Fractalization of a period-two curve in a piecewise-linear quasiperiodically forced map

Rafael Martinez Vergara and Joan Carles Tatjer
Departament de Matemàtiques i Informàtica, Universitat de Barcelona,
Gran Via de les Corts Catalanes, 585. 08007 Barcelona. Spain.

Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

CEDYA, July 9, 2026.

Index

Quasiperiodically forced systems

Basic definitions

Definition 1

Given a topological space X, a **quasiperiodically forced system (qpf)** is a continuous map $T : \mathbb{T} \times X \to \mathbb{T} \times X$ of the form

$$T(\theta, x) = (\theta + \omega \mod 2\pi, T_\theta(x)), \tag{1.1}$$

with $\omega \notin 2\pi\mathbb{Q}$.

Basic definitions

Definition 1

Given a topological space X, a **quasiperiodically forced system (qpf)** is a continuous map $T : \mathbb{T} \times X \to \mathbb{T} \times X$ of the form

$$T(\theta, x) = (\theta + \omega \mod 2\pi, T_\theta(x)), \tag{1.1}$$

with $\omega \notin 2\pi\mathbb{Q}$.

We will restrict to the case where $X = \mathbb{R}$, and the fibre maps T_θ are monotonically increasing on $\mathbb{R}$. Let $\mathbb{T}^{\mathbb{R}}$ denote the set of functions from $\mathbb{T}$ to $\mathbb{R}$. To a qpf system T on $\mathbb{T} \times \mathbb{R}$, we associate the operator $\mathcal{F}\colon \mathbb{T}^{\mathbb{R}} \to \mathbb{T}^{\mathbb{R}}$ given by

$$\mathcal{F}(\varphi)(\theta) := T_{\theta-\omega}(\varphi(\theta - \omega)).$$

Invariant curves

Definition 2

Let T be a qpf monotone map T on $\mathbb{T} \times \mathbb{R}$.

- A T-**invariant curve** is a measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies

$$\mathcal{F}(\varphi) = \varphi. \tag{1.2}$$

- A T-**two-periodic curve** is a measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies

$$\mathcal{F}^2(\varphi) = \varphi, \tag{1.3}$$
$$\mathcal{F}(\varphi) \neq \varphi. \tag{1.4}$$

- The equation 1.2 implies that the graph of φ, which will be denoted by Φ, is forward invariant by T.

Invariant curves

Definition 2

Let T be a qpf monotone map T on $\mathbb{T} \times \mathbb{R}$.

- A T-**invariant curve** is a measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies

$$\mathcal{F}(\varphi) = \varphi. \tag{1.2}$$

- A T-**two-periodic curve** is a measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies

$$\mathcal{F}^2(\varphi) = \varphi, \tag{1.3}$$
$$\mathcal{F}(\varphi) \neq \varphi. \tag{1.4}$$

- The equation 1.2 implies that the graph of φ, which will be denoted by Φ, is forward invariant by T.
- If all the fibre maps are differentiable and we denote their derivatives by DT_θ, then the stability of an invariant curve is measured by its (vertical) **Lyapunov exponent**, given by

$$\lambda(\varphi) := \int_{\mathbb{T}} \left(\log DT_\theta(\mathcal{F}(\varphi)(\theta)) + \log DT_\theta(\varphi(\theta))\right) d\theta. \tag{1.5}$$

Two-periodic Strange Non-chaotic Attractor

We now define what is (for us) a strange non-chaotic attractor.

Definition 3

A **two-periodic strange non-chaotic attractor (TSNA)**, in a quasiperiodically forced system T, is the union of a non-continuous T-two-periodic curve φ with $\mathcal{F}(\varphi)$ such that $\lambda(\varphi) < 0$.

It is important to remark the following:

- The point set corresponding to a non-continuous invariant curve is not a compact invariant set, which is usually required in the definition of "attractor".
- However, in some cases, it can be proved that an SNA attracts and determines the behavior of a set of initial conditions with full Lebesgue measure (Keller 1996 [Kel96], Bjerklöv 2009 [Bje09] and Jorba et al. [JTZ24]).

