The emergence of nonsmooth bifurcations in quasiperiodically forced systems

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DocTorV Seminar, March 27, 2026.

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Quasiperiodically forced systems

Basic definitions

Definition 1

Given a topological space X, a **quasiperiodically forced system (qpf)** is a continuous map $T : \mathbb{T} \times X \to \mathbb{T} \times X$ of the form

$$T(\theta, x) = (\theta + \omega \mod 2\pi, T_\theta(x)), \tag{1.1}$$

with $\omega \notin 2\pi\mathbb{Q}$.

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with $\omega \notin 2\pi\mathbb{Q}$.

We will restrict to the case where $X = \mathbb{R}$, and the fibre maps T_θ are monotonically increasing on $\mathbb{R}$. Let $\mathbb{T}^{\mathbb{R}}$ denote the set of functions from $\mathbb{T}$ to $\mathbb{R}$. To a qpf system T on $\mathbb{T} \times \mathbb{R}$, we associate the operator $\mathcal{F}\colon \mathbb{T}^{\mathbb{R}} \to \mathbb{T}^{\mathbb{R}}$ given by

$$\mathcal{F}(\varphi)(\theta) := T_{\theta-\omega}(\varphi(\theta - \omega)).$$

Invariant curves

Definition 2

Let T be a qpf monotone map T on $\mathbb{T} \times \mathbb{R}$.

- A T-**invariant curve** is a measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies

$$\mathcal{F}(\varphi) = \varphi. \tag{1.2}$$

- A T-**two-periodic curve** is a measurable function $\varphi : \mathbb{T} \to \mathbb{R}$ which satisfies

$$\mathcal{F}^2(\varphi) = \varphi, \tag{1.3}$$
$$\mathcal{F}(\varphi) \neq \varphi. \tag{1.4}$$

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- The equation 1.2 implies that the graph of φ, which will be denoted by Φ, is forward invariant by T.
- If all the fibre maps are differentiable and we denote their derivatives by DT_θ, then the stability of an T-**two-periodic curve** is measured by its (vertical) **Lyapunov exponent**, given by

$$\lambda(\varphi) := \int_{\mathbb{T}} \left(\log DT_\theta(\mathcal{F}(\varphi)(\theta)) + \log DT_\theta(\varphi(\theta))\right) d\theta. \tag{1.5}$$

Two-periodic Strange Non-chaotic Attractor

We now define what is (for us) a strange non-chaotic attractor.

Definition 3

A **two-periodic strange non-chaotic attractor (TSNA)**, in a quasiperiodically forced system T, is the union of a non-continuous T-two-periodic curve φ with $\mathcal{F}(\varphi)$ such that $\lambda(\varphi) < 0$.

It is important to remark the following:

- The point set corresponding to a non-continuous invariant curve is not a compact invariant set, which is usually required in the definition of "attractor".
- However, in some cases, it can be proved that an SNA attracts and determines the behavior of a set of initial conditions with full Lebesgue measure (Keller 1996 [Kel96], Bjerklöv 2009 [Bje09] and Jorba et al. [JTZ24]).

Finite time Lyapunov exponents

Let T be a qpf monotone map T on $\mathbb{T} \times \mathbb{R}$. We introduce several notions about Lyapunov exponents. For a given $n > 0$ and $(\theta, x) \in \mathbb{T} \times \mathbb{R}$, the **pointwise (vertical) finite-time forward** and **finite-time backward Lyapunov exponents** of T are defined as $\lambda^+(\theta, x, n)$ and $\lambda^-(\theta, x, n)$ are defined by

$$\lambda^+(\theta, x, n) := \frac{1}{n} \sum_{i=0}^{n-1} \log(DT(\theta + i\omega, T_\theta^i(x)),$$

$$\lambda^-(\theta, x, n) := \frac{1}{n} \sum_{i=0}^{n-1} \log(DT(\theta - i\omega, T_\theta^{-i}(x)).$$

If we have a family T_b of qpf monotone maps on $\mathbb{T} \times \mathbb{R}$ that depend on b, then $\lambda^\pm(\theta, x, n) = \lambda^\pm(b, \theta, x, n)$ will depend also on b.

