Computing p-adic periods of abelian varieties from automorphic forms

Xavier Guitart and Marc Masdeu

Abstract. We give an overview of [**GM16**], in which we exploit Darmon's p-adic $\mathcal{L}$-invariants to compute p-adic uniformizations of certain motives attached to modular forms. We illustrate our methods with new examples.

1. Introduction

Let E be an elliptic curve of conductor pN, with p a prime not dividing N, and let f be the newform for $\Gamma_0(pN)$ corresponding to E under the Modularity Theorem. Let also K be a real quadratic field in which p is inert and all the primes dividing N split. In this setting, Darmon introduced in his seminal paper [**Dar01**] a construction of local points on $E(K_p)$, which he called Stark–Heegner points, and which are conjecturally global and defined over ring class fields of K.

Taking advantage of the isomorphism $E(K_p) \simeq K_p^\times/q_E^{\mathbb{Z}}$, where q_E is the Tate period of E, Darmon defined Stark–Heegner points as suitable values of certain $K_p^\times$-valued integrals of f. By construction, however, such values are only well defined (i.e., independent of any choices) modulo a certain p-adic lattice $q_f^{\mathbb{Z}}$, where q_f is a p-adic number whose definition depends on f and is a priori unrelated to q_E. The quantity $\mathcal{L}_p(f) = \log_p(q_f)/\operatorname{ord}_p(q_f)$ can be interpreted as a p-adic $\mathcal{L}$-invariant. Indeed, one of the main results of [**Dar01**] is the following relation between special values of classical and p-adic L-functions:

$$L_p'(f,1) = \mathcal{L}_p(f)L(f,1).$$

Therefore, as a consequence of the exceptional zero conjecture of Mazur–Tate–Teitelbaum proven by Greenberg-Stevens [**GS93**], Darmon's p-adic $\mathcal{L}$-invariant coincides with $\log_p(q_E)/\operatorname{ord}_p(q_E)$ and, in particular, $K_p^\times/q_E^{\mathbb{Z}}$ and $K_p^\times/q_f^{\mathbb{Z}}$ are isogenous elliptic curves. Thus the construction of Stark–Heegner points does produce (perhaps after composing with an isogeny) well defined points in $E(K_p)$.

Over the years there have been a number of generalizations and variants of Stark–Heegner points attached to more general motives (e.g., elliptic curves over number fields [**Gre09, Tri06, Gär12, GM16, GMS15**], modular abelian varieties [**Das05, LRV12**], motives attached to higher weight modular forms [**RS12**]), that have come to be known as Darmon points or, even more generally, Darmon

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cycles. A common feature of these constructions is that the points (or cycles) are defined as p-adic integrals which are only well defined modulo a Darmon-like $\mathcal{L}$-invariant given by certain period integrals of automorphic forms. That these integrals yield well defined cycles on the motive depends on the (in some cases still conjectural) relationship between the p-adic $\mathcal{L}$-invariant and the periods of the p-adic uniformization of the motive.

In this note we give an overview of [**GM16**], in which we explore the possibility of using Darmon's p-adic $\mathcal{L}$-invariants, in the form of suitable automorphic periods, and their conjectured properties to compute p-adic uniformizations of certain motives attached to modular forms for which no unconditional construction is yet known. More precisely, we consider the case of modular forms of weight two over number fields of mixed signature. The associated motives are expected to be abelian varieties of GL_2-type, although the Eichler–Shimura-type construction that associates to any such modular form an abelian variety is only known for totally real number fields. In some cases for which no Eichler–Shimura construction is known, such as number fields with one complex place, we have computed approximations to the p-adic automorphic periods conforming Darmon's $\mathcal{L}$-invariants. Granting the conjecture that these periods are isogenous to the p-adic periods of the abelian variety, we have been able to recover the global algebraic equations.

In Section 2 we recall the conjecture that associates to any modular form of weight two for GL_2 an abelian variety, and we collect a few facts on their p-adic uniformization. In Section 3 we describe the p-adic integrals attached to the modular forms and the construction of the periods that, conjecturally, coincide with the p-adic periods of the associated abelian variety. Finally, in Section 4, we illustrate the method by reporting on two new examples of the computation of $\mathcal{L}$-invariants and abelian varieties.