Quasiperiodically forced piecewise-linear maps

A quasiperiodically forced piecewise-linear map

Let $a < 0$, $b > 0$. We consider the quasiperiodically forced map $F_{a,b}\colon \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ defined by $F_{a,b}(\theta, x) = (\bar{\theta}, \bar{x})$ where

$$\begin{cases} \bar{\theta} = \theta + \omega \mod 2\pi, \\ \bar{x} = h_a(x) + bg(\theta), \end{cases} \tag{2.1}$$

for all $(\theta, x) \in \mathbb{T} \times \mathbb{R}, \omega \notin 2\pi\mathbb{Q}$. The map $g\colon \mathbb{T} \to \mathbb{R}$ is a sufficiently smooth function and the map $h_a\colon \mathbb{R} \to \mathbb{R}$ is given by

$$h_a(x) = \begin{cases} ax & \text{if } x > \frac{1}{a}, \\ 1 & \text{if } x \leqslant \frac{1}{a}. \end{cases} \tag{2.2}$$

Space of Lipschitz continuous functions in $\mathbb{T}$

Consider $Lip(\mathbb{T}, \mathbb{R})$ the space of Lipschitz continuous functions from $\mathbb{T}$ to $\mathbb{R}$. For any $f \in Lip(\mathbb{T}, \mathbb{R})$ let

$$L(f) := \sup \left\{ \frac{|f(\theta) - f(\theta')|}{|\theta - \theta'|} \mid \theta, \theta' \in \mathbb{T}, \theta \neq \theta' \right\}.$$

Now we define $\|f\|_L := \|f\|_\infty + L(f)$.

Space of Lipschitz continuous functions in $\mathbb{T}$

Consider $Lip(\mathbb{T}, \mathbb{R})$ the space of Lipschitz continuous functions from $\mathbb{T}$ to $\mathbb{R}$. For any $f \in Lip(\mathbb{T}, \mathbb{R})$ let

$$L(f) := \sup \left\{ \frac{|f(\theta) - f(\theta')|}{|\theta - \theta'|} \mid \theta, \theta' \in \mathbb{T}, \theta \neq \theta' \right\}.$$

Now we define $\|f\|_L := \|f\|_\infty + L(f)$.

- The function $\|\cdot\|_L \colon Lip(\mathbb{T}, \mathbb{R}) \to \mathbb{R}^+$ is a norm.
- The space $Lip(\mathbb{T}, \mathbb{R})$ endowed with the norm $\|\cdot\|_L$ is a Banach space.

Space of Lipschitz continuous functions in $\mathbb{T}$

Consider $Lip(\mathbb{T}, \mathbb{R})$ the space of Lipschitz continuous functions from $\mathbb{T}$ to $\mathbb{R}$. For any $f \in Lip(\mathbb{T}, \mathbb{R})$ let

$$L(f) := \sup \left\{ \frac{|f(\theta) - f(\theta')|}{|\theta - \theta'|} \mid \theta, \theta' \in \mathbb{T}, \theta \neq \theta' \right\}.$$

Now we define $\|f\|_L := \|f\|_\infty + L(f)$.

- The function $\|\cdot\|_L \colon Lip(\mathbb{T}, \mathbb{R}) \to \mathbb{R}^+$ is a norm.
- The space $Lip(\mathbb{T}, \mathbb{R})$ endowed with the norm $\|\cdot\|_L$ is a Banach space.

Proposition 4 (Arcelà-Ascoli)

Let $f_n \colon \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}$, be Lipschitz functions such that there exists an M such that $L(f_n) \leqslant M$ for all $n \in \mathbb{N}$. Then if for every $\theta \in \mathbb{T}$ the exists the limit

$$f(\theta) := \lim_{n \to \infty} f_n(\theta),$$

then f is Lipschitz with $L(f) \leqslant M$ and $\{f_n\}_n$ converges uniformly to f on $\mathbb{T}$.

Uniform contraction case (case $-1 < a < 0$)

Let $-1 < a < 0$. The map $\mathcal{F}\colon Lip(\mathbb{T},\mathbb{R}) \to Lip(\mathbb{T},\mathbb{R})$ is given by

$$\mathcal{F}(\varphi)(\theta) = h_a(\varphi(\theta-\omega)) - bg(\theta-\omega).$$

Uniform contraction case (case $-1 < a < 0$)

Let $-1 < a < 0$. The map $\mathcal{F}\colon Lip(\mathbb{T},\mathbb{R}) \to Lip(\mathbb{T},\mathbb{R})$ is given by

$$\mathcal{F}(\varphi)(\theta) = h_a(\varphi(\theta-\omega)) - bg(\theta-\omega).$$

