Explicit computation of ATR Darmon points

Computation joint with Marc Masdeu

<u>Goal</u>: provide some numerical evidence towards Darmon's conjectural ATR method

<u>talk</u>

1. Heegner points
2. ATR points
3. Numerical evidence & computational issues

1. Heegner points

$E/\mathbb{Q}$ elliptic curve, $\text{cond}(E)=N$, $r_{an}(E/\mathbb{Q}) := \text{ord}_{s=1} L(E/\mathbb{Q}, s)$, $r_{alg} := \text{rank}(E(\mathbb{Q}))$.

<u>Theorem</u> (Gross–Zagier, Kolyvagin)

If $r_{an}(E/\mathbb{Q}) \leq 1$ then $r_{an}(E/\mathbb{Q}) = r_{an}(E/\mathbb{Q})$

• If $r_{an}(E/\mathbb{Q}) = 1 \Rightarrow$ point of ∞ order can be computed with the Heegner point method

• <u>Thm</u> E is modular: $\exists$ newform $f_E \in S_2(\Gamma_0(N))$ s.t. $L(E/\mathbb{Q}, s) = L(f_E, s)$

• $f_E(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$, $a_p = p+1 - |E(\mathbb{F}_p)|$

• $\omega_{f_E} = (2\pi i) f_E(z)\, dz$ holomorphic 1-form on $X_0(N)$

modular curve $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N) \backslash \mathcal{H}$

• $K \subseteq \mathbb{C}$ quadratic imaginary, $\tau \in K \cap \mathcal{H}$

$$J_\tau = \int_{i\infty}^{\tau} \omega_{f_E} \in \mathbb{C}/\Lambda_{f_E} \overset{\eta}{\sim} E(\mathbb{C}) \quad ; \quad P_\tau = \eta(J_\tau) \in E(H_\tau)$$

H_τ/K finite abelian

$$\Lambda_{f_E} = \left\{ \int_\gamma \omega_{f_E} \;\middle|\; \gamma \in H_1(X_0(N), \mathbb{Z}) \right\}$$

• Heegner point $P_K = \text{Tr}_K^{H_\tau}(P_\tau) \in E(K)$

if $r_{an}(E/K) = 1 \Rightarrow P_K$ is non-torsion ~~and $P_K \in E(\mathbb{Q})$~~

GZ formula

<u>Key property</u>: E is <u>geometrically modular</u>:

$$\exists\, \pi_E : X_0(N) \longrightarrow E \quad \text{exh. and } /\mathbb{Q}$$

$$K \cap \mathcal{H} \ni \tau \in X_0(N)(H_\tau) \text{ (moduli + CM theory)} \Rightarrow P_\tau = \pi_E(\tau)$$

• E/F, F totally real number field, E/F cond$(E) = \mathcal{N} \subset F$

<u>Thm</u>: E is modular; i.e, there exists a Hilbert modular form $f_E \in S_2(\mathcal{N})$ s.t. $L(E/F, s) = L(f_E, s)$ (well, under some technical conditions)

<u>Theorem</u> (GZK + Zhang) Suppose that E satisfies:

(JL) either $[F:\mathbb{Q}]$ is odd or $\exists\, \mathfrak{p} \subset F$ s.t. ord$_\mathfrak{p}(\mathcal{N})$ is odd.

then $r_{an}(E/F) \leq 1 \Rightarrow r_{an}(E/F) = r_{alg}(E/F)$.

<u>Why (JL)?</u> If E satisfies (JL) then it is geometrically modular:

$$\exists \quad \pi_E : \mathrm{Jac}(X) \longrightarrow E \text{ over } F,$$

X/F a Shimura curve, has CM points, HP on E, $\mathbb{Q}$ lab

• If E does not satisfy (JL), it is not known to be geometrically modular unless it is a $\mathbb{Q}$-curve ($\sigma_E \sim E \ \forall \sigma \in \mathrm{Gal}(F/\mathbb{Q})$), and no Heegner Points are available.

• $E \in JL$, E not $\mathbb{Q}$-curve $\Rightarrow$ no systematic way of producing alg. non-torsion points.