Upper Lyapunov exponents

We do not have granted that $\lambda^{\pm}(\theta, x, n)$ converge as $n \to \infty$. Then, we define the **upper forward Lyapunov exponent** and **upper backward Lyapunov exponent** by

$$\lambda_s^+(\theta, x) := \limsup_{n\to\infty} \lambda^+(\theta, x, n),$$
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$$\lambda_s^{-}(\theta, x) := \limsup_{n\to\infty} \lambda^{-}(\theta, x, n).$$

Consider the situation where ψ is an unstable and φ is a stable continuous invariant graph, and there is no other invariant graph in between. There is four cases:

1. Points on the repeller ψ will have a positive forward and a negative backward Lyapunov exponent.
2. Points on the attractor φ will have a negative forward and a positive backward Lyapunov exponent.
3. All points between ψ and φ will converge to φ forwards and to ψ backwards in time. Hence, their exponents will be negative.

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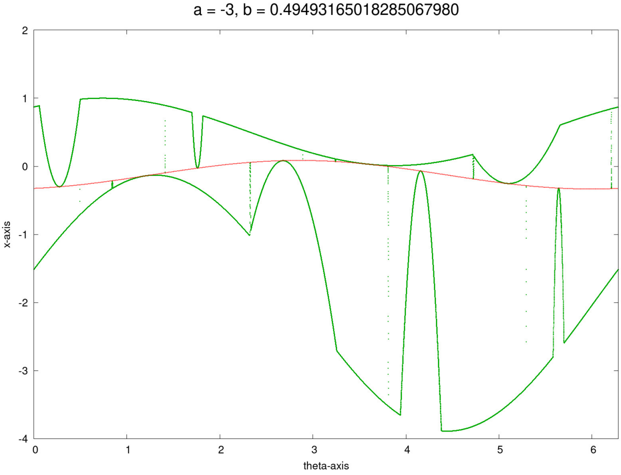
3. All points between ψ and φ will converge to φ forwards and to ψ backwards in time. Hence, their exponents will be negative.
4. The remaining possibility is that both Lyapunov exponents are positive!!

System with two attracting invariant curve and one repelling in the middle



Sink-source orbits

The main idea for this orbits is the following:

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Definition 4 (Jäger 2008)

For a given $(\theta, x) \in \mathbb{T} \times \mathbb{R}$, we say that an orbit $\{T^i(\theta, x)\}_{i\in\mathbb{Z}}$ is a **sink-source orbit** when

$$\lambda^+(\theta, x) := \lim_{n\to\infty} \lambda^+(\theta, x, n),$$
$$\lambda^-(\theta, x) := \lim_{n\to\infty} \lambda^-(\theta, x, n),$$

exists and $\lambda^+(\theta, x) > 0$ and $\lambda^-(\theta, x) > 0$. We say that an orbit $\{T^i(\theta, x)\}_{i\in\mathbb{Z}}$ is a **weak sink-source orbit** when $\lambda_s^+(\theta, x) > 0$ and $\lambda_s^-(\theta, x) > 0$.

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- In certain qpf maps, the numerical computation of the finite-time Lyapunov exponents showed a large proportion of positive values for very big $n > 0$.

Existence of SNA using orbits

Definition 5

A **strange non-chaotic attractor (SNA)**, in a quasiperiodically forced system T, is a non-continuous curve φ such that $\mathcal{F}(\varphi) = \varphi$ and that

$$\int_{\mathbb{T}} \log |DT(\varphi(\theta))| d\theta < 0.$$

Definition 6

A **strange non-chaotic repeller (SNR)**, in a quasiperiodically forced system T, is a non-continuous curve φ such that $\mathcal{F}(\varphi) = \varphi$ and that

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Theorem 7 (Jäger 2008)

If there exists a weak sink-source orbit then there exists a strange non-chaotic attractor and a strange non-chaotic repeller.

Quasiperiodically forced piecewise-linear maps

A quasiperiodically forced piecewise-linear map

Let $a < 0$, $b > 0$. We consider the quasiperiodically forced map $F_{a,b}\colon \mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R}$ defined by $F_{a,b}(\theta,x) = (\bar{\theta},\bar{x})$ where

$$\begin{cases} \bar{\theta} = \theta + \omega \mod 2\pi, \\ \bar{x} = h_a(x) + bg(\theta), \end{cases} \tag{2.1}$$

for all $(\theta,x)\in\mathbb{T}\times\mathbb{R}, \omega\notin 2\pi\mathbb{Q}$. The map $g\colon\mathbb{T}\to\mathbb{R}$ is a sufficiently smooth function and the map $h_a\colon\mathbb{R}\to\mathbb{R}$ is given by

$$h_a(x) = \begin{cases} ax & \text{if } x > \frac{1}{a}, \\ 1 & \text{if } x \leqslant \frac{1}{a}. \end{cases} \tag{2.2}$$

