Modular abelian varieties over number fields

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Modular abelian varieties over $\mathbb{Q}$

Shimura's construction

Associates to each newform $f \in S_2(\Gamma_1(N))$ an abelian variety $A_f/\mathbb{Q}$:

- $A_f/\mathbb{Q}$ is $\mathbb{Q}$-isogenous to a simple factor of $J_1(N)/\mathbb{Q}$
- $L(A_f/\mathbb{Q}; s) \sim \prod_{\sigma: E_f \hookrightarrow \mathbb{C}} L({}^\sigma f; s)$

$A/\mathbb{Q}$ is modular if it is $\mathbb{Q}$-isogenous to some A_f

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Theorem (Ribet + Serre's Conjecture)

A simple variety $A/\mathbb{Q}$ is modular if and only if it is of GL_2-type (i.e $\mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ is a number field E with $[E : \mathbb{Q}] = \dim A$).

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Here, modular means either of these two equivalent conditions:

- $A/\mathbb{Q}$ is $\mathbb{Q}$-isogenous to some factor of $J_1(N)$, for some N.
- $L(A/\mathbb{Q}; s) \sim$ product of L-series of newforms $f \in S_2(\Gamma_1(N))$.

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If we replace $\mathbb{Q}$ for K these conditions are no longer equivalent.

Modular abelian varieties over a number field K

B/K a non-CM abelian variety ($\bar{\mathbb{Q}}$-simple, $\mathrm{End}(B) = \mathrm{End}_K(B)$)
$K/\mathbb{Q}$ Galois

Definition

- B/K is modular if it is K-isogenous to a simple factor of $J_1(N)_K$.
- B/K es strongly modular if $L(B/K;s) \sim \prod_f L(f;s)$, for some newforms $f \in S_2(\Gamma_1(N_f))$.

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Theorem (Ribet-Pyle)

B/K is modular if and only if

- B is a $\mathbb{Q}$-variety: for each $s \in Gal(K/\mathbb{Q})$ there exists an isogeny $\mu_s : {}^sB \rightarrow B$ compatible with the endomorphisms of B.
- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ is:
    - A totally real number field F with $[F:\mathbb{Q}] = \dim B$
    - A quaternion algebra over F with $2[F:\mathbb{Q}] = \dim B$

These modular varieties are also called building blocks.

Strongly modular abelian varieties

Aim

To characterize the abelian varieties B/K that are strongly modular.

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Observation: strongly modular $\Rightarrow$ modular

$$L(B/K;s) = L((\mathrm{Res}_{K/\mathbb{Q}}B)/\mathbb{Q};s)$$

$$\begin{aligned} B/K \text{ strongly modular} \quad &\Leftrightarrow \quad (\mathrm{Res}_{K/\mathbb{Q}}B)/\mathbb{Q} \text{ strongly modular} \\ &\Leftrightarrow \quad \mathrm{Res}_{K/\mathbb{Q}}B \sim_{\mathbb{Q}} \prod A_f \end{aligned}$$

$$(\mathrm{Res}_{K/\mathbb{Q}}B)_K \sim_K \prod_{s\in\mathrm{Gal}(K/\mathbb{Q})} {}^sB$$

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$\mathbb{Q}$-varieties and Galois Cohomology

B/K building block, $K/\mathbb{Q}$ Galois.

- B is $\bar{\mathbb{Q}}$-simple and $\mathrm{End}^0(B) = \mathrm{End}^0_K(B)$.
- $\mathrm{End}^0(B) = F$, $\mathrm{End}^0(B) = D$ (quaternion algebra over F)
- For each $s \in \mathrm{Gal}(K/\mathbb{Q})$ we have $\mu_s : {}^sB \rightarrow B$.

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Definition: $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)$

- $s, t \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow c_{B/K}(s,t) = \mu_s \circ {}^s\mu_t \circ \mu_{st}^{-1} \in Z(\mathrm{End}^0(B)) = F$
- $[c_{B/K}] \in H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*)[2]$
- $[c_{B/K}]$ only depends on the K-isogeny class of B.

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Definition: $[c_B] \in H^2(G_{\mathbb{Q}}, F^*)$

- $[c_B] = \mathrm{Inf}[c_{B/K}]$, $\mathrm{Inf} : H^2(\mathrm{Gal}(K/\mathbb{Q}), F^*) \to H^2(G_{\mathbb{Q}}, F^*)$
- $[c_B]$ only depends on the $\bar{\mathbb{Q}}$-isogeny class of B.

Strongly modular varieties: Results

Proposition

$\mathrm{End}^0_{\mathbb{Q}}(\mathrm{Res}_{K/\mathbb{Q}} B) \simeq \mathrm{End}^0(B) \otimes_F F^{c_{B/K}}[\mathrm{Gal}(K/\mathbb{Q})]$

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Theorem (Characterization of strongly modular varieties)

A non-CM building block B/K is strongly modular if and only if

- $K/\mathbb{Q}$ is abelian
- $[c_{B/K}]$ is symmetric.

