APPROXIMATION OF ADELIC DIVISORS AND EQUIDISTRIBUTION OF SMALL POINTS

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Abstract. We study the asymptotic distribution of Galois orbits of algebraic points with small height in a projective variety over a number field. Our main result is a generalization of Yuan's equidistribution theorem that does not require Zhang's lower bound for the essential minimum to be attained. It extends to all projective varieties a theorem of Burgos Gil, Philippon, Rivera-Letelier and the second author for toric varieties, and implies the semiabelian equidistribution theorem of Kühne. We also generalize previous work of Chambert-Loir and Thuillier to prove new logarithmic equidistribution results. In the last section, we extend our main theorem to the quasi-projective setting recently introduced by Yuan and Zhang.

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1. Introduction

In their seminal work [SUZ97], Szpiro, Ullmo and Zhang used Arakelov theory to prove an equidistribution theorem for the Galois orbits of algebraic points on a projective variety over a number field. This theorem applies in the case of an abelian variety for generic sequences of points with Néron-Tate heights converging to zero, and is at the heart of the Bogomolov conjecture for abelian varieties [Ull98, Zha98]. It has been widely studied and generalized by many authors. On a projective variety over a number field, these generalizations culminated with the celebrated equidistribution theorem of Yuan [Yua08]. In this paper we extend the scope of Yuan's theorem to allow more flexibility in the choices of the height function and of the generic sequences of algebraic points.

Let X be a projective variety of dimension $d \geq 1$ over a number field K, and let $\overline{K}$ be an algebraic closure of K. Arakelov geometry provides a powerful and very general framework to define and study the height of algebraic points. Classical heights in Diophantine geometry such as Néron-Tate heights on abelian varieties are special cases of height functions associated to adelic metrized line bundles in the sense of Zhang [Zha95b]. We work with the essentially equivalent notion of adelic divisors, which can be understood as usual Cartier divisors together with a structure of adelic line bundle on the associated invertible sheaf.

Let $\overline{D}$ be an adelic divisor on X with geometric divisor D and height function $h_{\overline{D}}\colon X(\overline{K}) \to \mathbb{R}$. We denote by $\mu^{\text{ess}}(\overline{D})$ the essential minimum of this function. For every generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ we have $\liminf_{\ell\to\infty} h_{\overline{D}}(x_\ell) \geq \mu^{\text{ess}}(\overline{D})$, and there exist generic sequences for which equality holds. Following [BPRS19], we say that a generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ is $\overline{D}$-*small* if the heights of the points converge to the smallest possible value, that is $\lim_{\ell\to\infty} h_{\overline{D}}(x_\ell) = \mu^{\text{ess}}(\overline{D})$.

A celebrated inequality of Zhang [Zha95a] asserts that if D is ample and $\overline{D}$ is semipositive then

$$\mu^{\text{ess}}(\overline{D}) \geq \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}, \tag{1.1}$$

where $(\overline{D}^{d+1})$ denotes the top arithmetic intersection number of $\overline{D}$.

For every place $v \in \mathfrak{M}_K$, we denote by X_v^{an} the v-adic analytification of X. It is a Berkovich space over $\mathbb{C}_v$, the completion of an algebraic closure of the local field K_v. For every algebraic point $x \in X(\overline{K})$, we let $\delta_{O(x)_v}$ be the uniform probability measure on the orbit $O(x)_v \subset X_v^{\text{an}}$ of x under the action of $\text{Gal}(\overline{K}/K)$.

With this notation, Yuan's equidistribution theorem [Yua08] reads as follow.

Theorem 1.1 (Yuan)**.** *Let $\overline{D} \in \widehat{\text{Div}}(X)$ be a semipositive adelic divisor on X with ample geometric divisor D. Assume that*

$$\mu^{\text{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}. \tag{1.2}$$

Then for every $v \in \mathfrak{M}_K$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$, the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to $c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

In other words, this theorem asserts that if Zhang's lower bound (1.1) is attained then $\overline{D}$ has the equidistribution property, in the sense that the Galois orbits of points in generic $\overline{D}$-small sequences equidistribute in X_v^{an} for every place v. Moreover, it shows that in this case the equidistribution measure is the normalized v-adic Monge-Ampère measure of $\overline{D}$. We note that when X is a curve, this theorem is due to Autissier [Aut01] and Chambert-Loir [Cha06].

Yuan's theorem is a cornerstone result in the study of the equidistribution property, that encompasses in a unified way the theorem of Szpiro, Ullmo and Zhang for abelian varieties [SUZ97], its non-Archimedean analogue due to Chambert-Loir [Cha06], as well as the theorems of Bilu for canonical heights on toric varieties [Bil97] and Chambert-Loir for canonical heights on isotrivial semi-abelian varieties [Cha00].

The positivity assumptions in Theorem 1.1 can be weakened, and in fact Yuan's proof remains valid when D is big but not necessarily ample. Moreover, Berman and Boucksom [BB10] and later Chen [Che11] generalized Yuan's theorem at Archimedean places to the non-semipositive case.

In general the equality (1.2) is not satisfied, and in fact it holds if and only if the essential minimum coincides with the absolute minimum $\mu^{\text{abs}}(\overline{D}) = \inf_{x\in X(\overline{K})} h_{\overline{D}}(x)$

[Bal24, Theorem 6.6]. This is a very restrictive condition on the behavior of the height function, that is nevertheless satisfied in the important case of canonical heights on polarized dynamical systems [Zha95b]. This includes canonical heights on toric varieties and Néron-Tate heights on abelian varieties.

To our knowledge, apart from the positivity assumptions Yuan's theorem is the most general result ensuring the equidistribution property on a projective variety over a number field. In particular, in general nothing is known at the moment for an arbitrary adelic divisor when Zhang's inequality is strict. However, there are two remarkable situations that fall outside the scope of Yuan's theorem but where equidistribution phenomena are known to occur.

In the toric case, Burgos Gil, Philippon, Rivera-Letelier and the second author [BPRS19] achieved a systematic description of the equidistribution property. Their result gives a criterion for the equidistribution property in terms of convex analysis, and shows that there are plenty of toric adelic divisors for which Zhang's inequality is strict but that nevertheless satisfy the equidistribution property. This provides a wealth of new equidistribution phenomena, previously out of reach. On the other hand, not every toric adelic divisor has the equidistribution property.

In [Küh22], Kühne proved the long-standing semiabelian equidistribution conjecture, showing that the equidistribution property holds for canonical heights on semiabelian varieties. As shown by Chambert-Loir [Cha00], Yuan's theorem does not apply in this case as (1.2) can fail when the semiabelian variety is not isotrivial. Kühne's theorem allowed him to give a purely Arakelov-geometric proof of the semiabelian Bogomolov conjecture, previously established by David and Philippon with other methods [DP00].

1.1. **Main theorem.** The results of [BPRS19] and [Küh22] raise the following question: on an arbitrary projective variety, what can be said regarding the equidistribution property for an adelic divisor when Zhang's inequality is strict? More precisely, can we identify a condition weaker than (1.2) that guarantees the equidistribution property for a given adelic divisor? Our main contribution is a generalization of Yuan's theorem that answers positively this question.

As in [SUZ97, Yua08, Cha06, Che11], positivity properties of adelic divisors play a central role in our approach. We work with the more general notion of adelic $\mathbb{R}$-divisors developed by Moriwaki, as it provides a particularly efficient framework to study positivity [BMPS16, Mor16]. Adelic $\mathbb{R}$-divisors are better behaved on normal varieties, so we assume that X is normal from now on. This assumption is harmless since our main result is invariant under modification.

Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X. A *semipositive approximation* of $\overline{D}$ is a pair $(\phi, \overline{Q})$ consisting of a normal modification $\phi\colon X' \to X$ and a semipositive adelic $\mathbb{R}$-divisor $\overline{Q}$ on X' such that the geometric $\mathbb{R}$-divisor Q underlying $\overline{Q}$ is big and $\phi^*\overline{D} - \overline{Q}$ is pseudo-effective. The definition of semipositive approximations is very similar to the one of admissible decompositions in the sense of Chen [Che11, Definition 2], but here we do not require $\overline{Q}$ to be nef.

