadratic character of conductor N.
- Let $f_{\mathbb{Q}} \in S_2(N, \varepsilon)$ be a modular form over $\mathbb{Q}$ such that $\dim A_{f_{\mathbb{Q}}} = 2$.
- Then $A_{f_{\mathbb{Q}}} \sim_F E^2$, where E/F is a $\mathbb{Q}$-curve of conductor 1.
- In this case E is geometrically modular:

$$\pi_E \colon J_1(N) = \mathrm{Jac}(X_1(N)) \longrightarrow A_{f_{\mathbb{Q}}} \longrightarrow E.$$

Heegner points on $\mathbb{Q}$-curves

- N a squarefree odd integer, and let $F = \mathbb{Q}(\sqrt{N})$
- ε be a quadratic character of conductor N.
- Let $f_{\mathbb{Q}} \in S_2(N, \varepsilon)$ be a modular form over $\mathbb{Q}$ such that $\dim A_{f_{\mathbb{Q}}} = 2$.
- Then $A_{f_{\mathbb{Q}}} \sim_F E^2$, where E/F is a $\mathbb{Q}$-curve of conductor 1.
- In this case E is geometrically modular:

$$\pi_E \colon J_\varepsilon(N) = \mathrm{Jac}(X_\varepsilon(N)) \longrightarrow A_{f_{\mathbb{Q}}} \longrightarrow E.$$

Heegner points on $\mathbb{Q}$-curves

- N a squarefree odd integer, and let $F = \mathbb{Q}(\sqrt{N})$
- ε be a quadratic character of conductor N.
- Let $f_{\mathbb{Q}} \in S_2(N, \varepsilon)$ be a modular form over $\mathbb{Q}$ such that $\dim A_{f_{\mathbb{Q}}} = 2$.
- Then $A_{f_{\mathbb{Q}}} \sim_F E^2$, where E/F is a $\mathbb{Q}$-curve of conductor 1.
- In this case E is geometrically modular:

$$\pi_E \colon J_\varepsilon(N) = \mathrm{Jac}(X_\varepsilon(N)) \longrightarrow A_{f_{\mathbb{Q}}} \longrightarrow E.$$

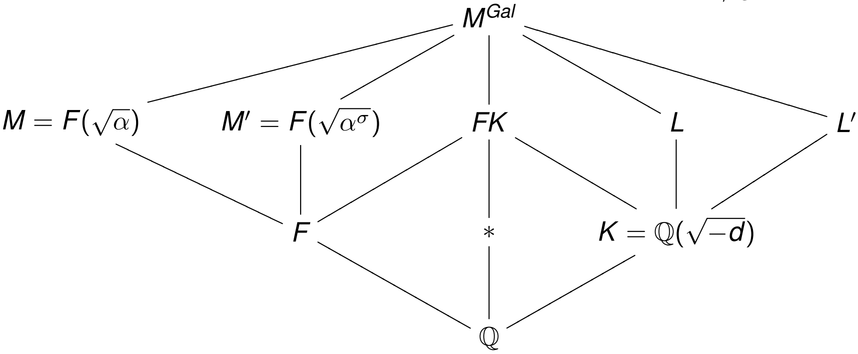
- Let $M = F(\sqrt{\alpha})$ be an ATR extension, and $-d = \mathrm{Nm}_{F/\mathbb{Q}}(\alpha) < 0$.

Heegner points on $\mathbb{Q}$-curves

- N a squarefree odd integer, and let $F = \mathbb{Q}(\sqrt{N})$
- ε be a quadratic character of conductor N.
- Let $f_{\mathbb{Q}} \in S_2(N, \varepsilon)$ be a modular form over $\mathbb{Q}$ such that $\dim A_{f_{\mathbb{Q}}} = 2$.
- Then $A_{f_{\mathbb{Q}}} \sim_F E^2$, where E/F is a $\mathbb{Q}$-curve of conductor 1.
- In this case E is geometrically modular:

$$\pi_E \colon J_\varepsilon(N) = \mathrm{Jac}(X_\varepsilon(N)) \longrightarrow A_{f_{\mathbb{Q}}} \longrightarrow E.$$

- Let $M = F(\sqrt{\alpha})$ be an ATR extension, and $-d = \mathrm{Nm}_{F/\mathbb{Q}}(\alpha) < 0$.



- The idea is to use Heegner points on $X_\varepsilon(N)$ attached to K to construct point on $E(M^{ab})$.