2. ATR points

• E/F, F real quadratic, cond$(E) = 1$, $h^+(F) = 1$, $v_0, v_1 : F \hookrightarrow \mathbb{R}$

• K/F quadratic extension s.t. $\begin{cases} v_0 \text{ extends to a complex place of } K \\ v_1 \text{ extends to a pair of real places of } K \end{cases}$

• Such K is called Almost totally Real (ATR)

<u>Rmk</u>: M/F quadratic, sign $L(E/M, s) = -1 \Leftrightarrow M$ is ATR

<u>Aim</u>: construct ~~P_K~~ points on $E(K^{ab})$ as integrals of f_E (the HMF attached to E)

$$f_E(z_0, z_1) = \sum_{n \in \mathcal{O}_F^+} a_{(n)} e^{2\pi i \left(\frac{n_0 z_0}{d_0} + \frac{n_1 z_1}{d_1}\right)}, \quad (d) = \text{different of } F, \ n_i = v_i(n)$$

$x \in F$, $v_i(x) = x_i$

$$a_\mathfrak{p} = \mathrm{Nm}(\mathfrak{p}) + 1 - \#E(\mathcal{O}_F/\mathfrak{p}), \quad \sum_{\mathfrak{n} \in \mathcal{O}_F} a_\mathfrak{n} \mathrm{Nm}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - a_\mathfrak{p} \mathrm{Nm}(\mathfrak{p})^{-s} + \mathrm{Nm}(\mathfrak{p})^{1-2s}}$$

$\omega_{f_E} = \frac{(2\pi i)^2}{\sqrt{D_F}} f(z_0, z_1)\, dz_0\, dz_1$ holomorphic 2-form on $SL_2(\mathcal{O}_F)\backslash \mathcal{H}\times\mathcal{H} = X$ (open) Hilbert modular surface attached to $SL_2(\mathcal{O}_F)$.

$$\omega^+_{f_E} = \frac{2\pi i}{\sqrt{D_F}}\left(f(z_0, z_1)\, dz_0\, dz_1 + f(u_0 z_0, u_1 z_1)\, d(u_0 z_0)\, d(u_1 z_1)\right)$$

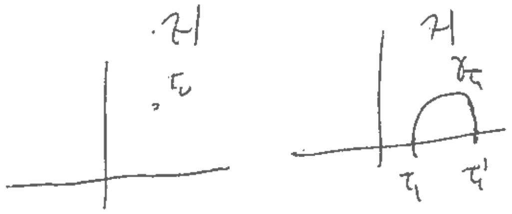
$\mathcal{O}_F^\times = \{\pm 1\}\times\langle u\rangle$, $u_0 > 0$, $u_1 < 0$

Conjecture (Oda) $\Lambda^+_{f_E} = \left\{ \int_\gamma \omega^+_{f_E} \mid \gamma \in H_2(X, \mathbb{Z}) \right\} \subseteq \mathbb{C}$ is a lattice

and $\mathbb{C}/\Lambda^+_{f_E} \underset{\eta}{\sim} E$.

Let $\varphi : K \hookrightarrow M_2(F)$ optimal embeddings ($\varphi(K)\cap M_2(\mathcal{O}_F) = \varphi(\mathcal{O}_K)$)

K ATR $\Rightarrow$ $\begin{cases} \varphi(K^\times) \text{ has a single fixed point } \tau_0 \in \mathcal{H} \text{ acting via } v_0 \\ \varphi(K^\times) \text{ has 2 fixed points } \tau_1, \tau_1' \in \mathbb{R} = \partial\mathcal{H} \text{ acting via } v_1 \end{cases}$



$\{\tau_0\}\times\gamma_{\tau_1} \subseteq \mathcal{H}\times\mathcal{H}$

R_φ = image of $\{\tau_0\}\times\gamma_{\tau_1}$ ~~under~~ in $X = SL_2(\mathcal{O}_F)\backslash\mathcal{H}\times\mathcal{H}$

