Computing equations of elliptic curves over number fields via p-adic methods

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- Modularity: elliptic curves (should) correspond to modular forms

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1. Modularity of elliptic curves over number fields
2. $K = \mathbb{Q}$ (and K totally real)
3. K non-totally real: A p-adic construction
4. Explicit computations and tables

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- Rational eigenclass $f \in H^{n+s}(Y_0(\mathcal{N}), \mathbb{C})$ such that

$$T_{\mathfrak{l}} f = a_{\mathfrak{l}} f \text{ with } a_{\mathfrak{l}} \in \mathbb{Z} \text{ for all } \mathfrak{l}$$

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Conjecture

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- K totally real: Eicher–Shimura generalizes (at least in some cases). The geometric object is a Shimura curve.

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Outline

The $\mathfrak{p}$-adic integration pairing

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- More generally: $\Gamma\subset B^\times$ non-split quaternion algebras
  - $n+s\rightsquigarrow$ number of infinite places of K at which B splits

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- For $K \neq \mathbb{Q}$ it is open
  - Λ_f is explicitly computable in some cases
  - extensive numerical evidence for the conjecture
  - in practice, this can be used to compute E_f

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  - Teitelbaum: $\Omega^1_{\mathcal{H}_{\mathfrak{p}}}(\mathbb{Z}) \simeq \mathrm{Meas}_0(\mathbb{P}^1(K_{\mathfrak{p}}), \mathbb{Z})$
  - Need integrals of the form $⨍_{\mathbb{P}^1(K_{\mathfrak{p}})} \left(\frac{t-\tau_1}{t-\tau_2}\right) d\mu_f(t)$
  - Riemann products $\rightsquigarrow$ exponential algorithm
  - use overconvergent cohomology instead $\rightsquigarrow$ polynomial algorithm (generalization of Steven's overconvergent modular symbols)

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- Similarly: over 300 curves over fields of degree 2, 3, 4, 5.

Tables (Cubic fields of signature $(1,1)$)

$\lvert\Delta_K\rvert$	$f_K(x)$	$\mathrm{Nm}(\mathfrak{N})$	$\mathfrak{p}\mathfrak{D}\mathfrak{m}$	$c_4(E), c_6(E)$
23	$[1,0,-1]$	185	$(r^2+1)_5(3r^2-r+1)_{37}(1)$	$643318r^2-1128871r+852306$, $925824936r^2-1624710823r+1226456111$
31	$[-1,1,0]$	129	$(-r-1)_3(-3r^2-2r-1)_{43}(1)$	$-4787r^2+10585r+3349$, $1268769r^2-371369r+424764$
44	$[1,1,-1]$	121	$(2r-1)_{11}(r^2+2)_{11}(1)$	$4097022r^2-6265306r+7487000$, $14168359144r^2-21861492432r+260391407$
44	$[1,1,-1]$	121	$(2r-1)_{11}(r^2+2)_{11}(1)$	$1774r^2-1434r-1304$, $-42728r^2-123104r-54300$
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59	$[-1,2,0]$	34	$(-r^2-1)_2(-r^2-2r-2)_{17}(1)$	$262r^2+513r+264$, $-2592r^2+448r+13231$
59	$[-1,2,0]$	34	$(-r^2-2r-2)_{17}(-r^2-1)_2(1)$	$16393r^2+20228r-12524$, $4430388r^2-5579252r+1619039$
59	$[-1,2,0]$	46	$(-2r^2+r-2)_{23}(-r^2-1)_2(1)$	$18969r^2+8532r+41788$, $4216716r^2+1911600r+9298151$
59	$[-1,2,0]$	74	$(-r^2-1)_2(2r^2+2r+1)_{37}(1)$	$33054r^2+15049r+72776$, $9702640r^2+4400116r+21401723$
59	$[-1,2,0]$	88	$(-r^2-1)_2(r-2)_{11}(r^2+r+1)_4$	$16609r^2+7084r+37332$, $3522136r^2+1613876r+7760395$
59	$[-1,2,0]$	187	$(2r^2+r+2)_{17}(r-2)_{11}(1)$	$-32r^2-848r+432$, $-7600r^2+23368r-8704$
76	$[-2,-2,0]$	117	$(2r^2-r-3)_{13}(-r^2+2r+1)_9(1)$	$48r+16$, $-128r^2-224r-216$
83	$[-2,1,-1]$	65	$(r+1)_5(-2r+1)_{13}(1)$	$3089r^2+1086r+4561$, $333604r^2+117840r+493059$

Computing equations of elliptic curves over number fields via p-adic methods

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