- The idea is to use Heegner points on $X_\varepsilon(N)$ attached to K to construct point on $E(M^{ab})$.
- $X_\varepsilon(N) \xrightarrow{2:1} X_0(N)$

- The idea is to use Heegner points on $X_\varepsilon(N)$ attached to K to construct point on $E(M^{ab})$.
- $X_\varepsilon(N) \xrightarrow{2:1} X_0(N)$
  $P_\tau \in X_0(N)(H)$ gives rise to points $\tilde{P}_\tau \in X_\varepsilon(N)(\tilde{L}), \tilde{P}'_\tau \in X_\varepsilon(N)(\tilde{L}')$

- The idea is to use Heegner points on $X_\varepsilon(N)$ attached to K to construct point on $E(M^{ab})$.
- $X_\varepsilon(N) \xrightarrow{2:1} X_0(N)$
  $P_\tau \in X_0(N)(H)$ gives rise to points $\tilde{P}_\tau \in X_\varepsilon(N)(\tilde{L}), \tilde{P}'_\tau \in X_\varepsilon(N)(\tilde{L}')$
  Project $\tilde{P}_\tau$, $\tilde{P}'_\tau$ via $X_\varepsilon(N) \longrightarrow J_\varepsilon(N) \longrightarrow A_{f_{\mathbb{Q}}} \longrightarrow E$ and add them

- The idea is to use Heegner points on $X_\varepsilon(N)$ attached to K to construct point on $E(M^{ab})$.
- $X_\varepsilon(N) \xrightarrow{2:1} X_0(N)$
  $P_\tau \in X_0(N)(H)$ gives rise to points $\tilde{P}_\tau \in X_\varepsilon(N)(\tilde{L}), \tilde{P}'_\tau \in X_\varepsilon(N)(\tilde{L}')$
  Project $\tilde{P}_\tau$, $\tilde{P}'_\tau$ via $X_\varepsilon(N) \longrightarrow J_\varepsilon(N) \longrightarrow A_{f_\mathbb{Q}} \longrightarrow E$ and add them
- The "Heegner Point" on E is defined as

$$z_\tau = \int_\tau^\infty (\omega_{f_\mathbb{Q}} + \omega_{f_\mathbb{Q}|W_N})dz + \int_{\tau'}^\infty (\omega_{f_\mathbb{Q}} + \omega_{f_\mathbb{Q}|W_N})dz \in \mathbb{C}/\Lambda_{f_\mathbb{Q}},$$

  where now

$$\Lambda_{f_\mathbb{Q}} = \left\langle \int_\gamma (\omega_{f_\mathbb{Q}} + \omega_{f_\mathbb{Q}|W_N})dz \mid \gamma \in H_1(X_1(N), \mathbb{Z}) \right\rangle$$

- The idea is to use Heegner points on $X_\varepsilon(N)$ attached to K to construct point on $E(M^{ab})$.
- $X_\varepsilon(N) \xrightarrow{2:1} X_0(N)$
  $P_\tau \in X_0(N)(H)$ gives rise to points $\tilde{P}_\tau \in X_\varepsilon(N)(\tilde{L})$, $\tilde{P}'_\tau \in X_\varepsilon(N)(\tilde{L}')$
  Project $\tilde{P}_\tau$, $\tilde{P}'_\tau$ via $X_\varepsilon(N) \longrightarrow J_\varepsilon(N) \longrightarrow A_{f_{\mathbb{Q}}} \longrightarrow E$ and add them
- The "Heegner Point" on E is defined as

$$z_\tau = \int_\tau^\infty (\omega_{f_{\mathbb{Q}}} + \omega_{f_{\mathbb{Q}}|W_N})dz + \int_{\tau'}^\infty (\omega_{f_{\mathbb{Q}}} + \omega_{f_{\mathbb{Q}}|W_N})dz \in \mathbb{C}/\Lambda_{f_{\mathbb{Q}}},$$

where now

$$\Lambda_{f_{\mathbb{Q}}} = \left\langle \int_\gamma (\omega_{f_{\mathbb{Q}}} + \omega_{f_{\mathbb{Q}}|W_N})dz \mid \gamma \in H_1(X_1(N), \mathbb{Z}) \right\rangle$$

Theorem (Darmon-Rotger-Zhao)

There exist $\tau \in M$ and $\eta \colon \mathbb{C}/\Lambda_{f_{\mathbb{Q}}} \to E$ such that $\eta(z_\tau) \in E(M^{\mathrm{ab}})$.

Outline

More general fields

- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.