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2. Automorphic Forms and abelian varieties

Let F be a number field of signature (r,s), and consider an ideal $\mathfrak{N}\subset\mathcal{O}_F$, which we will call a *level*. In order to avoid technical difficulties, we will assume that F has narrow class number one, and that $\mathfrak{N}$ is squarefree. After fixing $r+s$ embeddings of F into $\mathbb{C}$ corresponding to the places of F, the group $\Gamma_0(\mathfrak{N})=\{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathcal{O}_F)\colon\mathfrak{N}\mid c\}$ acts discretely on $\mathbb{H}^r\times\mathbb{H}_3^s$, where $\mathbb{H}$ (respectively $\mathbb{H}_3$) denotes the hyperbolic upper half plane (respectively upper half space). The cohomology of the quotient orbifold $Y_0(\mathfrak{N})=\Gamma_0(\mathfrak{N})\backslash(\mathbb{H}^n\times\mathbb{H}_3^s)$ can be computed via group cohomology. This cohomology comes equipped with an action of the commutative Hecke algebra $\mathbb{T}$, generated by the Hecke operators $T_\mathfrak{l}$ for primes $\mathfrak{l}\nmid\mathfrak{N}$. Let $f\in H^{n+s}(\Gamma_0(\mathfrak{N}),\mathbb{C})$ be an eigenvector for all the Hecke operators, say $T_\mathfrak{l}f=a_\mathfrak{l}f$ for all $\mathfrak{l}\nmid\mathfrak{N}$. Then $K_f=\mathbb{Q}(\{a_\mathfrak{l}\}_\mathfrak{l})$ is a number field, say of degree d. Suppose that f is cuspidal, new, and without complex multiplication.

CONJECTURE 2.1 (Taylor, ICM 1994). *There is a simple abelian variety A_f/F of dimension d, of conductor $\mathfrak{N}^d$, and with* $\mathrm{End}(A_f) \otimes \mathbb{Q} \supseteq K_f$*, such that*

$$L(A_f, s) = \prod_{\sigma\colon K_f \hookrightarrow \mathbb{C}} L({}^{\sigma}f, s).$$

One should remark that in the original conjecture there is another possibility for the variety associated to f; namely, it could happen that it is of dimension $2d$ and has quaternionic multiplication. This possibility is excluded in our setting, thanks to the assuption of $\mathfrak{N}$ being square-free (in fact, it would be enough to ask for $\mathfrak{N}$ not being square-full, see [**GM16**] for more details).

Taylor's conjecture is true when $F = \mathbb{Q}$, thanks to the Eichler–Shimura construction. It is known to be true in other cases, namely when F is totally real and there exists a Jacquet–Langlands lift of f to a Shimura curve (e.g., this is always the case if $[F : \mathbb{Q}]$ is odd).

The aim of this short note is to give an overview of the explicit construction that we propose in [**GM16**] which gives a conjectural description of the variety A_f as a p-adic torus. At this point the reader might get unsettled by the possibility that A_f would not admit a p-adic uniformization. After all, here is one example of this situation: the elliptic curve E_1 given by the equation $y^2 = x^3 + 1$ has conductor $N_1 = 36$. The elliptic curve E_2 given by the equation $y^2 + xy = x^3 - x^2 - 2x - 1$ has conductor 49. Therefore the abelian surface $A = E_1 \times E_2$ has conductor $1764 = (2 \cdot 3 \cdot 7)^2$. However, since neither E_1 nor E_2 is p-adically uniformizable (because neither has multiplicative reduction at any p), the abelian surface A cannot be p-adically uniformizable. In our setting, however, the endomorphism structure of A will make the uniformization possible.

THEOREM 2.2 ([**GM16**], Prop. 2.4). *Let A/F be an abelian variety of dimension d and let $\mathfrak{p}$ be a prime of F. Denote by $\mathbb{Q}_{\mathfrak{p}}$ the completion of F at $\mathfrak{p}$. Suppose that*

- $\mathfrak{p}^d \parallel \mathrm{conductor}(A)$, *and*
- $\mathrm{End}(A) \otimes \mathbb{Q}$ *contains a totally real field K of degree d.*

Then there exists a discrete lattice $\Lambda \subset (\mathbb{Q}_{\mathfrak{p}}^{\times})^d$ such that

$$A(\bar{\mathbb{Q}}_{\mathfrak{p}}) \cong (\bar{\mathbb{Q}}_{\mathfrak{p}}^{\times})^d / \Lambda.$$

3. Periods of automorphic forms

As in the previous section let $f \in H^{n+s}(\Gamma_0(\mathfrak{N}), \mathbb{C})$ be a cuspidal newform without complex multiplication, and suppose that $\mathfrak{p}$ is a prime that divides $\mathfrak{N}$ exactly. The periods are constructed by means of a $\mathfrak{p}$-adic integration pairing between certain homology and cohomology groups of a $\{\mathfrak{p}\}$-arithmetic subgroup $\Gamma \subset \mathrm{PGL}_2(\mathbb{Q}_{\mathfrak{p}})$ related to $\Gamma_0(\mathfrak{N})$.