Given two big $\mathbb{R}$-Cartier divisors P, A on X, the *inradius* of P with respect to A is the real number

$$r(P; A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is big}\}.$$

This geometric invariant was introduced by Teissier [Tei82] and measures the bigness of P, see §2.2 for more details. Our main result is the following.

Theorem 1.2. *Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X with D big. Assume that there exists a sequence $(\phi_n\colon X_n \to X, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that*

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} = 0. \tag{1.3}$$

Let $v \in \mathfrak{M}_K$, and for each $n \geq 1$ let $\nu_{n,v}$ be the probability measure on X_v^{an} given by the pushforward of the normalized v-adic Monge-Ampère measure $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$ on $X_{n,v}^{\mathrm{an}}$. Then

(1) the sequence $(\nu_{n,v})_n$ converges weakly to a probability measure ν_v on X_v^{an},

(2) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$, the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to ν_v.

Theorem 1.1 follows immediately from Theorem 1.2 applied with the constant sequence $(\phi_n, \overline{Q}_n) = (\mathrm{Id}_X, \overline{D})$, $n \in \mathbb{N}$. It also implies Chen's equidistribution theorem (Corollary 6.5).

We actually prove a stronger result (Theorem 6.1), showing that under the assumption of Theorem 1.2 we have that for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ and every $\overline{E} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$,

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D}) \times (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)}. \tag{1.4}$$

In particular, both limits exist in $\mathbb{R}$ and the second one does not depend on the choice of the sequence $(\phi_n, \overline{Q}_n)_n$. Theorem 1.2 follows by specifying (1.4) to adelic divisors $\overline{E}$ constructed from continuous functions on the analytic spaces X_v^{an}, $v \in \mathfrak{M}_K$.

In Theorem 6.7 we give an alternative formulation of Theorem 1.2, which is more intrinsic in the sense that it does not rely on the choice of a specific sequence of semipositive approximations. In fact, we show that the existence of a sequence $(\phi_n, \overline{Q}_n)_n$ as in Theorem 1.2 is equivalent to an equality

$$\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t}{\rho(\overline{D}(t))} = 0,$$

where $\rho(\overline{D}(t))$ is an invariant measuring the inradii of the geometric divisors of arithmetic Fujita approximations for an adelic $\mathbb{R}$-divisor $\overline{D}(t)$ obtained by twisting the Archimedean Green functions of $\overline{D}$. In this case, for each $v \in \mathfrak{M}_K$ the equidistribution measure can be computed using positive arithmetic intersection numbers introduced by Chen [Che11] (Remark 6.11).

Using (1.4), we also obtain a generalization of Chambert-Loir and Thuillier's theorem on logarithmic equidistribution [CLT09]. More precisely, we prove that in the situation of Theorem 1.2 the equidistribution property extends to test functions with logarithmic singularities along hypersurfaces satisfying some numerical condition (Theorem 8.4 and Remark 8.5).

1.2. **Toric varieties.** Theorem 1.2 allows to recover the main result of [BPRS19]. Our approach is different from that in *loc. cit.*, and gives a new perspective on the equidistribution problem in the toric setting.

We recall some of the basic constructions from the Arakelov geometry of toric varieties. Let X be a projective toric variety over K with torus $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m}}^d$, and $\overline{D}$ a toric adelic $\mathbb{R}$-divisor on X with big underlying $\mathbb{R}$-divisor D. Denote by Δ_D the

d-dimensional polytope associated to this divisor. Then the family of Green functions of $\overline{D}$ define a family of concave functions $\vartheta_{\overline{D},v}\colon \Delta_D \to \mathbb{R}$, $v \in \mathfrak{M}_K$, called the *local roof functions*, and whose weighted sum give the *global roof function*

$$\vartheta_{\overline{D}}\colon \Delta_D \to \mathbb{R}.$$

These concave functions convey a lot of information about the height function of $\overline{D}$. For instance, its essential minimum can be computed as the maximum of the global roof function [BPS15].

In the toric setting, the condition that $\overline{D}$ satisfies the equality in Zhang's lower bound (1.1) is very restrictive, as it is equivalent to the condition that the global roof function is constant [BPRS19, Proposition 5.3], see also Example 9.2. Thus the only toric adelic divisor to which Yuan's theorem applies are those for which this concave function is constant.

The global roof function $\vartheta_{\overline{D}}$ is said to be *wide* if the width of its sup-level sets remains relatively large as the level approaches its maximum, see Appendix A for details. When this is the case, there is a unique balanced family of vectors u_v, $v \in \mathfrak{M}_K$ such that each u_v is a sup-gradient of the v-adic roof function $\vartheta_{\overline{D},v}$. Moreover, for each $v \in \mathfrak{M}_K$ we consider a probability measure η_{v,u_v} on X_v^{an} associated to u_v (Definition 9.7). When v is Archimedean, it is the Haar probability measure on a translate of the compact torus $\mathbb{S}_v \simeq (S^1)^d$ of $\mathbb{T}_v^{\mathrm{an}}$, whereas in the non-Archimedean case it is the Dirac measure at a translate of the Gauss point of this v-adic analytic torus.

The following is our main result in the toric setting.

Theorem 1.3. *Let $\overline{D}$ be a toric adelic $\mathbb{R}$-divisor on X with D big. Assume that $\vartheta_{\overline{D}}$ is wide and let u_v, $v \in \mathfrak{M}_K$, be its balanced family of sup-gradients. Then each $v \in \mathfrak{M}_K$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$, the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to η_{v,u_v}.*

We actually show a stronger result (Theorem 9.8): when $\vartheta_{\overline{D}}$ is wide the limit in (1.4) implies that for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ the limit $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell)$ exists in $\mathbb{R}$ and can be explicitly computed. When E is toric we have

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u_v}, \tag{1.5}$$

where $g_{\overline{E},v}$ denotes the v-adic Green function of $\overline{E}$.

Furthermore, we show that the converse of this theorem holds when $\overline{D}$ is semipositive: in this situation, the condition that $\vartheta_{\overline{D}}$ is wide is necessary for the equidistribution of the Galois orbits of the $\overline{D}$-small generic sequences of points (Corollary 9.12). These results allow us to recover the criterion for the equidistribution property in the toric setting from [BPRS19]. As a byproduct they also show that the sufficient condition in this criterion remains valid without the semipositivity hypothesis and extend its scope to cover the height convergence property in (1.5), see Remark 9.13 for more details.

The proof of Theorem 1.3 is an application of Theorem 6.7, which as explained is a reinterpretation of Theorem 1.2 in terms of positive arithmetic intersection numbers. In addition, it uses the computation of the positive arithmetic intersection numbers of toric adelic divisors in terms of the mixed integrals of their local roof functions.

Finally, as an application of Theorem 8.4 we strengthen our toric equidistribution theorem to allow test functions with logarithmic singularities along the boundary divisor and special translates of subtori. For simplicity, here we state it for the semipositive setting, and refer to Theorem 9.15 for the general case.

Theorem 1.4. *Let $\overline{D}$ be a semipositive toric adelic $\mathbb{R}$-divisor on X with D big and such that $\vartheta_{\overline{D}}$ is wide. Let E be an effective divisor on X such that each of its geometric irreducible components is either contained in the boundary $X \setminus X_0$, or is the closure V of the translate of a subtorus of X_0 with $\mu^{\mathrm{ess}}(\overline{D}|_V) = \mu^{\mathrm{ess}}(\overline{D})$. Then for each $v \in \mathfrak{M}_K$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi \, d\eta_{v,u_v}.$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along E.

1.3. **Semiabelian varieties.** As another application, we study the equidistribution of the Galois orbits of algebraic points of small height in semiabelian varieties. In particular, we recover Kühne's theorem in this direction.

Let G be a semiabelian variety over K that is the extension of an abelian variety A of dimension g by the split torus $\mathbb{G}_{\mathrm{m}}^r$. Consider the compactification $\overline{G}$ induced by toric compactification $(\mathbb{P}^1)^r$ of $\mathbb{G}_{\mathrm{m}}^r$, which is a smooth variety containing G as a dense open subset. The projection $G \to A$ extends to a morphism $\pi\colon \overline{G} \to A$ allowing to consider this compactification as a $(\mathbb{P}^1)^r$-bundle over A. Moreover, for any given integer $\ell > 1$ the multiplication-by-ℓ on G extends to an endomorphism $[\ell]\colon \overline{G} \to \overline{G}$.