Fact: $R_\varphi \in H_1(X, \mathbb{Z})$ and $\exists\, T_\varphi \in C_2(X, \mathbb{Z})$ s.t. $\partial T_\varphi = R_\varphi$ (well, up to torsion)

$$J_\varphi = \int_{T_\tau} \omega^+_{f_E} \in \mathbb{C}/\Lambda^+_{f_E} \sim E(\mathbb{C})$$

Conjecture (Darmon): $P_\tau = \eta(J_\varphi) \in E(H)$, H = Hilbert class field of K

$P_K = \mathrm{Tr}^H_K(P_\varphi) \in E(K)$ is non-torsion if and only if $L'(E/K) \neq 0$

$P = \sum_\varphi P_\varphi$, φ runs over a set of ~~inequivalent~~ rep's of optimal embeddings mod $SL_2(\mathcal{O}_F)$.

- J_φ explicit except for T_φ ⇝ Darmon–Logan algorithm
- DL verified (numerically) the conjecture for 3 elliptic curves
- They were $\mathbb{Q}$-curves: they have Heegner points (Darmon–Rotger–Zhao)
- Goal: compute an ATR point and check that it is non-torsion algebrai

3. Numerical evidence & computational issues

$E: \; y^2 - xy - \omega y = x^3 + (2+2\omega)x^2 + (162+3\omega)x + (71+34\omega)$ (Pinch), $\omega = \frac{1+\sqrt{509}}{2}$

$K = F(\sqrt{\underbrace{9144\omega + 98577}_{\alpha}})$, $P = \left(\omega + 17, \frac{\sqrt{\alpha}}{2} + \frac{\sqrt{509}}{2} + 9\right) \in E(K)$ non-torsion

$\updownarrow$

$z \in \mathbb{C}/\Lambda_E$

$[H:K] = 2$

$J_K = J_{\epsilon_1} + J_{\epsilon_2}$, ϵ_1 and ϵ_2 optimal embeddings $K \hookrightarrow M_2(F)$

~~Dk~~ Computed J_K up to 12 digits of precision and

$\cancel{J_K} \; z \approx -4z$ holds up to the precision

- $a_{\mathfrak{n}}$ for $|\mathfrak{n}| \leq 4 \cdot 10^8$
- took 2-days in ~~the~~ (?), the 32 processor machine with 32.0 GB at the MP

·) 2 computational difficulties:

① DL Algorithm: given $c \in F$, need to find $q_1, \ldots, q_n \in \mathcal{O}_F$ s.t. $c = 1 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots + \cfrac{1}{q_n}}}$

- If F is norm-euclidean $\to$ OK
- only a finite # of norm euclidean real quadratic fields
- real quadratic fields with $h^+(F) = 1$ are conjectured to be 2-stage eu[clidean]

$\forall\; a, b \in \mathcal{O}_F$, $b \neq 0$, there exist

·) $q, r \in \mathcal{O}_F$ s.t. $a = bq + r$, $Nm(r) < Nm(b)$, or

·) q_1, q_2, r_1, r_2 s.t. $a = bq_1 + r_1$, $b = r_1 q_2 + r_2$, $Nm(r_2) < Nm($…

Cooke proved that some real quad. fields are E_2 ($\mathbb{Q}(\sqrt{853})$ the biggest, …

__Thm__ (G-Masden) All real quadratic fields of disc ≤ 8000 are 2-stage euclidean

② to compute $\int_a^b \int_c^d \omega_{f_E}^+$, $a, b, c, d \in \mathcal{H}$, # of a_n's depends on $Im(a), Im(b), Im(c), Im(d)$

__Theorem__: There exists a constant ξ (depending only on F) s.t.

$$\int_a^b \int_c^d \omega_{f_E}^+ = \int_{a_1}^{b_1} \int_{c_1}^{d_1} \omega_{f_E}^+ + \cdots + \int_{a_n}^{b_n} \int_{c_n}^{d_n} \omega_{f_E}^+ \text{ with } Im(a_i), Im(b_i), Im(c_i), Im(d_i) \geq \ldots$$

__Idea__: use $SL_2(\mathcal{O}_F)$-invariance.