3.1. The $\{\mathfrak{p}\}$-arithmetic subgroup.

We now proceed to define a $\{\mathfrak{p}\}$-arithmetic group Γ. We start by choosing a factorization of the form $\mathfrak{N} = \mathfrak{p}\mathfrak{D}\mathfrak{m}$ and a choice of $n \leq r$ real places, say $v_1, v_2, \ldots, v_r$, in such a way that the set of places $\{\mathfrak{q} \mid \mathfrak{D}\} \cup \{v_{n+1}, \ldots, v_r\}$ has even cardinal. Let $B_{/F}$ be the quaternion algebra whose ramification locus is precisely this set. Fix Eichler orders $R_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) \subset R_0^{\mathfrak{D}}(\mathfrak{m}) \subset B$, and an embedding $\iota_{\mathfrak{p}}\colon R_0^{\mathfrak{D}}(\mathfrak{m}) \to M_2(\mathbb{Z}_{\mathfrak{p}})$ (here $\mathbb{Z}_{\mathfrak{p}}$ stands for the ring of integers of $\mathbb{Q}_{\mathfrak{p}}$). Define $\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) = R_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})^\times/\mathcal{O}_F^\times$ and $\Gamma_0^{\mathfrak{D}}(\mathfrak{m}) = R_0^{\mathfrak{D}}(\mathfrak{m})^\times/\mathcal{O}_F^\times$. Finally, define

$$\Gamma = \iota_{\mathfrak{p}}(R_0^{\mathfrak{D}}(\mathfrak{m})[1/\mathfrak{p}]^\times/\mathcal{O}_F[1/\mathfrak{p}]^\times) \subset \mathrm{PGL}_2(\mathbb{Q}_{\mathfrak{p}}).$$

In the following subsections we will see how an integration pairing in the (co)homology of Γ gives rise to naturally defined periods of automorphic forms.

3.2. Integration on $\mathcal{H}_{\mathfrak{p}}$. Let $\mathbb{Q}_{\mathfrak{p}^2}$ be the quadratic unramified extension of $\mathbb{Q}_{\mathfrak{p}}$, and consider $\mathcal{H}_{\mathfrak{p}} = \mathbb{P}^1(\mathbb{Q}_{\mathfrak{p}^2})\smallsetminus\mathbb{P}^1(\mathbb{Q}_{\mathfrak{p}})$. It is a $\mathfrak{p}$-adic analogue to $\mathbb{H}$ in many ways: it has a rigid-analytic structure, and an action of $\mathrm{PGL}_2(\mathbb{Q}_{\mathfrak{p}})$ by fractional linear transformations. There is a theory of rigid-analytic 1-forms $\omega \in \Omega^1_{\mathcal{H}_{\mathfrak{p}}}$, which replaces the complex-analytic theory. We denote by $\Omega^1_{\mathcal{H}_{\mathfrak{p}},\mathbb{Z}}$ the forms having $\mathbb{Z}$-valued residues. Coleman integration allows us to make sense of $\int_{\tau_1}^{\tau_2}\omega \in \mathbb{C}_{\mathfrak{p}}$. Moreover, Darmon [**Dar01**] constructed a $\mathrm{PGL}_2(\mathbb{Q}_{\mathfrak{p}})$-equivariant pairing

$$⨍\colon \Omega^1_{\mathcal{H}_{\mathfrak{p}},\mathbb{Z}} \times \mathrm{Div}^0\,\mathcal{H}_{\mathfrak{p}} \to \mathbb{Q}_{\mathfrak{p}^2}^\times \subset \mathbb{C}_{\mathfrak{p}}^\times,$$

which refines Coleman integration into a multiplicative variant. Cap product induces a pairing