More general fields

- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.
- If M is ATR, Darmon's theory can be adapted.

More general fields

- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.
- If M is ATR, Darmon's theory can be adapted.
- Now, $\mathrm{sign}\, L(E/M, s) = -1$ in many situations where M in not ATR.

More general fields

- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.
- If M is ATR, Darmon's theory can be adapted.
- Now, $\mathrm{sign}\, L(E/M, s) = -1$ in many situations where M in not ATR.

Goal

To analytically construct points on $E(M^{ab})$, for a class of fields M which are not ATR. We want it to be explicitly computable.

More general fields

- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.
- If M is ATR, Darmon's theory can be adapted.
- Now, $\operatorname{sign} L(E/M, s) = -1$ in many situations where M in not ATR.

Goal

To analytically construct points on $E(M^{ab})$, for a class of fields M which are not ATR. We want it to be explicitly computable.

- We can do it under the following hypothesis:

More general fields

- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.
- If M is ATR, Darmon's theory can be adapted.
- Now, $\mathrm{sign}\, L(E/M, s) = -1$ in many situations where M in not ATR.

Goal

To analytically construct points on $E(M^{ab})$, for a class of fields M which are not ATR. We want it to be explicitly computable.

- We can do it under the following hypothesis:
  - There exists $F_0 \subseteq F$ with $[F : F_0] = 2$ such that E is an F_0-curve (i.e. E is F-isogenous to its $\mathrm{Gal}(F/F_0)$-conjugate)

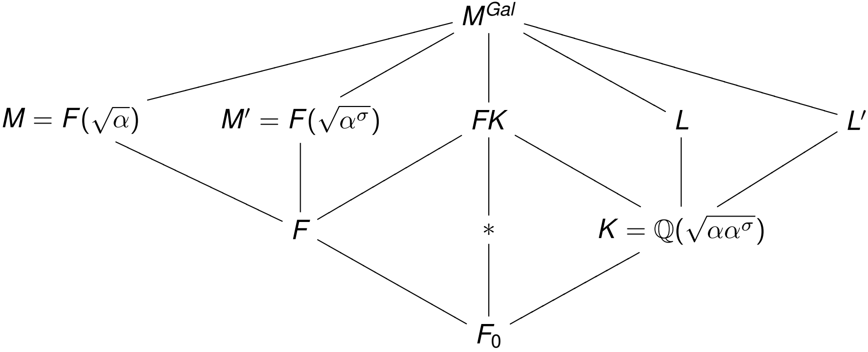
More general fields

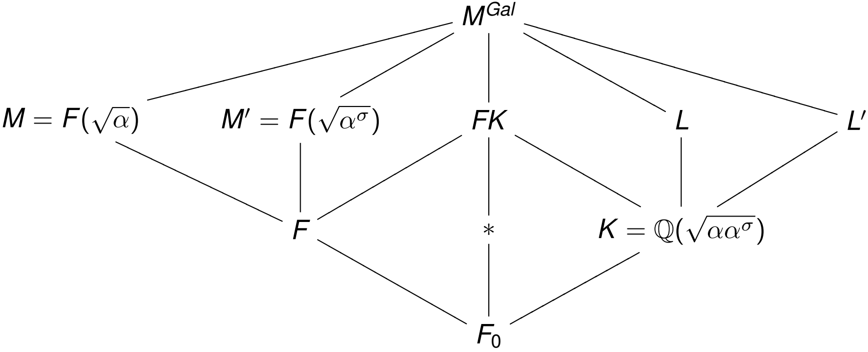
- Returning to the general case:
  - F totally real number field of arbitrary degree (and narrow class number 1),
  - E/F not satisfying (JL),
  - M/F a quadratic extension.
- If M is ATR, Darmon's theory can be adapted.
- Now, $\mathrm{sign}\, L(E/M, s) = -1$ in many situations where M in not ATR.

Goal

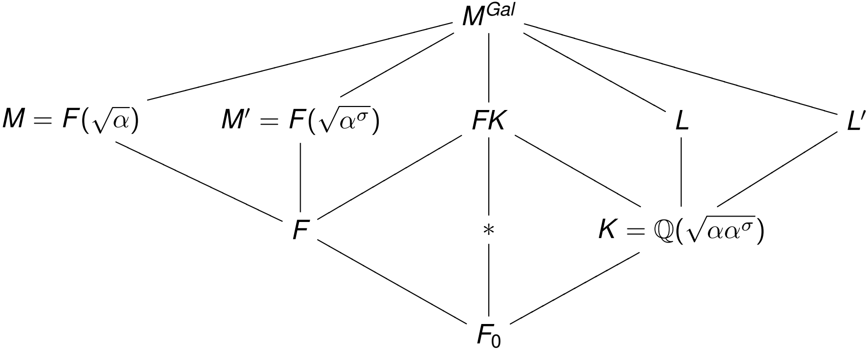
To analytically construct points on $E(M^{ab})$, for a class of fields M which are not ATR. We want it to be explicitly computable.