Let $\overline{M}^{\mathrm{can}}$ be the boundary divisor $\overline{G} \setminus G$ equipped with its canonical structure of adelic divisor with respect to $[\ell]$ in the sense of arithmetic dynamics. In addition, denote by $\overline{N}^{\mathrm{can}}$ the canonical adelic divisor associated to an ample symmetric divisor on A. Then we consider the adelic divisor on $\overline{G}$ defined as

$$\overline{D} = \overline{M}^{\mathrm{can}} + \pi^*\overline{N}^{\mathrm{can}}.$$

The associated height function is nonnegative on $G(\overline{K})$ and vanishes on the torsion points, and so $\mu^{\mathrm{ess}}(\overline{D}) = 0$. On the other hand, it might take negative values on $(\overline{G} \setminus G)(\overline{K})$ whenever G is not isotrivial, that is isogenous to the product $\mathbb{G}_{\mathrm{m}}^r \times A$. In this case we have $\mu^{\mathrm{ess}}(\overline{D}) > \mu^{\mathrm{abs}}(\overline{D})$ and so $\overline{D}$ is outside the scope of Yuan's theorem. In spite of this, we show that it is possible to recover the semiabelian equidistribution theorem from Theorem 1.2.

Corollary 1.5 (Kühne [Küh22])**.** *Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $\overline{G}(\overline{K})$. Then for every $v \in \mathfrak{M}_K$ the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on X_v^{an} converges weakly to $c_1(\overline{D}_v)^{\wedge r+g}/(D^{r+g})$.*

This corollary is obtained by considering the sequence of adelic $\mathbb{R}$-divisors defined as

$$\overline{Q}_n = \ell^{-n}\overline{M} + \pi^*\overline{N}, \quad n \in \mathbb{N}.$$

Using the dynamical properties of $\overline{M}^{\mathrm{can}}$ and $\pi^*\overline{N}^{\mathrm{can}}$ one can check that $(\mathrm{Id}_{\overline{G}}, \overline{Q}_n)$ is a semipositive approximation of $\overline{D}$ whose absolute minimum decreases much faster than its inradius as $n \to \infty$, and that the normalized v-adic Monge-Ampère measure of $\overline{Q}_n$ coincides with that of $\overline{D}$. The result then follows as a direct application of Theorem 1.2.

More generally, the limit in (1.4) similarly implies that for every generic $\overline{D}$-small sequence $(x_\ell)_\ell$ in $\overline{G}(\overline{K})$ and every $\overline{E} \in \widehat{\mathrm{Div}}(\overline{G})$ we have

$$\lim_{\ell \to \infty} h_{\overline{E}}(x_\ell) = \frac{((\overline{M}^{\mathrm{can}})^r \cdot (\pi^* \overline{N}^{\mathrm{can}})^g \cdot \overline{E})}{(M^r \cdot \pi^* N^g)}, \tag{1.6}$$

see Theorem 10.11 and Remark 10.12.

Our approach is similar in spirit to Kühne's, in the sense that it ultimately relies on an asymptotic use of the arithmetic Siu inequality. However, our technical implementation is different since we do not need to modify G through a sequence of isogenies as in [Küh22], but rather produce a suitable sequence of semipositive approximations sitting on $\overline{G}$.

Finally, as an application of Theorem 8.4 we strengthen the semiabelian equidistribution theorem to allow test functions with logarithmic singularities along the closure of a torsion hypersurface (Theorem 10.15).

Theorem 1.6. *Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $\overline{G}(\overline{K})$. Then for every $v \in \mathfrak{M}_K$ and any function $\varphi \colon \overline{G}_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along the closure of a torsion hypersurface of G we have*

$$\lim_{\ell \to \infty} \int_{\overline{G}_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{\overline{G}_v^{\mathrm{an}}} \varphi \, \frac{c_1(\overline{D}_v)^{\wedge r+g}}{(D^{r+g})}.$$

We formalize our approach in the dynamical setting for sums of canonical adelic $\mathbb{R}$-divisors, obtaining general versions of the above results which might be of interest beyond the semiabelian case (Theorems 10.8 and 10.10).

1.4. **Further results and questions.** In [YZ23, Theorem 5.4.3], Yuan and Zhang generalized Yuan's Theorem 1.1 to quasi-projective varieties using their new theory of adelic line bundles. In §11 we present a generalization of Theorem 1.2 in this context, that recovers [YZ23, Theorem 5.4.3].

As already explained, in the toric case the condition of Theorem 1.2 translates into a convex analysis statement involving the global roof function. It turns out that on an arbitrary projective variety we have a similar result involving arithmetic Okounkov bodies in the sense of Boucksom and Chen [BC11]. Since this is beyond the scope of this article, it will appear in a subsequent manuscript.

The condition of Theorem 1.2 is optimal in the semipositive toric case, in the sense that for a semipositive toric adelic $\mathbb{R}$-divisor the equidistribution property at every place is equivalent to the existence a sequence of semipositive approximations with (1.3). It is natural to ask whether this remains true in general, that is if Theorem 1.2 actually gives a criterion for the equidistribution property. We refer to Question 6.12 for a precise formulation involving the invariants $\rho(\overline{D}(t))$. In Proposition 6.13 we give an affirmative answer under an additional technical condition, which is always satisfied for semipositive toric adelic $\mathbb{R}$-divisors.

1.5. **Outline of the proof.** Our approach is based on the variational principle of [SUZ97]. Let $v \in \mathfrak{M}_K$ be a place and φ a real-valued function on X_v^{an}. With the notation and assumptions of Theorem 1.2, we need to prove that

$$\lim_{\ell \to \infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \lim_{n \to \infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\nu_{n,v}. \tag{1.7}$$

By a standard density argument, it suffices to consider the case where φ is $\mathrm{Gal}(\mathbb{C}_v/K_v)$-invariant (see §3.5). This allows to associate to φ an adelic divisor $\overline{E} = \overline{0}^{\varphi}$ with trivial geometric divisor $E = 0$, such that (1.7) translates into

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)}. \tag{1.8}$$

The proof is significantly simpler when X is a curve, and we restrict to this case here. Then $\phi_n = \mathrm{Id}_X$ for every n, and the inradius of two nef and big $\mathbb{R}$-Cartier divisors A, B is simply $r(A;B) = (A)/(B)$ (where (A), (B) denote the degrees of the $\mathbb{R}$-Weil divisors associated to A and B). Therefore, in that case the assumption of Theorem 1.2 boils down to

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{(Q_n)} = 0. \tag{1.9}$$

For simplicity, we furthermore assume that $\overline{E}$ can be written as the difference of two semipositive adelic divisors, and that $\mu^{\mathrm{ess}}(\overline{D}) > 0$. Then $\mu^{\mathrm{abs}}(\overline{Q}_n) \geq 0$ for n large enough, which means that $\overline{Q}_n$ is nef. A classical consequence of Yuan's arithmetic version of Siu's inequality [Yua08] then implies that there exists a constant $C \in \mathbb{R}$ such that

$$\widehat{\mathrm{vol}}(\overline{Q}_n + \lambda\overline{E}) \geq (\overline{Q}_n^2) + 2\lambda \overline{Q}_n \cdot \overline{E} - C\lambda^2 \tag{1.10}$$

for every $\lambda \in (0,1)$ and every sufficiently large n (see [Che11, Lemma 4.2] and Lemma 4.5). Here $\widehat{\mathrm{vol}}$ denotes the arithmetic volume (§3.3). On the other hand, standard variants of Zhang's theorem on minima give

$$\mu^{\mathrm{ess}}(\overline{Q}_n + \lambda\overline{E}) \geq \frac{\widehat{\mathrm{vol}}(\overline{Q}_n + \lambda\overline{E})}{2(Q_n)} \quad \text{and} \quad \frac{(\overline{Q}_n^2)}{2(Q_n)} \geq \mu^{\mathrm{abs}}(\overline{Q}_n). \tag{1.11}$$