$$\begin{array}{ccc} \mathrm{H}^i(\Gamma,\Omega^1_{\mathcal{H}_{\mathfrak{p}},\mathbb{Z}}) \times \mathrm{H}_i(\Gamma,\mathrm{Div}^0\,\mathcal{H}_{\mathfrak{p}}) & \xrightarrow{\langle\cdot,\cdot\rangle} & \mathbb{C}_{\mathfrak{p}}^\times \\ \left(\phi, \sum_{\underline{\gamma}} \underline{\gamma}\otimes D_{\underline{\gamma}}\right) & \longmapsto & \sum_{\underline{\gamma}} ⨍_{D_{\underline{\gamma}}} \phi(\underline{\gamma}). \end{array}$$

3.3. The conjecture. Denote with a subindex f the f-isotypical part of a Hecke-module: the submodule on which the Hecke operator T_ℓ acts as multiplication by $a_\ell(f)$. An application of the Jacquet–Langlands transfer, together with Shapiro's lemma, yields the following result.

Theorem 3.1. *There is a natural isomorphism of Hecke modules*

$$\mathrm{H}^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}),\mathbb{Z})_f \cong \mathrm{H}^{n+s}(\Gamma,\Omega^1_{\mathcal{H}_{\mathfrak{p}},\mathbb{Z}})_f.$$

Next, consider the Γ-equivariant short exact sequence that defines the degree 0 divisors on $\mathcal{H}_{\mathfrak{p}}$

$$0 \to \mathrm{Div}^0\,\mathcal{H}_{\mathfrak{p}} \to \mathrm{Div}\,\mathcal{H}_{\mathfrak{p}} \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Taking Γ-coinvariants yields a long exact sequence in homology, from which we isolate the following piece:

$$\cdots \to \mathrm{H}_{n+s+1}(\Gamma,\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}_{n+s}(\Gamma,\mathrm{Div}^0\,\mathcal{H}_{\mathfrak{p}}) \to \mathrm{H}_{n+s}(\Gamma,\mathrm{Div}\,\mathcal{H}_{\mathfrak{p}}) \to \mathrm{H}_{n+s}(\Gamma,\mathbb{Z}) \to \cdots$$

Set $\underline{\omega}_f$ to be a fixed basis of the non-torsion part of $\mathrm{H}^{n+s}(\Gamma,\Omega^1_{\mathcal{H}_{\mathfrak{p}},\mathbb{Z}})_f$.

Conjecture 3.2. *Set*

$$\Lambda_f = \left\{\langle\underline{\omega}_f,\delta(c)\rangle\colon c \in \mathrm{H}_{n+s+1}(\Gamma,\mathbb{Z})\right\} \subset (\bar{\mathbb{Q}}_{\mathfrak{p}}^\times)^d.$$

Then Λ_f is a lattice in $(\mathbb{Q}_{\mathfrak{p}}^\times)^d$ and $A_f(\bar{\mathbb{Q}}_{\mathfrak{p}})$ is isogenous to $(\bar{\mathbb{Q}}_{\mathfrak{p}}^\times)^d/\Lambda_f$.

This conjecture was proven by Darmon [**Dar01**] in the case $F = \mathbb{Q}$, $B = \mathrm{M}_2(\mathbb{Q})$ and $d = 1$; by Dasgupta [**Das05**] when $F = \mathbb{Q}$, $B = \mathrm{M}_2(\mathbb{Q})$, and $d > 1$; by Dasgupta–Greenberg [**DG12**] and Longo–Rotger–Vigni ([**LRV12**]) in the case $F = \mathbb{Q}$ and B a division algebra; by Spiess [**Spi14**] when F is totally real, $B = M_2(F)$, $\mathbb{Q}_{\mathfrak{p}} = \mathbb{Q}_p$ and $d = 1$. To the best of our knowledge, it is open in all other cases.

We have developed and implemented algorithms to compute approximations to the periods defining Λ_f. In the next section we illustrate how these can be used to find equations for the putative A_f, at least in favourable situations.