- We can do it under the following hypothesis:
  - There exists $F_0 \subseteq F$ with $[F : F_0] = 2$ such that E is an F_0-curve (i.e. E is F-isogenous to its $\mathrm{Gal}(F/F_0)$-conjugate)
  - $M = F(\sqrt{\alpha})$ a quadratic Almost Totally Complex extension (ATC) (in this case $\mathrm{sign}(L(E/M, s) = -1)$)

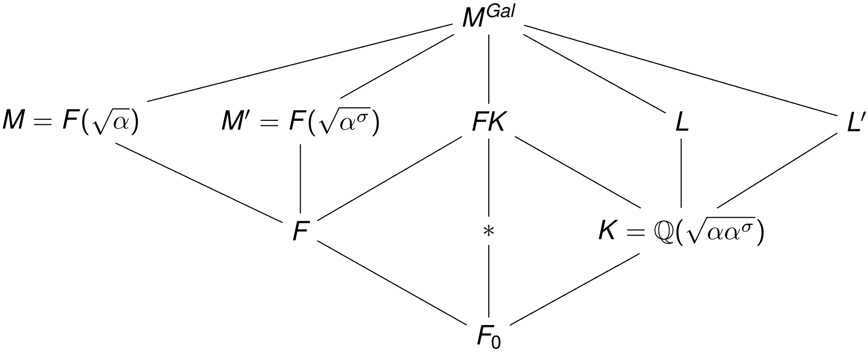

M^{Gal}
$M = F(\sqrt{\alpha})$
$M' = F(\sqrt{\alpha^\sigma})$
FK
L
L'
F
$*$
$K = \mathbb{Q}(\sqrt{\alpha\alpha^\sigma})$
F_0



- Now K is ATR: we can play the same game as before, with Darmon's ATR points replacing Heegner points

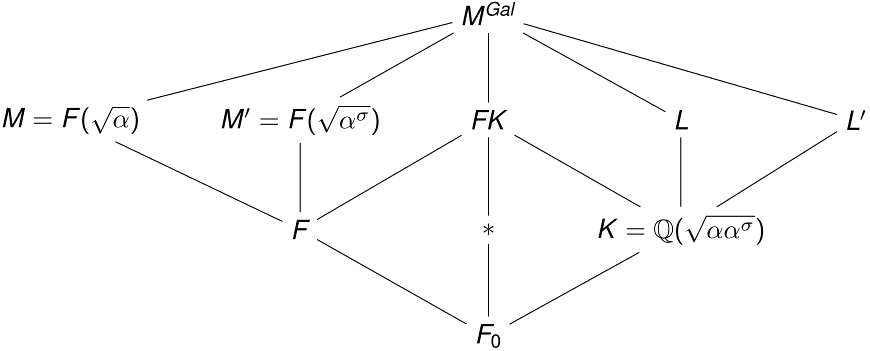


- Now K is ATR: we can play the same game as before, with Darmon's ATR points replacing Heegner points
- There exists a HMF over F_0 such that $A_f \sim_F E^2$ (E-S, S-T)



- Now K is ATR: we can play the same game as before, with Darmon's ATR points replacing Heegner points
- There exists a HMF over F_0 such that $A_f \sim_F E^2$ (E-S, S-T)
- Idea: generalize Darmon's construction to obtain ATR points on A_f, and project them to E to get points on $E(M^{ab})$: if $K = F_0(\tau)$

$$z_\tau = \int_{\Delta_\tau} \omega_f + \omega_{f|W_N} + \int_{\Delta_{\tau'}} \omega_f + \omega_{f|W_N} \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$



- Now K is ATR: we can play the same game as before, with Darmon's ATR points replacing Heegner points
- There exists a HMF over F_0 such that $A_f \sim_F E^2$ (E-S, S-T)
- Idea: generalize Darmon's construction to obtain ATR points on A_f, and project them to E to get points on $E(M^{ab})$: if $K = F_0(\tau)$

$$z_\tau = \int_{\Delta_\tau} \omega_f + \omega_{f|W_N} + \int_{\Delta_{\tau'}} \omega_f + \omega_{f|W_N} \in \mathbb{C}/\Lambda_f \overset{\iota}{\sim} E$$