The map $\mathcal{F}$ is a Lipschitz map with Lipschitz constant $|a|$. Actually,

$$|\mathcal{F}(\varphi)(\theta) - \mathcal{F}(\psi)(\theta)| \leqslant |a||\varphi(\theta-\omega) - \psi(\theta-\omega)|.$$

Uniform contraction case (case $-1 < a < 0$)

Let $-1 < a < 0$. The map $\mathcal{F}\colon Lip(\mathbb{T},\mathbb{R}) \to Lip(\mathbb{T},\mathbb{R})$ is given by

$$\mathcal{F}(\varphi)(\theta) = h_a(\varphi(\theta-\omega)) - bg(\theta-\omega).$$

The map $\mathcal{F}$ is a Lipschitz map with Lipschitz constant $|a|$. Actually,

$$|\mathcal{F}(\varphi)(\theta) - \mathcal{F}(\psi)(\theta)| \leqslant |a||\varphi(\theta-\omega) - \psi(\theta-\omega)|.$$

So we obtain that

$$\|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\|_L \leqslant |a|\|\varphi - \psi\|_L.$$

Uniform contraction case (case $-1 < a < 0$)

Let $-1 < a < 0$. The map $\mathcal{F}\colon Lip(\mathbb{T},\mathbb{R}) \to Lip(\mathbb{T},\mathbb{R})$ is given by

$$\mathcal{F}(\varphi)(\theta) = h_a(\varphi(\theta-\omega)) - bg(\theta-\omega).$$

The map $\mathcal{F}$ is a Lipschitz map with Lipschitz constant $|a|$. Actually,

$$|\mathcal{F}(\varphi)(\theta) - \mathcal{F}(\psi)(\theta)| \leqslant |a||\varphi(\theta-\omega) - \psi(\theta-\omega)|.$$

So we obtain that

$$\|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\|_L \leqslant |a|\|\varphi - \psi\|_L.$$

As a consequence of the Banach fixed point theorem we can state the following result.

Theorem 5

Let $-1 < a < 0$ and $b > 0$. The dynamical system $F_{a,b}$ has a unique Lipschitz continuous invariant curve.

Sequences of functions

Consider the following curves

$$\varphi_0(\theta) = 1 - bg(\theta - \omega),$$

and μ the continuous solution of the equation

$$\mu(\theta) = a\mu(\theta - \omega) - bg(\theta).$$

A straightforward computation shows that, for $b > 0$ small enough, the curve φ_0 is a two-periodic curve and μ is an invariant curve.

Sequences of functions

Consider the following curves

$$\varphi_0(\theta) = 1 - bg(\theta - \omega),$$

and μ the continuous solution of the equation

$$\mu(\theta) = a\mu(\theta - \omega) - bg(\theta).$$

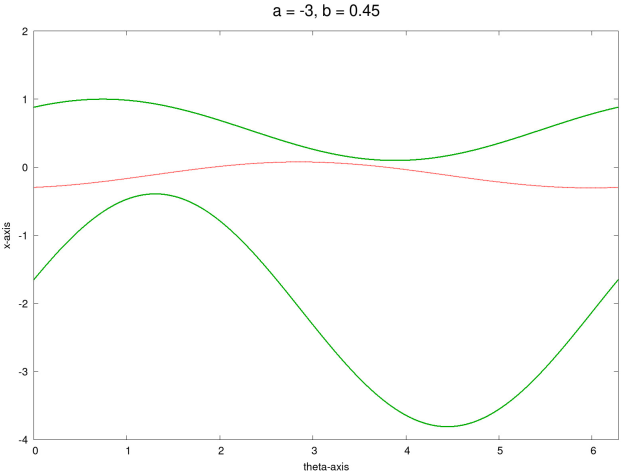
A straightforward computation shows that, for $b > 0$ small enough, the curve φ_0 is a two-periodic curve and μ is an invariant curve.

Lemma 6

Suppose $a < 0$. Then we have that

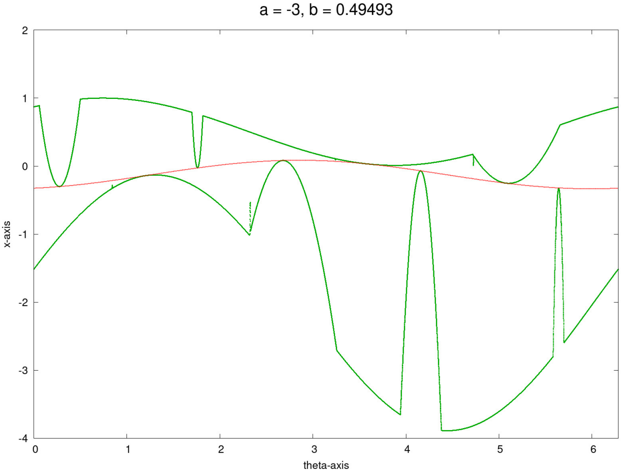
$$d_{\varphi_0} := \min_{\theta \in \mathbb{T}} (\varphi_0(\theta) - \mu(\theta)) = a \min_{\theta \in \mathbb{T}} \left(\frac{1}{a} - \mu(\theta - \omega) \right).$$