4. Examples

4.1. Elliptic Curves. We explain first how to compute the equations of elliptic curves, from their p-adic periods. First, the knowledge of $\Lambda_f = \langle q_f \rangle$ gives us the p-adic period q_f, which we conjecture should be the same (up to taking a rational power whose denominator can be bounded by the p-adic valuation of q_f) as the period q_E attached to $E = E_f$. By possibly replacing q_f with q_f^{-1} we can assume that $v_{\mathfrak{p}}(q_f) > 0$. By using the q-expansion of the j-function we can recover

$$j(q_f) = q_f^{-1} + 744 + 196884 q_f + \cdots \in \mathbb{Q}_{\mathfrak{p}}^{\times}.$$

Instead of trying to recognize $j(q_f)$ algebraically, it is often better to proceed in a more indirect way, since $j(q_f)$ has much larger height than the minimal equation for the sought E. Instead, from the knowledge of $\mathfrak{N}$ (and therefore of the primes of bad reduction of E) we can guess the discriminant Δ_E of a minimal model. This just amounts to trying finitely many possibilities. One can then use the equation $j_E = c_4^3/\Delta_E$ to recover a p-adic approximation to c_4 (after taking a p-adic cube root). One can then recognize c_4 algebraically. Similarly, from the equation $1728\Delta_E = c_4^3 - c_6^2$ one may recover c_6. The candidate equation is then

$$E\colon Y^2 = X^3 - \frac{c_4}{48}X - \frac{c_6}{864}.$$

If E has the right conductor $\mathfrak{N}$ (which is easily computed using Tate's algorithm) one can then proceed to compare some terms of the L-series, until one is convinced that E is really attached to f. If one requires a proof, then one could use the Faltings'–Serre method, although we have not done this in practice.

Here is a new example of such a calculation. Let $F = \mathbb{Q}(\alpha)$, where α has minimal polynomial $f_\alpha(x) = x^6 - x^5 - 4x^4 + x^3 + 4x^2 + x - 1$. The discriminant of F is $\mathrm{disc}(F) = -367792$, and it has signature $(4, 1)$. Consider the level $\mathfrak{N} = (\alpha^3 - \alpha^2 - 2\alpha)$, which has norm 7. We take $\mathfrak{p} = \mathfrak{N}$, and consider the quaternion algebra B/F whose locus of ramification is the set of real places of F. There is a rational eigenclass $f \in S_2(\Gamma_0^{\mathcal{O}_F}(\mathfrak{N}))$. From f we compute $\omega_f \in \mathrm{H}^1(\Gamma, \Omega^1_{\mathcal{H}_p,\mathbb{Z}})$ and an approximation to $\Lambda_f = \langle q_f \rangle$. This yields the quantity

$$q_E \stackrel{?}{=} q_f = 7^{-2} \cdot 68530475965426443260903897030400405725776366704466935859 44 + O(7^{67}).$$

Using the method described above we find

$$c_4 = 16\alpha^5 + 16\alpha^4 - 48\alpha^3 - 32\alpha^2 + 32\alpha + 32$$
$$c_6 = 160\alpha^5 + 264\alpha^4 - 32\alpha^3 - 336\alpha^2 - 256\alpha - 56$$

yielding the elliptic curve

$$E_0\colon y^2 = x^3 + \frac{1}{3}\left(-\alpha^5 - \alpha^4 + 3\alpha^3 + 2\alpha^2 - 2\alpha - 2\right)x + \frac{1}{108}\left(-20\alpha^5 - 33\alpha^4 + 4\alpha^3 + 42\alpha^2 + 32\alpha + 7\right)$$

with conductor $\mathfrak{N}$ and seemingly correct L-series.

Note that in this case the working precision allowed us to also recognize $j(q_f^{-1})$ directly, as the algebraic number

$$j_E = \frac{1}{49}\left(163840\alpha^5 - 180224\alpha^4 - 557056\alpha^3 + 139264\alpha^2 + 360448\alpha + 368640\right).$$

One could have then immediately written down the curve

$$E_1\colon y^2 = x^3 - 3j_E(j_E - 1728)x - 2j_E(j_E - 1728)^2,$$

which has the right j-invariant, but its conductor is[1]

$$\mathfrak{N}\cdot(\alpha^2-\alpha-1)_4^8\cdot(3)^2\cdot(\alpha^4-3\alpha^2-3\alpha)_{43}^2\cdot(\alpha^5-5\alpha^3-\alpha^2+3\alpha+1)_{53}^2\cdot(\alpha^5-4\alpha^3-2\alpha^2+\alpha+2)_{63}^2.$$

After performing a quadratic twist by the element

$$-\frac{3931}{78178816}\alpha^5 - \frac{12315}{156357632}\alpha^4 + \frac{77849}{234536448}\alpha^3 + \frac{182557}{469072896}\alpha^2 - \frac{3}{9772352}\alpha - \frac{64003}{469072896}$$

we would still recover the curve E_0.