Moreover, we have

$$\mu^{\mathrm{ess}}(\overline{D}) + \lambda \liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \liminf_{\ell\to\infty} h_{\overline{D}+\lambda\overline{E}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) \geq \mu^{\mathrm{ess}}(\overline{Q}_n + \lambda\overline{E}),$$

where the last inequality comes from the fact that $\overline{D} - \overline{Q}_n$ is pseudo-effective, see Lemma 3.15. Combining this with (1.10) and (1.11) gives

$$\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \frac{1}{\lambda}(\mu^{\mathrm{abs}}(\overline{Q}_n) - \mu^{\mathrm{ess}}(\overline{D})) + \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)} - C\frac{\lambda}{(Q_n)}.$$

Applying this to a suitable sequence of real numbers $(\lambda_n)_n$ and using (1.9) we obtain $\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \limsup_{n\to\infty} (\overline{Q}_n \cdot \overline{E})/(Q_n)$, and the result follows in that case by applying this inequality to $-\overline{E}$. Similar arguments remain valid for an arbitrary adelic $\mathbb{R}$-divisor $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, leading to the height convergence property in (1.4). This requires a bit more work as we have to take care of a term $\mathrm{vol}(Q_n + \lambda E)$ appearing in Zhang's inequality.

In higher dimension the argument breaks down as a lower bound of the form (1.10) is not precise enough to take advantage of the assumption (1.3). The core of our proof is a finer consequence of Yuan's bigness theorem, with an error term involving the inradius of Q_n (§7.1). It relies on precise estimates of arithmetic intersection numbers based on the arithmetic Hodge index theorem of Yuan and Zhang [YZ17].

As emphasized in [Che11], the variational principle can be translated into differentiability statements for invariants associated to adelic divisors. For example, Yuan's theorem is a consequence of the differentiability of arithmetic χ-volumes [Yua08], and Chen's equidistribution theorem follows from the differentiability of arithmetic volumes [Che11]. Under the assumption of Theorem 1.2 we actually show the differentiability of the essential minimum at $\overline{D}$, which is equivalent to the height convergence property in (1.4) and thus implies Theorem 1.2.

1.6. **Organization of the paper.** Sections 2 and 3 contain the material we need concerning $\mathbb{R}$-divisors and adelic $\mathbb{R}$-divisors. In section 4 we collect basic results on the differentiability of concave functions and we recall Chen's theorem on the differentiability of arithmetic volumes. Section 5 relates the equidistribution problem to the differentiability of the essential minimum function. We state our main theorem in section 6 and we prove it in section 7. In section 8 we deduce a generalization of Chambert-Loir and Thuillier's logarithmic equidistribution theorem. In sections 9 and 10 we apply our results in the setting of toric varieties and of (non-necessarily polarized) dynamical systems, including semi-abelian varieties. Finally, in section 11 we extend our main theorem to quasi-projective varieties. Appendix A contains auxiliary results on convex analysis that are used in section 9.

1.7. **Notation and conventions.**

1.7.1. A *variety* is a separated and integral scheme of finite type over a field. We let K be a number field and we fix an algebraic closure $\overline{K}$ of K. Throughout the text, we let X be a normal projective variety over K of dimension $d = \dim X \geq 1$. The elements of $X(\overline{K})$ are called the *algebraic points* of X. We let $X_{K'} = X \times_K \operatorname{Spec} K'$ for any field extension $K \subset K'$. A *modification* of X is a birational projective morphism $\phi\colon X' \to X$. We say that it is normal if so is X'.

1.7.2. We let $\mathcal{O}_K$ be the ring of integers of K, $\mathfrak{M}_K$ its set of places, and $\mathfrak{M}_K^\infty \subset \mathfrak{M}_K$ the subset of Archimedean places. For every $v \in \mathfrak{M}_K$, we let K_v be the completion of K with respect to v and $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ with respect to the restriction of v to $\mathbb{Q}$. We fix an algebraic closure $\overline{K}_v$ of K_v and an embedding $\overline{K} \hookrightarrow \overline{K}_v$. We equip $\overline{K}_v$ with the unique absolute value $|\cdot|_v$ extending the usual absolute value on $\mathbb{Q}_v$: $|p|_v = p^{-1}$ if v is a finite place over a prime number p, and $|\cdot|_v$ is the usual absolute value on $\mathbb{C}$ if v is Archimedean. We denote by $\mathbb{C}_v$ the completion of $\overline{K}_v$ with respect to this absolute value. We let

$$n_v = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}.$$

For any place $v_0 \in \mathfrak{M}_\mathbb{Q}$ we have $\sum_{v|v_0} n_v = 1$, where the sum is over the places $v \in \mathfrak{M}_K$ above v_0. Moreover, we have the product formula for K: for any $a \in K^\times$,

$$\sum_{v \in \mathfrak{M}_K} n_v \log |a|_v = 0.$$

1.7.3. A *measure* is a regular Borel measure on a locally compact Hausdorff space. A *signed measure* is a difference of two measures.

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2. $\mathbb{R}$-divisors and inradius

We recall some facts that we shall frequently use about $\mathbb{R}$-divisors and their positivity in the geometric setting. We refer the reader to [Laz04] and [Mor16, §1.2] for more background. In §2.2 we review the definition and basic properties of the inradius of two $\mathbb{R}$-divisors.

2.1. **$\mathbb{R}$-divisors.** We denote by $\mathrm{Div}(X)$ the set of Cartier divisors on X and by $\mathrm{Rat}(X)$ the set of rational functions on X. We let $\mathrm{Rat}(X)^\times_\mathbb{R} = \mathrm{Rat}(X)^\times \otimes_\mathbb{Z} \mathbb{R}$. We denote by $\mathrm{Div}(X)_\mathbb{Q} = \mathrm{Div}(X)\otimes_\mathbb{Z}\mathbb{Q}$ the set of $\mathbb{Q}$-Cartier divisors, and by $\mathrm{Div}(X)_\mathbb{R} = \mathrm{Div}(X)\otimes_\mathbb{Z}\mathbb{R}$ the set of $\mathbb{R}$-Cartier divisors. Since X is normal, there exists an injective morphism from $\mathrm{Div}(X)$ to the free Abelian group of Weil divisors of X. Therefore $\mathrm{Div}(X)$ has no torsion and we have inclusions

$$\mathrm{Div}(X) \subset \mathrm{Div}(X)_\mathbb{Q} \subset \mathrm{Div}(X)_\mathbb{R}.$$

In this text, we are mainly concerned with Cartier divisors and $\mathbb{R}$-Cartier divisors and so we call them just divisors and $\mathbb{R}$-divisors, for short. Given a rational function $f \in \mathrm{Rat}(X)^\times$, we let $\mathrm{div}(f) \in \mathrm{Div}(X)$ be the corresponding divisor. The map $\mathrm{div}\colon \mathrm{Rat}(X)^\times \to \mathrm{Div}(X)$ extends by linearity to a map $\mathrm{Rat}(X)^\times_\mathbb{R} \to \mathrm{Div}(X)_\mathbb{R}$, which we also denote by div. Given two $\mathbb{R}$-divisors $D, D' \in \mathrm{Div}(X)_\mathbb{R}$, we write $D \equiv D'$ if they are linearly equivalent, that is $D' = D + \mathrm{div}(f)$ for some $f \in \mathrm{Rat}(X)^\times_\mathbb{R}$.

For $D \in \mathrm{Div}(X)_\mathbb{R}$, we let $\mathrm{supp}(D) \subset X$ be the support of D, that is the support of the associated $\mathbb{R}$-Weil divisor. It is a closed subset of X. We say that D is effective if so is its associated $\mathbb{R}$-Weil divisor.