Plots of the system



For the plot we used $g(\theta) = 1 + \cos(\theta)$.

Plots of the system



Distance of the curves

Now we define two sequences of curves: $\lambda_0 = \varphi_0 - \mu$ and for all $n \geqslant 0$

$$\varphi_{n+1}(\theta) := \mathcal{F}^2(\varphi_n)(\theta) = h(h(\varphi_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - bg(\theta - \omega),$$
$$\lambda_{n+1}(\theta) := h(h(\mu(\theta - 2\omega) + \lambda_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - a\mu(\theta - \omega).$$

Distance of the curves

Now we define two sequences of curves: $\lambda_0 = \varphi_0 - \mu$ and for all $n \geqslant 0$

$$\varphi_{n+1}(\theta) := \mathcal{F}^2(\varphi_n)(\theta) = h(h(\varphi_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - bg(\theta - \omega),$$
$$\lambda_{n+1}(\theta) := h(h(\mu(\theta - 2\omega) + \lambda_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - a\mu(\theta - \omega).$$

We can prove the following properties of the sequences:

- $\lambda_n(\theta) = \varphi_n(\theta) - \mu(\theta)$ for all $n \geqslant 0$.
- The sequences $\{\varphi_n\}_n$ and $\{\lambda_n\}_n$ are monotone decreasing.
- The sequences $\{\varphi_n\}_n$ and $\{\lambda_n\}_n$ are lower bounded by μ and $x = 0$ respectively. Additionally, $\{\varphi_n\}_n$ and $\{\lambda_n\}_n$ have an upper semicontinuous pointwise limit φ_∞ and λ_∞ respectively.
- There exists a $b > 0$ such that $d_{\varphi_0} = 0$. Moreover, we define $b^*(a)$ as $\sup_{b>0} d_{\varphi_0} > 0$.

Strange Nonchaotic Attractor

Main Theorem 7

Let $a < -1$ and $b = b^*(a)$. Then

1. φ_∞ is a two-periodic upper semicontinuous curve such that $\lambda(\varphi_\infty) < 0$.
2. The set

$$A = \{\theta \in \mathbb{T} \mid \varphi_\infty(\theta) = \mu(\theta)\}$$

   is a residual dense subset of $\mathbb{T}$ and has zero Lebesgue measure.

Strange Nonchaotic Attractor

Main Theorem 7

Let $a < -1$ and $b = b^*(a)$. Then

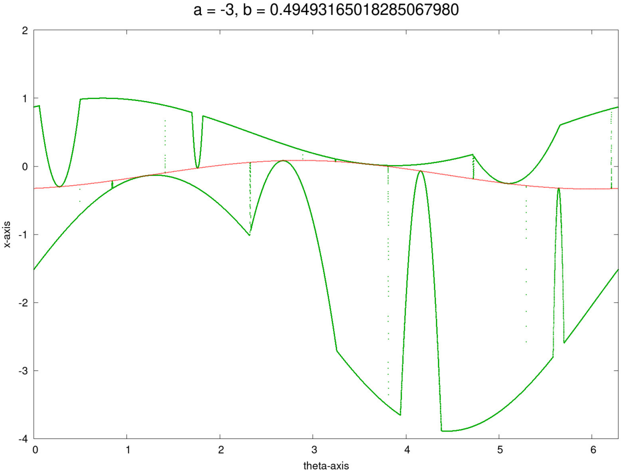
1. φ_∞ is a two-periodic upper semicontinuous curve such that $\lambda(\varphi_\infty) < 0$.
2. The set

$$A = \{\theta \in \mathbb{T} \mid \varphi_\infty(\theta) = \mu(\theta)\}$$

   is a residual dense subset of $\mathbb{T}$ and has zero Lebesgue measure.