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There exists a variety B_0/K strongly modular in the $\bar{\mathbb{Q}}$-isogeny class of B if and only if K contains a splitting field for $[c_B]$.

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Tate: $\exists\alpha : G_{\mathbb{Q}} \to \bar{F}^*$ s.t. $c_B(\sigma,\tau) = \alpha(\sigma)\alpha(\tau)\alpha(\sigma\tau)^{-1}$.
The field $\bar{\mathbb{Q}}^{\ker(\alpha \mod F^*)}$ is a splitting field for $[c_B]$

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A family of curves of Genus 2

Baba-Granath family:

$$
\begin{aligned}
C_j: \quad Y^2 = &\left(-4+3\sqrt{-6j}\right)X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28+9\sqrt{-6j}\right)X^4 \\
&+ 16(27j+16)^2X^3 + 12(27j+16)2\left(28-9\sqrt{-6j}\right)X^2 \\
&- 48(27j+16)^3X + 8(27j+16)3\left(4+3\sqrt{-6j}\right)
\end{aligned}
$$

- $B_j = \mathrm{Jac}(C_j)$. Then B_j/K is modular and $\mathrm{End}^0(B_j) \simeq (2,3)_{\mathbb{Q}}$
- $K = \mathbb{Q}(\sqrt{-6j}, \sqrt{j}, \sqrt{-(27j+16)}, \sqrt{-2(27j+16)})$
- We have computed $[c_{B_j}] \in H^2(G_{\mathbb{Q}}, \mathbb{Q}^*)[2]$
- $H^2(G_{\mathbb{Q}}, \mathbb{Q}^*)[2] \simeq \mathrm{Hom}(G_{\mathbb{Q}}, \mathbb{Q}^*/\{\pm 1\}\mathbb{Q}^{*2}) \times H^2(G_{\mathbb{Q}}, \{\pm 1\})$
  - $\overline{\mathrm{Gal}(\mathbb{Q}(\sqrt{-(27j+16)}, \sqrt{-j(27j+16)})/\mathbb{Q})} = \langle \sigma, \tau \rangle$
  - $\overline{[c_{B_j}]} : \sigma \mapsto 3 \quad \tau \mapsto 2$
  - $[c_{B_j}]_{\pm} = (-(27j+16), 3)_{\mathbb{Q}} \cdot (-j(27j+16), 2)_{\mathbb{Q}} \cdot (2,3)_{\mathbb{Q}}$

A concrete example: j=-4/27

- $K = \mathbb{Q}(\sqrt{-6}, \sqrt{-3})$ but B_j is not strongly modular over K.
- $L = K(\sqrt{-1})$ contains a splitting field for $[c_{B_j}]$.
- $[c_{B_j/L}]$ not symmetric $\rightarrow B_j/L$ not strongly modular: we should twist
- $\gamma = \sqrt{6} + \sqrt{18}$

$$C_\gamma: \ \gamma Y^2 = \left(-4 + 3\sqrt{-6j}\right) X^6 - 12(27j+16)X^5 - 6(27j+16)\left(28 + 9\sqrt{-6j}\right) X^4 + 16(27j+16)^2 X^3 + 12(27j+16)2\left(28 - 9\sqrt{-6j}\right) X^2 - 48(27j+16)^3 X + 8(27j+16)3\left(4 + 3\sqrt{-6j}\right)$$

- $[c_{B_\gamma/L}]$ symmetric $\rightarrow B_\gamma/L$ strongly modular
- γ is the solution of the embedding problem corresponding to the non-symmetric part of $[c_{B_j/L}]_\pm$.

A concrete example: j=-4/27

We find $f \in S_2(\Gamma_1(2^4 \cdot 3^4), \chi)$

$$\begin{aligned} f &= q - \sqrt{3}\, q^5 + 3i\, q^7 - 3\sqrt{3}\, q^{11} + q^{13} - 2i\sqrt{3}\, q^{17} - 6i\, q^{19} \\ &+ 3\sqrt{3}\, q^{23} + 2\, q^{25} - 5\sqrt{3}i\, q^{29} - 3i\, q^{31} + \cdots \end{aligned}$$

and $g \in S_2(\Gamma_1(2^6 \cdot 3^4), \varepsilon)$

$$\begin{aligned} g &= q - \sqrt{3}\, q^5 + 3i\, q^7 - 3\sqrt{3}\, q^{11} - q^{13} + 2i\sqrt{3}\, q^{17} + 6i\, q^{19} \\ &- 3\sqrt{3}\, q^{23} + 2\, q^{25} - 5\sqrt{3}i\, q^{29} - 3i\, q^{31} + \cdots \end{aligned}$$

such that

$$L(B_\gamma/L, s) = L(A_f, s)^2 \cdot L(A_g, s)^2$$

$$\mathrm{Res}_{L/\mathbb{Q}} B_\gamma \sim_{\mathbb{Q}} A_f^2 \times A_g^2$$

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