Space of Lipschitz continuous functions in $\mathbb{T}$

Consider $Lip(\mathbb{T},\mathbb{R})$ the space of Lipschitz continuous functions from $\mathbb{T}$ to $\mathbb{R}$. For any $f \in Lip(\mathbb{T},\mathbb{R})$ let

$$L(f) := \sup\left\{ \frac{|f(\theta) - f(\theta')|}{|\theta - \theta'|} \mid \theta, \theta' \in \mathbb{T}, \theta \neq \theta' \right\}.$$

Now we define $\|f\|_L := \|f\|_\infty + L(f)$.

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- The function $\|\cdot\|_L \colon Lip(\mathbb{T},\mathbb{R}) \to \mathbb{R}^+$ is a norm.
- The space $Lip(\mathbb{T},\mathbb{R})$ endowed with the norm $\|\cdot\|_L$ is a Banach space.

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Proposition 8 (Arcelà-Ascoli)

Let $f_n \colon \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}$, be Lipschitz functions such that there exists an M such that $L(f_n) \leqslant M$ for all $n \in \mathbb{N}$. Then if for every $\theta \in \mathbb{T}$ the exists the limit

$$f(\theta) := \lim_{n\to\infty} f_n(\theta),$$

then f is Lipschitz with $L(f) \leqslant M$ and $\{f_n\}_n$ converges uniformly to f on $\mathbb{T}$.

Uniform contraction case (case $-1 < a < 0$)

Let $-1 < a < 0$. The map $\mathcal{F}\colon Lip(\mathbb{T}, \mathbb{R}) \to Lip(\mathbb{T}, \mathbb{R})$ is given by

$$\mathcal{F}(\varphi)(\theta) = h_a(\varphi(\theta - \omega)) - bg(\theta - \omega).$$

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The map $\mathcal{F}$ is a Lipschitz map with Lipschitz constant $|a|$. Actually,

$$|\mathcal{F}(\varphi)(\theta) - \mathcal{F}(\psi)(\theta)| \leqslant |a||\varphi(\theta-\omega) - \psi(\theta-\omega)|.$$

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So we obtain that

$$\|\mathcal{F}(\varphi) - \mathcal{F}(\psi)\|_L \leqslant |a|\|\varphi - \psi\|_L.$$

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As a consequence of the Banach fixed point theorem we can state the following result.

Theorem 9

Let $-1 < a < 0$ and $b > 0$. The dynamical system $F_{a,b}$ has a unique Lipschitz continuous invariant curve.

Sequences of functions

Consider the following curves

$$\varphi_0(\theta) = 1 - bg(\theta - \omega),$$

and μ the continuous solution of the equation

$$\mu(\theta) = a\mu(\theta - \omega) - bg(\theta).$$

A straightforward computation shows that, for $b > 0$ small enough, the curve φ_0 is a two-periodic curve and μ is an invariant curve.

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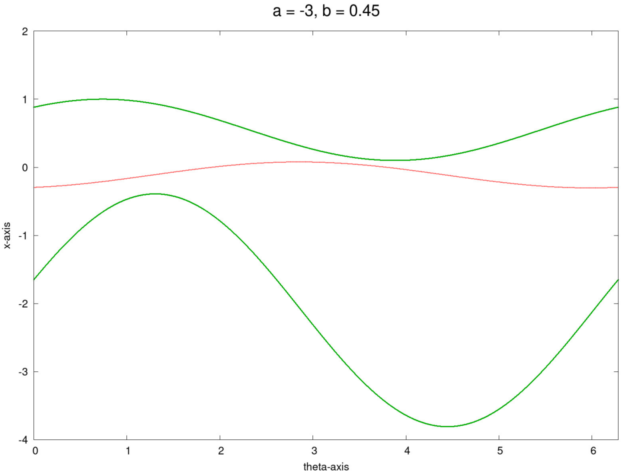
Lemma 10