Given a family $D_1, \dots, D_d$ in $\mathrm{Div}(X)_\mathbb{R}$ and an r-dimensional cycle Z on X ($0 \le r \le d$), we denote by $(D_1 \cdots D_r \cdot Z)$ the intersection number. When $Z = X$, we simply write $(D_1 \cdots D_d)$. This number only depends on the linear equivalence classes of the $\mathbb{R}$-divisors and it is invariant under normal modifications. If D_i, $i = 1, \dots, d-1$, are nef and $D_d - D'_d$ is pseudo-effective, then the corresponding intersection numbers compare as

$$(D_1 \cdots D_{d-1} \cdot D'_d) \le (D_1 \cdots D_{d-1} \cdot D_d). \tag{2.1}$$

If $D \in \mathrm{Div}(X)_\mathbb{R}$ is nef, the intersection number (D^d) coincides with the volume $\mathrm{vol}(D)$ of D.

The set of global sections of an $\mathbb{R}$-divisor D is defined as

$$\Gamma(X, D) = \{(f, D) \mid f \in \mathrm{Rat}(X)^\times,\ D + \mathrm{div}(f) \ge 0\} \cup \{0\}.$$

For $s = (f, D) \in \Gamma(X, D)\setminus\{0\}$, we let $\mathrm{div}(s) = D + \mathrm{div}(f) \in \mathrm{Div}(X)_\mathbb{R}$. If $s_1 = (f_1, D_1)$ and $s_2 = (f_2, D_2)$ are global sections of $\mathbb{R}$-divisors $D_1, D_2 \in \mathrm{Div}(X)_\mathbb{R}$, we denote by

$$s_1 \otimes s_2 = (f_1 f_2, D_1 + D_2) \in \Gamma(X, D_1 + D_2)$$

their product.

2.2. **Inradius of two $\mathbb{R}$-divisors.** Following Teissier [Tei82], given two $\mathbb{R}$-divisors $P, A \in \mathrm{Div}(X)_{\mathbb{R}}$ with A big, we define the *inradius* of P with respect to A as

$$r(P;A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is big}\} = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is pseudo-effective}\}.$$

In general, $r(P;A) \in \mathbb{R} \cup \{-\infty\}$. However, if P is big then $r(P;A) > 0$ by the continuity of the volume function. Moreover, we have $r(\delta P;A) = \delta\, r(P,A)$ and $r(P;\delta A) = \delta^{-1}\, r(P,A)$ for any $\delta \in \mathbb{R}_{>0}$, and $r(\phi^* P, \phi^* A) = r(P,A)$ for any normal modification $\phi\colon X' \to X$. This follows from the fact that the volume is invariant under birational morphisms.

Lemma 2.1. *Let P, A_1, A_2 be big $\mathbb{R}$-divisors on X. Then*

$$r(A_2;A_1) \times r(P;A_2) \le r(P;A_1) \le \frac{1}{r(A_1;A_2)} \times r(P;A_2).$$

Proof. Since $P - r(P;A_2)A_2$ and $A_2 - r(A_2;A_1)A_1$ are pseudo-effective,

$$P - (r(P;A_2) \times r(A_2;A_1))A_1$$

is pseudo-effective. Therefore $r(A_2;A_1) \times r(P;A_2) \le r(P;A_1)$. The second inequality follows by interchanging the roles of A_1 and A_2. □

Lemma 2.2. *Let P and A be nef and big $\mathbb{R}$-divisors on X. Then*

$$\frac{(P^d)}{d \times (P^{d-1} \cdot A)} \le r(P;A) \le \frac{(P^d)}{(P^{d-1} \cdot A)}.$$

Proof. Let $\lambda \in \mathbb{R}$. By Siu's inequality [Laz04, Theorem 2.2.15], $P - \lambda A$ is big whenever

$$\lambda\, d \times (P^{d-1} \cdot A) < (P^d).$$

The first inequality follows. For the second one, note that $P - r(P;A)A$ is pseudo-effective. Since P is nef, we have

$$(P^d) - r(P;A)(P^{d-1} \cdot A) = (P^{d-1} \cdot (P - r(P;A)A)) \ge 0.$$

□

Lemma 2.3. *Let P and A be nef and big $\mathbb{R}$-divisors on X such that $A - P$ is pseudo-effective. Then*

$$r(P;A) \ge \frac{(P^d)}{d(A^d)}.$$

Proof. Since $A - P$ is pseudo-effective and A, P are nef, we have that $(P^{d-1} \cdot A) \le (A^d)$ by (2.1). Therefore the results follows from Lemma 2.2. □

Lemma 2.4. *Let $P, E \in \mathrm{Div}(X)_{\mathbb{R}}$ with P big. Let A be a big $\mathbb{R}$-divisor such that $A \pm E$ are pseudo-effective. For every $\lambda \ge 0$, we have*

$$\left(1 - \frac{\lambda}{r(P;A)}\right)^d \mathrm{vol}(P) \le \mathrm{vol}(P + \lambda E) \le \left(1 + \frac{\lambda}{r(P;A)}\right)^d \mathrm{vol}(P).$$

Proof. Since $A - E$ is pseudo-effective,

$$\frac{1}{r(P;A)} P - E = \frac{1}{r(P;A)}(P - r(P;A)A) + A - E$$

is also pseudo-effective. Therefore

$$\mathrm{vol}(P + \lambda E) \le \mathrm{vol}(P + \frac{\lambda}{r(P;A)} P) = \left(1 + \frac{\lambda}{r(P;A)}\right)^d \mathrm{vol}(P).$$

Similarly,

$$\frac{1}{r(P;A)}P+E=\frac{1}{r(P;A)}(P-r(P;A)A)+A+E$$

is pseudo-effective since so is $A+E$. Therefore

$$\left(1-\frac{\lambda}{r(P;A)}\right)^{d}\operatorname{vol}(P)=\operatorname{vol}(P-\frac{\lambda}{r(P;A)}P)\leq\operatorname{vol}(P+\lambda E).$$

□

3. Adelic $\mathbb{R}$-divisors

In this section we recall the definition and important facts concerning adelic $\mathbb{R}$-divisors, that generalize adelic line bundles in the sense of Zhang [Zha95b]. Our main references are [BPS14], [BMPS16] and [Mor16].

3.1. **First definitions.** For every $v\in\mathfrak{M}_K$, we let $X_v=X_{\mathbb{C}_v}$. We denote by X_v^{an} the analytification of X_v in the sense of Berkovich (see [BPS14, §1.2] and [Mor16, §1.3] for short introductions sufficient for our purposes). We have an injective map $X(\mathbb{C}_v)\hookrightarrow X_v^{\mathrm{an}}$, which induces an inclusion

$$\iota_v\colon X(\overline{K})\hookrightarrow X_v^{\mathrm{an}}$$

via the chosen embedding $\overline{K}\hookrightarrow\overline{K}_v\subset\mathbb{C}_v$. For every algebraic point $x\in X(\overline{K})$, we denote by $O(x)$ the orbit of x in $X(\overline{K})$ under the action of $\mathrm{Gal}(\overline{K}/K)$, and by $O(x)_v=\iota_v(O(x))$ its image in X_v^{an}. The absolute Galois group $G_v=\mathrm{Gal}(\overline{K}_v/K_v)$ acts on X_v^{an}. We let $C(X_v^{\mathrm{an}})$ be the set of continuous real-valued functions on X_v^{an}, and $C(X_v^{\mathrm{an}})^{G_v}\subset C(X_v^{\mathrm{an}})$ the subspace of continuous functions that are G_v-invariant.

Let $D\in\mathrm{Div}(X)_{\mathbb{R}}$ and let $v\in\mathfrak{M}_K$. A *continuous v-adic Green function* on D is a G_v-invariant function

$$g_v\colon X_v^{\mathrm{an}}\setminus\mathrm{supp}(D)_v^{\mathrm{an}}\to\mathbb{R}$$

with the following property: for every open covering $X=\bigcup_{i=1}^{\ell}U_i$ of X such that D is defined by $f_i\in\mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ on U_i, $g_v+\log|f_{i,v}|_v$ extends to a continuous function on $U_{i,v}^{\mathrm{an}}$ for every $i\in\{1,\dots,\ell\}$, where $f_{i,v}$ is the pull-back of f_i to $U_{i,v}^{\mathrm{an}}$. In this article we only consider continuous v-adic Green functions, and we call them v-adic Green functions for short.