1. Since in the base we have an irrational rotation, A is dense. By ergodicity A has zero measure or total measure.
2. If $\{\theta \in \mathbb{T} \mid \mathcal{F}(\varphi)(\theta) = a\varphi_\infty(\theta) - bg(\theta) < \frac{1}{a}\}$ has positive measure then $\lambda(\varphi_\infty) < 0$.

Plots of the system



Global attractor

Theorem 8

Let $a < 0$ and $0 \leqslant b \leqslant b^*(a)$. Consider the sets

$$\begin{aligned} A_+ &:= \{(\theta, x) \in \mathbb{T} \times \mathbb{R} \mid \mu(\theta) \leqslant x \leqslant \varphi_0(\theta)\}, \\ \Lambda_+ &:= \bigcap_{n \geqslant 0} F_{a,b}^{2n}(A_+), \\ \Lambda_- &:= F_{a,b}(\Lambda_+), \\ \Lambda &:= \Lambda_+ \cup \Lambda_-. \end{aligned}$$

Then,

- The set Λ is a compact invariant set for the map $F_{a,b}$.
- There exists a set $E \subset \mathbb{T}$ with full Lebesgue measure such that for every $(\theta, x) \in E \times \mathbb{R}$ there exists an $n_0 > 0$ such that $F^{n_0}(\theta, x) \in \Lambda$.
- The set Λ_+ has positive two dimensional Lebesgue measure.
- The graph of φ_∞ is dense on Λ_+.

Fractalization

Definition 9

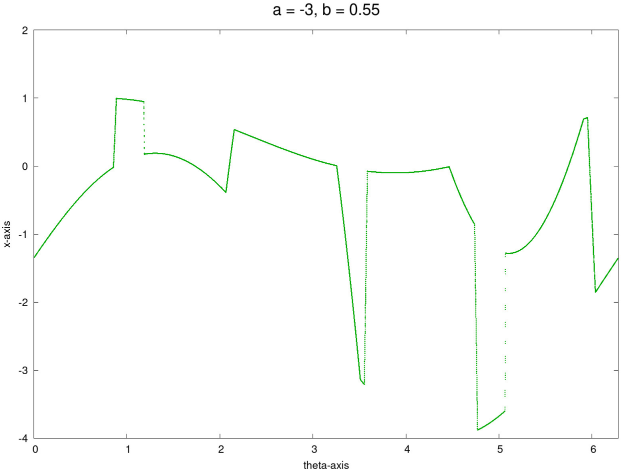
We say that a uniformly bounded family of Lipschitz continuous functions $\{f_b\}$, indexed by a parameter $b \in \mathbb{R}$, *left-fractalizes for some* $\hat{b}$ if

$$\limsup_{b \uparrow \hat{b}} L(f_b) = +\infty.$$

Proposition 10

Let $a < -1$ and $g(\theta) \geqslant 0$ for all $\theta \in \mathbb{T}$. The family of two-periodic curves $\{\varphi_\infty(b)\}_b$, parametrized by $b > 0$, left-fractalizes for the parameter $b = b^*(a)$.

Plots of the system



References

[Bje09] Kristian Bjerklöv.
SNA's in the quasi-periodic quadratic family.
Comm. Math. Phys., 286(1):137–161, 2009.

[Fuh16] Gabriel Fuhrmann.
Non-smooth saddle-node bifurcations I: existence of an SNA.
Ergodic Theory Dynam. Systems, 36(4):1130–1155, 2016.

[Jäg03] Tobias H Jäger.
Quasiperiodically forced interval maps with negative schwarzian derivative.
Nonlinearity, 16(4):1239, 2003.

[JT08] Àngel Jorba and Joan Carles Tatjer.
A mechanism for the fractalization of invariant curves in quasi-periodically forced 1-D maps.
Discrete Contin. Dyn. Syst. Ser. B, 10(2-3):537–567, 2008.

[JTZ24] Angel Jorba, Joan Carles Tatjer, and Yuan Zhang.
Nonsmooth pitchfork bifurcations in a quasi-periodically forced piecewise-linear map.
International Journal of Bifurcation and Chaos, 2024.

[Kel96] Gerhard Keller.
A note on strange nonchaotic attractors.
Fund. Math., 151(2):139–148, 1996.

[MVT26] Rafael Martinez-Vergara and Joan Carles Tatjer.
Nonsmooth bifurcations in families of one-dimensional piecewise-linear quasiperiodically forced maps.
Qual. Theory Dyn. Syst., 25(1):Paper No. 20, 41, 2026.