Suppose $a < 0$. Then we have that

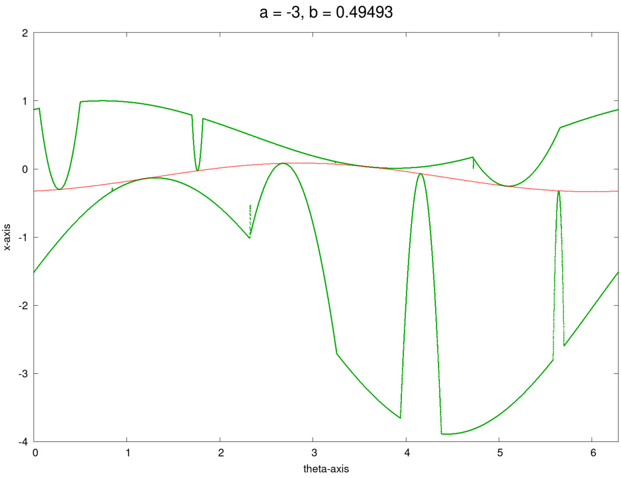
$$d_{\varphi_0} := \min_{\theta \in \mathbb{T}} (\varphi_0(\theta) - \mu(\theta)) = a \min_{\theta \in \mathbb{T}} \left(\frac{1}{a} - \mu(\theta - \omega) \right).$$

Plots of the system



For the plot we used $g(\theta) = 1 + \cos(\theta)$.

Plots of the system



Distance of the curves

Now we define two sequences of curves: $\lambda_0 = \varphi_0 - \mu$ and for all $n \geqslant 0$

$$\varphi_{n+1}(\theta) := \mathcal{F}^2(\varphi_n)(\theta) = h(h(\varphi_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - bg(\theta - \omega),$$
$$\lambda_{n+1}(\theta) := h(h(\mu(\theta - 2\omega) + \lambda_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - a\mu(\theta - \omega).$$

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$$\lambda_{n+1}(\theta) := h(h(\mu(\theta - 2\omega) + \lambda_n(\theta - 2\omega)) - bg(\theta - 2\omega)) - a\mu(\theta - \omega).$$

We can prove the following properties of the sequences:

- $\lambda_n(\theta) = \varphi_n(\theta) - \mu(\theta)$ for all $n \geqslant 0$.
- The sequences $\{\varphi_n\}_n$ and $\{\lambda_n\}_n$ are monotone decreasing.
- The sequences $\{\varphi_n\}_n$ and $\{\lambda_n\}_n$ are lower bounded by μ and $x = 0$ respectively. Additionally, $\{\varphi_n\}_n$ and $\{\lambda_n\}_n$ have an upper semicontinuous pointwise limit φ_∞ and λ_∞ respectively.
- There exists a $b > 0$ such that $d_{\varphi_0} = 0$. Moreover, we define $b^*(a)$ as $\sup_{b>0} d_{\varphi_0} > 0$.

Strange Nonchaotic Attractor

Main Theorem 11

Let $a < -1$ and $b = b^*(a)$. Then

1. φ_∞ is a two-periodic upper semicontinuous curve such that $\lambda(\varphi_\infty) < 0$.
2. The set

$$A = \{\theta \in \mathbb{T} \mid \varphi_\infty(\theta) = \mu(\theta)\}$$

is a residual dense subset of $\mathbb{T}$ and has zero Lebesgue measure.

Idea of the proof

1. Since in the base we have an irrational rotation, A is dense. By ergodicity A has zero measure or total measure.

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2. Define the following sequence of continuous functions

$$\psi_n(\theta) := \begin{cases} \frac{\lambda_{n+1}(\theta+2\omega)}{\lambda_n(\theta)} & \text{if } \lambda_n(\theta) \neq 0, \\ a^2 & \text{if } \lambda_n(\theta) = 0. \end{cases}$$

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3. The sequence $\{\log \circ \psi_n\}_n$ is a monotone increasing sequence of integrable functions such that $0 \leqslant \psi_n(\theta) \leqslant a^2$ for all $\theta \in \mathbb{T}$. By the Dominated Convergence Theorem we obtain that there exists an integrable function $\psi\colon \mathbb{T} \to \mathbb{R}$ such that

$$\int_{\mathbb{T}} \log(\psi(\theta)) = \lim_{n\to\infty} \int_{\mathbb{T}} \log(\psi_n(\theta)) \leqslant \int_{\mathbb{T}} \log(\lambda_n(\theta+2\omega)) - \int_{\mathbb{T}} \log(\lambda_n(\theta)) = 0.$$