Let $U\subset\mathrm{Spec}(\mathcal{O}_K)$ be a non-empty open subset. A model of X over U is a normal integral projective scheme $\mathcal{X}\to U$ such that $X=\mathcal{X}\times_U\mathrm{Spec}\,K=X$. Let $D\in\mathrm{Div}(X)_{\mathbb{R}}$. A model of (X,D) over U is a pair $(\mathcal{X},\mathcal{D})$ where $\mathcal{X}$ is a model of X over U and $\mathcal{D}$ is an $\mathbb{R}$-divisor on $\mathcal{X}$ such that $\mathcal{D}_{|X}=D$. Such a model induces v-adic Green functions on D for every place $v\in U$, that we denote by $g_{\mathcal{D},v}$ (see [Mor16, §2.1] for details).

Definition 3.1. An *adelic* $\mathbb{R}$-divisor on X is a pair $\overline{D}=(D,(g_v)_{v\in\mathfrak{M}_K})$ where $D\in\mathrm{Div}(X)_{\mathbb{R}}$ and g_v is a v-adic Green function on D for every $v\in\mathfrak{M}_K$, such that there exists a nonempty open subset $U\subset\mathrm{Spec}\,\mathcal{O}_K$ and a model $(\mathcal{X},\mathcal{D})$ of (X,D) over U with $g_v=g_{\mathcal{D},v}$ for every $v\in U$.

We say that $\overline{D}$ is an $\mathbb{R}$-divisor *over* D and conversely, we also say that D is the *geometric* $\mathbb{R}$*-divisor* of $\overline{D}$.

Unless otherwise stated, we use the following convention: given an adelic $\mathbb{R}$-divisor $\overline{D} \in \mathrm{Div}(X)_{\mathbb{R}}$, we always use the same letter D to denote its geometric $\mathbb{R}$-divisor and we denote by $(g_{\overline{D},v})_{v\in\mathfrak{M}_K}$ its family of Green functions.

The set of adelic $\mathbb{R}$-divisors forms a group, that we denote by $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. If $D \in \mathrm{Div}(X)$ (respectively $\mathrm{Div}(X)_{\mathbb{Q}}$), we say that $\overline{D}$ is an adelic divisor (respectively an adelic $\mathbb{Q}$-divisor). Adelic divisors and adelic $\mathbb{Q}$-divisors form subgroups of $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, which we denote by $\widehat{\mathrm{Div}}(X)$ and $\widehat{\mathrm{Div}}(X)_{\mathbb{Q}}$ respectively.

Example 3.2. Let $(\varphi_v)_{v\in\mathfrak{M}_K}$ be a family with $\varphi_v \in C^0(X_v^{\mathrm{an}})^{G_v}$ for every $v \in \mathfrak{M}_K$ and $\varphi_v = 0$ for all except finitely many $v \in \mathfrak{M}_K$. Then $(0, (\varphi_v)_{v\in\mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X)$. Every adelic divisor over $0 \in \mathrm{Div}(X)$ is of this form.

We let $[\infty] = (0, (\varphi_v)_{v\in\mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X)$ be the adelic divisor over 0 given by the constant functions $\varphi_v = 1$ if v is Archimedean, and $\varphi_v = 0$ otherwise.

Let $\overline{D} = (D, (g_v)_{v\in\mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, and let $\phi\colon X' \to X$ be a morphism from a normal projective variety X' such that the image of ϕ is not contained in $\mathrm{supp}(D)$. Then we define the pullback $\phi^*\overline{D} \in \widehat{\mathrm{Div}}(X')_{\mathbb{R}}$ as the $\mathbb{R}$-divisor ϕ^*D equipped, at each place $v \in \mathfrak{M}_K$, with the pullback of g_v to $(X')_v^{\mathrm{an}}$ by the v-adic analytification of ϕ.

Let $f \in \mathrm{Rat}(X)^\times$. For each $v \in \mathfrak{M}_K$, we let f_v be the pullback of f to X_v^{an}. We define an adelic divisor on X by

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), (-\log|f_v|_v)_{v\in\mathfrak{M}_K}).$$

This construction defines a group morphism $\mathrm{Rat}(X)^\times \to \widehat{\mathrm{Div}}(X)$. It extends by linearity to a group morphism $\mathrm{Rat}(X)^\times_{\mathbb{R}} \to \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, also denoted $\widehat{\mathrm{div}}$. Two adelic $\mathbb{R}$-divisors $\overline{D}, \overline{D}' \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ are said *linearly equivalent*, denoted $\overline{D} \equiv \overline{D}'$, if there exists $f \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$ such that $\overline{D}' = \overline{D} + \widehat{\mathrm{div}}(f)$.

Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. We say that $\overline{D}$ is *semipositive* if $g_{\overline{D},v}$ is of $C^0 \cap \mathrm{PSH}$-type in the sense of [Mor16, §1.4 and Definition 2.1.6] for every $v \in \mathfrak{M}_K$. We say that $\overline{D}$ is *DSP* (short for difference of semipositive) if there exist two semipositive adelic $\mathbb{R}$-divisors $\overline{D}_1$, $\overline{D}_2$ such that $\overline{D} = \overline{D}_1 - \overline{D}_2$. We denote by $\widehat{\mathrm{DSP}}(X)_{\mathbb{R}} \subset \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ the subgroup of DSP adelic $\mathbb{R}$-divisors. Note that if $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ is semipositive, then its geometric $\mathbb{R}$-divisor D is nef and in particular $\mathrm{vol}(D) = (D^d)$.

Remark 3.3. It is standard that to an adelic divisor $\overline{D} \in \widehat{\mathrm{Div}}(X)$ one can associate an adelic line bundle $\overline{L} = (\mathcal{O}_X(D), (\|.\|_v)_{v\in\mathfrak{M}_K})$ in the sense of Zhang [Zha95b], and that every adelic line bundle can be constructed in this way. We refer the reader to [BMPS16, proof of Proposition 3.8] for details on this construction. The adelic line bundle $\overline{L}$ is semipositive in the sense of [Zha95b] if and only if $\overline{D}$ is semipositive, and it is integrable in the sense of [Zha95b] if and only if $\overline{D}$ is DSP.

3.2. **Heights of points and subvarieties.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and $x \in X(\overline{K})$. Let $f \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$ be such that $x \notin \mathrm{supp}(D + \mathrm{div}(f))$. The *height* of x with respect to $\overline{D}$ is the real number

$$h_{\overline{D}}(x) = \sum_{v\in\mathfrak{M}_K} \frac{n_v}{\#O(x)_v} \sum_{y\in O(x)_v} g_{\overline{D}+\widehat{\mathrm{div}}(f),v}(y).$$

This quantity does not depend on the choice of f by the product formula.

Let $Y \subset X$ be an r-dimensional subvariety, $r \in \{1, \dots, d\}$, and $(\overline{D}_i)_{1\le i\le r} \in \widehat{\mathrm{DSP}}(X)^r_{\mathbb{R}}$. For every $v \in \mathfrak{M}_K$, we have a signed measure

$$c_1(\overline{D}_{1,v}) \wedge \cdots \wedge c_1(\overline{D}_{r,v}) \wedge \delta_{Y_v^{\mathrm{an}}} \tag{3.1}$$

on X_v^{an}, supported on Y_v^{an} and with total mass $(D_1 \cdots D_r \cdot Y)$. In the case where all the $\overline{D}_i$'s are semipositive adelic divisors, this is the measure defined in [Cha06] (associated to the metrized line bundles corresponding to $\overline{D}_1, \dots, \overline{D}_r$). This construction is symmetric and multilinear in $\overline{D}_1, \dots, \overline{D}_r$, thus it extends to the general case of DSP adelic $\mathbb{R}$-divisors by multilinearity. For every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, the v-adic Green function $g_{\overline{E},v}$ is integrable for the signed measure (3.1). This is proved in [CLT09, Theorem 4.1] for adelic divisors, and remains true for adelic $\mathbb{R}$-divisors by multilinearity. In the case where $Y = X$ and $\overline{D}_1 = \overline{D}_2 = \ldots = \overline{D}_r$ are equal, we simply write $c_1(\overline{D}_{1,v})^{\wedge d}$. This measure is called the *v-adic Monge-Ampère measure* associated to $\overline{D}_1$.