Finally, the global minimal model (recall that F has class number one) of E_0 is

$$E_{/F}\colon y^2 + \left(\alpha^2 - 1\right)y = x^3 - \left(\alpha^5 - \alpha^4 - 3\alpha^3 + \alpha^2 - \alpha - 1\right)x^2 - \left(\alpha^5 + \alpha^4 - 3\alpha^3 - 6\alpha^2 - 2\alpha + 2\right)x - \alpha^5 - \alpha^4 + 3\alpha^3 + 4\alpha^2 - 1.$$

4.2. Abelian surfaces. Suppose that A_f is principally polarizable, so that $A_f = \mathrm{Jac}(X_f)$ for a genus 2 (hyperelliptic) curve X_f. In this section we explain a method for computing an equation of X_f.

We expect $A_f(\bar{\mathbb{Q}}_\mathfrak{p})$ to be isogenous to $(\bar{\mathbb{Q}}_\mathfrak{p}^\times)^2/\Lambda_f$. Let $\left(\begin{smallmatrix}A\\B\end{smallmatrix}\right), \left(\begin{smallmatrix}B\\D\end{smallmatrix}\right) \in (\mathbb{Q}_\mathfrak{p}^\times)^2$ be generators of Λ_f. From these generators, we may recover the so-called "half-periods" $p_1 = (BD)^{-1/2}$, $p_2 = (AB)^{-1/2}$, $p_3 = B^{1/2}$. Write

$$X_f\colon y^2 = x(x-1)(x-x_1)(x-x_2)(x-x_3),$$

and define

$$\lambda_1 = 1 - x_1^{-1}, \quad \lambda_2 = (1-x_2)^{-1}, \quad \lambda_3 = x_3.$$

Teitelbaum provides in [**Tei88**] three power series in the variables p_1, p_2, p_3 such that

$$\lambda_k = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}^{(k)} p_1^i p_2^j p_3^{(i-j)}.$$

From this, one can compute the absolute invariants i_1, i_2, i_3 of X_f, defined as

$$i_1 = I_2^5/I_{10}, i_2 = I_2^3 I_4/I_{10}, \text{ and } i_3 = I_2^2 I_6/I_{10},$$

where I_2, I_4, I_6, I_{10} are the Igusa invariants. Next, from $\mathfrak{N} = \mathfrak{p}\mathfrak{D}$ one can again guess the discriminant I_{10}, which will be of the form $u \cdot 2^a \cdot \mathfrak{N}^2$, for some $u \in \mathcal{O}_F^\times$ and some $a \in \mathbb{Z}_{\geq 0}$. An approximation to $i_1 = I_2^5/I_{10}$ then allows us (after taking

[1]The notation $(a)_n$ with $a \in \mathcal{O}_F$ denotes that the ideal $(a) \subset \mathcal{O}_F$ has norm n.

a fifth root) to recover I_2. The approximation to $i_2 = I_2^3 I_4/I_{10}$ gives us I_4, and finally $i_3 = I_2^2 I_6/I_{10}$ gives I_6.

The final step is to apply Mestre's algorithm to find a genus-2 hyperelliptic curve X_f with invariants $(I_2 : I_4 : I_6 : I_{10})$.

The problem we are faced with is that A_f is determined up to isogeny, so we should allow for "isogenous" Λ_f. This will allow us to find Igusa invariants of smaller height, which we have a chance to identify. Recall that in the elliptic curve case $\mathbb{C}_p^\times/q_1^{\mathbb{Z}} \sim \mathbb{C}_p^\times/q_2^{\mathbb{Z}}$ if and only if there exist $y, z \in \mathbb{Z} \setminus \{0\}$ with $q_1^y = q_2^z$. The right analogue in higher dimension is provided by the following result.