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4. If $\theta_0 \in \mathbb{T}$ is the only zero of λ_0 then
$$A \setminus \{\theta_0 - \omega\} \subset \{\theta_0 \in \mathbb{T} \mid \psi(\theta) = a^2\}.$$

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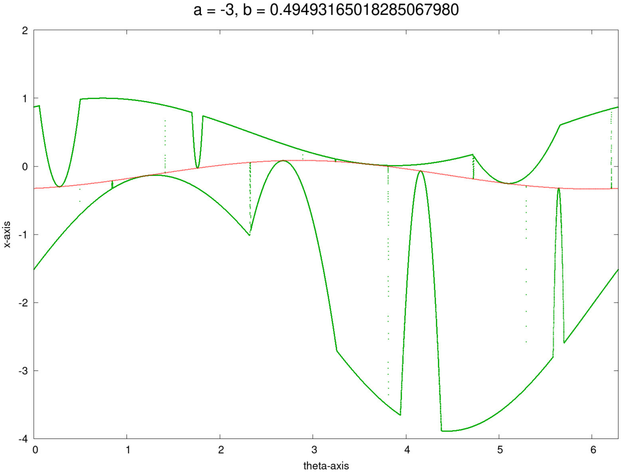
5. If $\{\theta \in \mathbb{T} \mid \mathcal{F}(\varphi)(\theta) = a\varphi_\infty(\theta) - bg(\theta) < \frac{1}{a}\}$ has positive measure then $\lambda(\varphi_\infty) < 0$.

Plots of the system



Global attractor

Theorem 12

Let $a < 0$ and $0 \leqslant b \leqslant b^*(a)$. Consider the sets

$$\begin{aligned} A_+ &:= \{(\theta, x) \in \mathbb{T} \times \mathbb{R} \mid \mu(\theta) \leqslant x \leqslant \varphi_0(\theta)\}, \\ \Lambda_+ &:= \bigcap_{n \geqslant 0} F_{a,b}^{2n}(A_+), \\ \Lambda_- &:= F_{a,b}(\Lambda_+), \\ \Lambda &:= \Lambda_+ \cup \Lambda_-. \end{aligned}$$

Then,

- The set Λ is a compact invariant set for the map $F_{a,b}$.
- There exists a set $E \subset \mathbb{T}$ with full Lebesgue measure such that for every $(\theta, x) \in E \times \mathbb{R}$ there exists an $n_0 > 0$ such that $F^{n_0}(\theta, x) \in \Lambda$.
- The set Λ_+ has positive two dimensional Lebesgue measure.
- The graph of φ_∞ is dense on Λ_+.

Fractalization

Definition 13

We say that a uniformly bounded family of Lipschitz continuous functions $\{f_b\}$, indexed by a parameter $b \in \mathbb{R}$, *left-fractalizes for some* $\hat{b}$ if

$$\limsup_{b \uparrow \hat{b}} L(f_b) = +\infty.$$

Proposition 14

Let $a < -1$ and $g(\theta) \geqslant 0$ for all $\theta \in \mathbb{T}$. The family of two-periodic curves $\{\varphi_\infty(b)\}_b$, parametrized by $b > 0$, left-fractalizes for the parameter $b = b^*(a)$.

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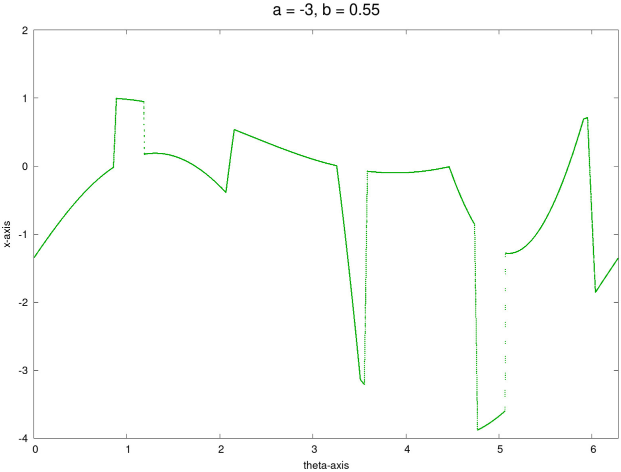

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Molte Grazie