Let $\overline{D}_{r+1} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, and let $\overline{D}'_{r+1} \equiv \overline{D}_{r+1}$ be such that D'_{r+1} intersects Y properly. The *height* $h_{\overline{D}_1,\dots,\overline{D}_{r+1}}(Y)$ of Y with respect to $(\overline{D}_i)_{1\le i\le r+1}$ is defined by induction using the following arithmetic *Bézout formula*: we let $h(\emptyset) = 0$, and

$$\begin{aligned} h_{\overline{D}_1,\dots,\overline{D}_{r+1}}(Y) = {} & h_{\overline{D}_1,\dots,\overline{D}_r}(Y \cdot D'_{r+1}) \\ & + \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{D}'_{r+1},v}\, c_1(\overline{D}_{1,v}) \wedge \cdots \wedge c_1(\overline{D}_{r,v}) \wedge \delta_{Y_v^{\mathrm{an}}}. \end{aligned} \tag{3.2}$$

This is well-defined, independent of the choice of $\overline{D}'_{r+1}$, and multilinear in $\overline{D}_1, \dots, \overline{D}_{r+1}$ [BPS14, §1.5], [BMPS16, page 225]. In the case where $\overline{D}_{r+1} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$, this construction is symmetric in $\overline{D}_1, \dots, \overline{D}_{r+1}$. The definition of $h_{\overline{D}_1,\dots,\overline{D}_{r+1}}(Y)$ extends to the case where Y is an r-dimensional cycle by multilinearity. When $\overline{D}_1 = \ldots = \overline{D}_{r+1}$, we simply write $h_{\overline{D}_1}(Y) = h_{\overline{D}_1,\dots,\overline{D}_{r+1}}(Y)$.

If $r = d$, the *arithmetic intersection number* of $\overline{D}_1, \dots, \overline{D}_{d+1}$ is

$$(\overline{D}_1 \cdots \overline{D}_{d+1}) = h_{\overline{D}_1,\dots,\overline{D}_{d+1}}(X).$$

This quantity only depends on the linear equivalence classes of $\overline{D}_1, \dots, \overline{D}_{d+1}$, and

$$(\phi^*\overline{D}_1 \cdots \phi^*\overline{D}_{d+1}) = (\overline{D}_1 \cdots \overline{D}_{d+1})$$

for every normal modification $\phi\colon X' \to X$. Moreover, it follows from the definition that

$$([\infty] \cdot \overline{D}_2 \cdots \overline{D}_{d+1}) = \sum_{v \in \mathfrak{M}_K^{\infty}} n_v \int_{X_v^{\mathrm{an}}} c_1(\overline{D}_{2,v}) \wedge \cdots \wedge c_1(\overline{D}_{d+1,v}) = (D_2 \cdots D_{d+1}). \tag{3.3}$$

When $\overline{D}_{d+1}$ is DSP, our definition of arithmetic intersection number coincides with the one in [Mor16, §4.5].

3.3. **Arithmetic volumes and positivity.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. For every $s = (f, D) \in \Gamma(X, D) \setminus \{0\}$ and $y \in X_v^{\mathrm{an}}$ with $y \notin \mathrm{supp}(\mathrm{div}(s))_v^{\mathrm{an}}$, we let

$$\|s(y)\|_v = \exp(-g_{\overline{D}+\widehat{\mathrm{div}}(f),v}(y)).$$

By [Mor16, Propositions 1.4.2 and 2.1.3], this definition can be extended to define a continuous map $y \in X_v^{\mathrm{an}} \mapsto \|s(y)\|_v \in \mathbb{R}$. We let

$$\|s\|_{v,\sup} = \sup_{y \in X_v^{\mathrm{an}}} \|s(y)\|_v,$$

and we say that s is *small* if $\|s\|_{v,\sup} \leq 1$ for every $v \in \mathfrak{M}_K$. By convention, we say that $0 \in \Gamma(X, D)$ is small. We denote by $\widehat{\Gamma}(X, \overline{D})$ the set of small global sections of $\overline{D}$. We remark that for every $m \in \mathbb{N}$ and non-zero $s \in \Gamma(X, mD)$ we have

$$h_{\overline{D}}(x) \geq -\frac{1}{m} \sum_{v \in \mathfrak{M}_K} n_v \log \|s\|_{v,\sup} \quad \text{for all } x \in X(\overline{K}) \setminus \operatorname{supp}(\operatorname{div}(s)). \tag{3.4}$$

Let $\mathbb{A}_K$ be the ring of adèles of K, and let

$$\mathbb{B}(\overline{D}) = \{(s_v)_{v \in \mathfrak{M}_K} \in \Gamma(X, D) \otimes_K \mathbb{A}_K \mid \|s_v\|_{v,\sup} \leq 1 \ \forall v \in \mathfrak{M}_K\}.$$

Note that $\Gamma(X, D) \otimes_K \mathbb{A}_K$ is a locally compact group, and $\Gamma(X, D)$ is a lattice in $\Gamma(X, D) \otimes_K \mathbb{A}_K$ via the diagonal embedding $K \hookrightarrow \mathbb{A}_K$. We denote by μ the unique Haar measure on $\Gamma(X, D) \otimes_K \mathbb{A}_K$ satisfying

$$\mu((\Gamma(X, D) \otimes_K \mathbb{A}_K)/\Gamma(X, D)) = 1,$$

and we let $\widehat{\chi}(X, \overline{D}) = \log \mu(\mathbb{B}(\overline{D}))$.

Definition 3.4. The *arithmetic volume* of $\overline{D}$ is the quantity

$$\widehat{\operatorname{vol}}(\overline{D}) = \frac{1}{[K : \mathbb{Q}]} \limsup_{m \to \infty} \frac{\log(\#\widehat{\Gamma}(X, m\overline{D}))}{m^{d+1}/(d+1)!}$$

The *arithmetic χ-volume* of $\overline{D}$ is

$$\widehat{\operatorname{vol}}_\chi(\overline{D}) = \frac{1}{[K : \mathbb{Q}]} \limsup_{m \to \infty} \frac{\widehat{\chi}(X, m\overline{D})}{m^{d+1}/(d+1)!}.$$

In the sequel we recall some important properties of arithmetic (χ-)volumes. We mainly refer to [Mor16] for details and proofs. We note that as opposed to [Mor16], we do not assume X to be geometrically irreducible. However, by Stein factorization there exists a finite field extension K' of K with the property that the structural morphism $X \to \operatorname{Spec}(K)$ factors through a morphism $X \to \operatorname{Spec}(K')$ and that X is geometrically connected over K'. Since X is normal, it is geometrically irreducible as a variety over K' and all the results we use from [Mor16] readily extend to our setting.

We have $\widehat{\operatorname{vol}}(\phi^*\overline{D}) = \widehat{\operatorname{vol}}(\overline{D})$ and $\widehat{\operatorname{vol}}_\chi(\phi^*\overline{D}) = \widehat{\operatorname{vol}}_\chi(\overline{D})$ for every normal modification $\phi \colon X' \to X$, and moreover $\widehat{\operatorname{vol}}(\overline{D}) \geq \widehat{\operatorname{vol}}_\chi(\overline{D})$ by Minkowski's inequality [Mor16, §4.3]. By [Mor16, Theorem 5.2.1], we have the following continuity property: for every $\overline{E} \in \widehat{\operatorname{Div}}(X)_\mathbb{R}$,

$$\widehat{\operatorname{vol}}(\overline{D}) = \lim_{\lambda \to 0} \widehat{\operatorname{vol}}(\overline{D} + \lambda \overline{E}) \quad \text{and} \quad \widehat{\operatorname{vol}}_\chi(\overline{D}) = \lim_{\lambda \to 0} \widehat{\operatorname{vol}}_\chi(\overline{D} + \lambda \overline{E}).$$

We now recall classical positivity notions for adelic $\mathbb{R}$-divisors.