Theorem: if Darmon's conjecture on ATR points holds, then there exists $\tau \in M$ such that $\iota(z_\tau)$ belongs to $E(M^{ab})$

Concrete example

- $F = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, $F_0 = \mathbb{Q}(\sqrt{2})$
- $E\colon y^2 = x^3 - 54(63 + 46\sqrt{2} + 27\sqrt{5} + 18\sqrt{10})x - 116(409 + 287\sqrt{2} + 189\sqrt{5} + 135\sqrt{10})$

Concrete example

- $F = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, $F_0 = \mathbb{Q}(\sqrt{2})$
- $E\colon y^2 = x^3 - 54(63 + 46\sqrt{2} + 27\sqrt{5} + 18\sqrt{10})x - 116(409 + 287\sqrt{2} + 189\sqrt{5} + 135\sqrt{10})$
- E is an F_0-curve, but it is also a $\mathbb{Q}$-curve (computed by J. Quer).
- The HMF f is base change to F_0 of a modular form $f_{\mathbb{Q}} \in S_2(40, \varepsilon_{10})$

Concrete example

- $F = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, $F_0 = \mathbb{Q}(\sqrt{2})$
- $E\colon y^2 = x^3 - 54(63 + 46\sqrt{2} + 27\sqrt{5} + 18\sqrt{10})x - 116(409 + 287\sqrt{2} + 189\sqrt{5} + 135\sqrt{10})$
- E is an F_0-curve, but it is also a $\mathbb{Q}$-curve (computed by J. Quer).
- The HMF f is base change to F_0 of a modular form $f_{\mathbb{Q}} \in S_2(40, \varepsilon_{10})$
- $M = F(\sqrt{\sqrt{10} + \sqrt{5} + \sqrt{2}})$ is ATC
- We can compute the ATC point $z_\tau \in \mathbb{C}/\Lambda_f$

Concrete example

- $F = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, $F_0 = \mathbb{Q}(\sqrt{2})$
- $E\colon y^2 = x^3 - 54(63 + 46\sqrt{2} + 27\sqrt{5} + 18\sqrt{10})x - 116(409 + 287\sqrt{2} + 189\sqrt{5} + 135\sqrt{10})$
- E is an F_0-curve, but it is also a $\mathbb{Q}$-curve (computed by J. Quer).
- The HMF f is base change to F_0 of a modular form $f_{\mathbb{Q}} \in S_2(40, \varepsilon_{10})$
- $M = F(\sqrt{\sqrt{10} + \sqrt{5} + \sqrt{2}})$ is ATC
- We can compute the ATC point $z_\tau \in \mathbb{C}/\Lambda_f$
- We (Magma) computed $z_{nt} \in \mathbb{C}/\Lambda_E$, a non-torsion point in $E(M)$.

Concrete example

- $F = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, $F_0 = \mathbb{Q}(\sqrt{2})$
- $E\colon y^2 = x^3 - 54(63 + 46\sqrt{2} + 27\sqrt{5} + 18\sqrt{10})x - 116(409 + 287\sqrt{2} + 189\sqrt{5} + 135\sqrt{10})$
- E is an F_0-curve, but it is also a $\mathbb{Q}$-curve (computed by J. Quer).
- The HMF f is base change to F_0 of a modular form $f_{\mathbb{Q}} \in S_2(40, \varepsilon_{10})$
- $M = F(\sqrt{\sqrt{10} + \sqrt{5} + \sqrt{2}})$ is ATC
- We can compute the ATC point $z_\tau \in \mathbb{C}/\Lambda_f$
- We (Magma) computed $z_{nt} \in \mathbb{C}/\Lambda_E$, a non-torsion point in $E(M)$.
- We numerically find the relation

$$7 \cdot 14 \cdot \iota(z_\tau) + 239 \cdot z_{nt} = 0 \mod \Lambda_E$$

(checked up to certain numerical precision), which gives evidence that z_τ belongs to $E(M)$ and it has infinite order.

Rational points on elliptic curves over almost totally complex quadratic extensions

Xevi Guitart[1] Víctor Rotger[2] Yu Zhao[3]

[1]Universitat Politècnica de Catalunya

[2]Universitat Politècnica de Catalunya

[3]McGill University

Comof 2011, Heidelberg