On the modularity level of modular abelian varieties over number fields

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- Every A of GL_2-type is modular: is a quotient of $J_1(N)$ for some N.
- By Carayol's formula: $N^{\dim A} = \mathcal{N}_{\mathbb{Q}}(A)$.
- GL_2-type varieties are simple over $\mathbb{Q}$, but they may factor over $\bar{\mathbb{Q}}$.
- Ribet determined their simple factors over $\bar{\mathbb{Q}}$.

Modular abelian varieties over number fields

Definition

A building block is an abelian variety $B/\bar{\mathbb{Q}}$ such that

- ${}^{\sigma}B \sim B$ for all $\sigma \in G_{\mathbb{Q}}$ (equivariant with respect to $\mathrm{End}_{\bar{\mathbb{Q}}}(B)$).
- $\mathrm{End}^0_{\bar{\mathbb{Q}}}(B)$ is a central division algebra over a field F of Schur index $t = 1$ or $t = 2$ and $t[F : \mathbb{Q}] = \dim B$.

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Theorem (Ribet and Pyle)

Let $A/\mathbb{Q}$ be a GL_2-type variety. Suppose that $A_{\bar{\mathbb{Q}}}$ does not have CM.

- $A \sim_{\bar{\mathbb{Q}}} B^n$, where $B/\bar{\mathbb{Q}}$ is a building block.
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- We aim to give an analogous of Carayol's formula: a formula relating the conductor of B and its level of modularity N.

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- Let $F = \mathbb{Q}(\{a_p^2/\varepsilon(p)\}_{p\nmid N}) \subseteq E$.
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Theorem (González-Lario)

$L = \bar{\mathbb{Q}}^{\cap \ker \chi_s}$ (identifying χ_s with a Galois character).

- Then we have

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1. If one of the following conditions is satisfied
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   b) $2 \cdot \mathcal{N}_L(B) \cdot \mathfrak{f}^{\dim B} = N^{\dim B}$, if $v_2(\mathfrak{f}_L) = 3$ and there exists $K \subseteq L$ with $v_2(\mathfrak{f}_K) = 2$.

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- Examples of $\mathbb{Q}$-curves where $\mathcal{N}_L(B)$ does not belong to $\mathbb{Z}$.
- Examples where b) does occur.

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Example

- $\dim(A_f) = 2$, $A_f \sim_L C^2$ and $[L:\mathbb{Q}] = 2$.
- $\mathrm{Res}_{L/\mathbb{Q}} C \sim_{\mathbb{Q}} A_f$.
- $\mathcal{N}_{\mathbb{Q}}(\mathrm{Res}_{L/\mathbb{Q}} C) = \mathcal{N}_{\mathbb{Q}}(A_f) = N^2$.
- Milne's formula for the conductor of the restriction of scalars:

$$N_{L/\mathbb{Q}}(\mathcal{N}_L(C)) d_{L/\mathbb{Q}}^2 = N^2$$

- $\mathcal{N}_L(C) \cdot \mathfrak{f}_L = N$

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Strategy

- Compute $\mathrm{Res}_{L/\mathbb{Q}} B$ as a product of A_g's
- Apply Milne and Carayol's formulas
- Do the computations to check the formula.

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Let $G = \langle \chi_s \mid s \in \mathrm{Gal}(E/F)\rangle$ and define an action

$$\begin{array}{ccc} \mathrm{Gal}(E/F) \times G & \longrightarrow & G \\ (s, \chi) & \longmapsto & s \cdot \chi = \chi_s\, {}^s\chi \end{array}$$

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- If $A_g \sim_L B^m$ and $A_f \sim_L B^n$ then $g = f \otimes \chi$ for some character of L.
- Dimension argument.

- $N_{L/\mathbb{Q}}(\mathcal{N}_L(B))d_{L/\mathbb{Q}}^{2\dim B} = \prod_{\chi\in G} N_\chi^{\dim B}$, N_χ = level of $f \otimes \chi$.

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- This gives a formula for $\mathcal{N}_L(B)$.
- Results of Atkin-Li give formulas for N_χ in terms of N and $\mathfrak{f}_\chi, f_\varepsilon$.
- Under the assumptions of the theorem we have a control of the N_χ and this formula simplifies to $\mathcal{N}_L(B)$

$$\mathcal{N}_L(B) \cdot \mathfrak{f}_L^{\dim B} = N^{\dim B}.$$

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