Theorem 4.1 (Kadziela, [**Kad07**]). *Let $V_1, V_2 \in M_d(\mathbb{Q}_{\mathfrak{p}}^\times)$ whose columns generate lattices Λ_1 and Λ_2. Then $(\bar{\mathbb{Q}}_{\mathfrak{p}}^\times)^d/\Lambda_1$ is isogenous to $(\bar{\mathbb{Q}}_{\mathfrak{p}}^\times)^d/\Lambda_2$ if and only if*

$$V_1^Y = {}^Z V_2, \qquad \textit{for some } Y, Z \in M_d(\mathbb{Z}).$$

Remark 4.2. The notation used in the theorem requires some explanation. Given a matrix $V \in M_d(\mathbb{Q}_{\mathfrak{p}}^\times)$ and a matrix $X \in M_d(\mathbb{Z})$, we define the matrix V^X to be the unique matrix $W \in M_d(\mathbb{Q}_{\mathfrak{p}}^\times)$ satisfying

$$\ell(W) = X\ell(V)$$

for all characters $\ell\colon \mathbb{Q}_{\mathfrak{p}}^\times \to \mathbb{Q}_{\mathfrak{p}}$. Here ℓ applied to a matrix is used to mean the matrix obtained by applying ℓ to its coefficients. Similarly, by ${}^X V$ we mean the unique matrix $W \in M_d(\mathbb{Q}_{\mathfrak{p}}^\times)$ satisfying

$$\ell(W) = \ell(V)X,$$

for all characters ℓ as above.

In [**GM16**] we report on two numerical calculations that illustrate how the above method can be used to compute equations of genus two curves C whose Jacobian is attached to a modular form over a number field. We end this note with an example of a surface defined over $\mathbb{Q}$. Recall that in this case Conjecture 3.2 is proven. Therefore, the above method can in fact be regarded as a means of computing the p-adic $\mathcal{L}$-invariant of Mazur–Tate–Teitelbaum [**MTT86**]. Explicit examples of such $\mathcal{L}$-invariants where computed in [**Tei88**] when C is a modular curve; we remark that the method presented in this note can be used more generally when $\mathrm{Jac}(C)$ is a simple factor of a modular Jacobian.

Consider the hyperelliptic curve

$$C\colon y^2 = x^6 + 6x^5 + 11x^4 + 14x^3 + 5x^2 - 12x,$$

whose Jacobian has conductor $165 = 3\cdot 5\cdot 11$ and is a simple factor of $\mathrm{Jac}(X_0(165))$. This curve can be found in [**GJG03**, Page 412].

Let $B/\mathbb{Q}$ be the (indefinite) quaternion algebra of disctriminant 15, which can be described as $\mathbb{Q}\langle i, j\rangle$, with $i^2 = -3$ and $j^2 = 5$.

At level 11 we find a two-dimensional Hecke-eigenspace for which the Hecke operator T_2 has characteristic polynomial $x^2 + 2x - 1$, of discriminant 8. The automorphic form f gives thus rise to the field $K_f \cong \mathbb{Q}(\sqrt{2})$. The integration pairing gives $\Lambda_f = \left(\begin{smallmatrix} A_0 & B_0 \\ C_0 & D_0 \end{smallmatrix}\right) = \mathbb{Z}[T_2] \cdot \left(\begin{smallmatrix} A_0 & B_0 \end{smallmatrix}\right)$, with

$$A_0 = 11^{-16} \cdot 5085873501488351836860609367615622367014 8823 + O(11^{42})$$
$$B_0 = 37393339478940759509269993612373109547450 + O(11^{41}).$$

By guessing appropriate Kadziela matrices we obtain a new set of periods:

$$A = A_0 B_0^2 \qquad\qquad B = B_0^{-2}.$$

They give rise to the absolute invariants

$$(i_1, i_2, i_3) = \left(\frac{I_2^5}{I_{10}}, \frac{I_2^3 I_4}{I_{10}}, \frac{I_2^2 I_6}{I_{10}}\right) = \left(\frac{28125651982744}{13476375}, -\frac{634841652013}{26952750}, \frac{163196533921}{35937000}\right)$$

which match with those of C.

Finally, the p-adic $\mathcal{L}$-invariant of $A_f = \mathrm{Jac}(C)$ is the element in $\mathbb{Z}[T_2] \otimes_{\mathbb{Z}} \mathbb{Q}_{11}$ written $a + bT_2$, where:

$$a = 37980088449048045735895106159967066662648944 + O(11^{41})$$

$$b = 44912627696644823043769914019754283487507211 + O(11^{41}).$$

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Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Catalonia
E-mail address: xevi.guitart@gmail.com

Mathematics Institute, University of Warwick, United Kingdom
E-mail address: M.Masdeu@warwick.ac.uk