Definition 3.5. Let $\overline{D} \in \widehat{\operatorname{Div}}(X)_\mathbb{R}$. We say that $\overline{D}$ is

- *effective* if D is effective and if $g_{\overline{D},v} \geq 0$ on $X_v^{\mathrm{an}} \setminus \operatorname{supp}(D)_v^{\mathrm{an}}$ for every $v \in \mathfrak{M}_K$,
- *big* if $\widehat{\operatorname{vol}}(\overline{D}) > 0$,
- *pseudo-effective* if $\overline{D} + \overline{B}$ is big for every big $\overline{B} \in \widehat{\operatorname{Div}}(X)_\mathbb{R}$,

- *nef* if $\overline{D}$ is semipositive and $h_{\overline{D}}(x) \geq 0$ for every $x \in X(\overline{K})$.

Remark 3.6. If there exists $s \in \widehat{\Gamma}(X, \overline{D}) \setminus \{0\}$, then for every big $\overline{B} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and $m \in \mathbb{N}_{>0}$ we have an inclusion $\widehat{\Gamma}(X, m\overline{B}) \hookrightarrow \widehat{\Gamma}(X, m(\overline{D} + \overline{B}))$ given by multiplication by $s^{\otimes m}$. It follows that $\widehat{\mathrm{vol}}(\overline{D} + \overline{B}) \geq \widehat{\mathrm{vol}}(\overline{B}) > 0$, and $\overline{D}$ is pseudo-effective. In particular, an effective adelic $\mathbb{R}$-divisor is pseudo-effective. Moreover, a nef adelic $\mathbb{R}$-divisor is pseudo-effective [Mor16, Proposition 4.4.2 (2)].

We recall a version of the well-known fact that adelic $\mathbb{R}$-divisors can be approximated by DSP adelic $\mathbb{R}$-divisors.

Lemma 3.7. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and let $\varepsilon > 0$. There exists $\overline{E}' \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$ over E such that $\overline{E}' - \overline{E}$ is effective and $\overline{E} + \varepsilon[\infty] - \overline{E}'$ is pseudo-effective.*

Proof. By [Mor16, Theorem 4.1.3], there exists a finite set $S \subset \mathfrak{M}_K$ containing $\mathfrak{M}_K^{\infty}$ with the following property. For every $\varepsilon' > 0$, there exists a model $\mathcal{E}$ of E over $\mathrm{Spec}\, \mathcal{O}_K$ such that if we set $g_{\overline{E}',v} = g_{\mathcal{E},v}$ and $\varphi_v = g_{\overline{E}',v} - g_{\overline{E},v}$ for every non-Archimedean place $v \in \mathfrak{M}_K$, then $\varphi_v = 0$ for every $v \notin S$ and $0 \leq \varphi_v \leq \varepsilon'$ for every $v \in S \setminus \mathfrak{M}_K^{\infty}$. By the Stone-Weierstrass theorem, for every Archimedean place $v \in \mathfrak{M}_K$ there exists a smooth v-adic Green function $g_{\overline{E}',v}$ on E such that $\varphi_v = g_{\overline{E}',v} - g_{\overline{E},v}$ satisfies $0 \leq \varphi_v \leq \varepsilon'$. By construction, $\overline{E}' = (E, (g_{\overline{E}',v})_{v \in \mathfrak{M}_K})$ is DSP and $\overline{E}' - \overline{E} = (0, (\varphi_v)_{v \in \mathfrak{M}_K})$ is effective. Moreover, it follows from [BMPS16, Lemma 1.11] that for ε' sufficiently small we have

$$\widehat{\Gamma}(X, \overline{E} + \varepsilon[\infty] - \overline{E}') = \widehat{\Gamma}(X, \varepsilon[\infty] - (0, (\varphi_v)_v)) \neq \emptyset,$$

hence $\overline{E} + \varepsilon[\infty] - \overline{E}'$ is pseudo-effective by Remark 3.6. □

Lemma 3.8. *Let $(\overline{D}_i)_{1 \leq i \leq d} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}^d$ be nef. For every pseudo-effective $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have*

$$(\overline{D}_1 \cdots \overline{D}_d \cdot \overline{E}) \geq 0.$$

Proof. In the case where $\overline{E} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$, this is [Mor16, Proposition 4.5.4 (3)]. The general case follows by (3.3) and Lemma 3.7. □

Arithmetic (χ-)volumes coincide with arithmetic intersection numbers under suitable positivity conditions.

Theorem 3.9 ([Mor16, Theorem 5.3.2])**.** *Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. If $\overline{D}$ is semipositive, then $\widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (\overline{D}^{d+1})$. If moreover $\overline{D}$ is nef, then $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (\overline{D}^{d+1})$.*

Remark 3.10. If $\overline{D} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$, then it can be written as $\overline{D} = \overline{N}_1 - \overline{N}_2$ with $\overline{N}_1, \overline{N}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ nef and big. Indeed, there exist semipositive $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{D} = \overline{D}_1 - \overline{D}_2$. Take any semipositive $\overline{A} \in \mathrm{Div}(X)_{\mathbb{R}}$ with A ample. For each $i \in \{1, 2\}$, $A + D_i$ is ample since D_i is nef. It follows from (3.4) that the height functions $h_{\overline{D}_i + \overline{A}} \colon X(\overline{K}) \to \mathbb{R}$ are both bounded from below by a real number. Taking a sufficiently large $c \in \mathbb{R}$ and letting $\overline{N}_i = \overline{D}_i + \overline{A} + c[\infty]$ for $i = 1, 2$, we have that $\overline{N}_i$ is nef and $(\overline{N}_i^{d+1}) > 0$. Then $\overline{D} = \overline{N}_1 - \overline{N}_2$ and $\overline{N}_1$ and $\overline{N}_2$ are big by Theorem 3.9.

The following is Yuan's arithmetic analogue of Siu's inequality [Yua08, Theorem 2.2]. The statement in [Yua08] extends to nef adelic $\mathbb{R}$-divisors by continuity, as shown in [CM15].

Theorem 3.11 ([CM15, Theorem 7.5])**.** *Let $\overline{P}_1$ and $\overline{P}_2$ be two nef adelic $\mathbb{R}$-divisors on X. Then*

$$\widehat{\mathrm{vol}}_\chi(\overline{P}_1 - \overline{P}_2) \geq (\overline{P}_1^{d+1}) - (d+1)(\overline{P}_1^d \cdot \overline{P}_2).$$

We end this paragraph with a consequence of the existence of arithmetic Fujita approximations, established independently by Yuan [Yua09] and by Chen [Che10].

Theorem 3.12 ([Mor16, Theorem 5.1.6])**.** *Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$ be big. For every $\varepsilon > 0$, there exists a normal modification $\phi\colon X' \to X$ and a nef $\overline{P} \in \widehat{\mathrm{Div}}(X')_\mathbb{R}$ such that $\phi^*\overline{D} - \overline{P}$ is pseudo-effective and*

$$(\overline{P}^{d+1}) = \widehat{\mathrm{vol}}(\overline{P}) \geq \widehat{\mathrm{vol}}(\overline{D}) - \varepsilon.$$

Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$. We associate to D the graded algebra

$$R(D) = \bigoplus_{m\in\mathbb{N}} \Gamma(X, mD).$$

When D is a divisor, this is the classical algebra of sections of the line bundle $\mathcal{O}_X(D)$ [Laz04, Definition 2.1.17].

For every $n \in \mathbb{N}$ and for every $t \in \mathbb{R}$, we let $R^t_m(\overline{D}) \subset \Gamma(X, mD)$ be the K-linear subspace generated by $\widehat{\Gamma}(X, m(\overline{D} - t[\infty]))$. Note that $\widehat{\Gamma}(X, m(\overline{D} - t[\infty])$ is the set of global sections $s \in \Gamma(X, mD)$ such that

$$\log \|s\|_{v,\sup} \leq \begin{cases} -mt & \text{if } v \text{ is Archimedean} \\ 0 & \text{otherwise.} \end{cases}$$

Then we set

$$R^t(\overline{D}) = \bigoplus_{m\in\mathbb{N}} R^t_m(\overline{D}),$$

which is a graded subalgebra of $R(D)$. The volume of $R^t(\overline{D})$ is the quantity

$$\mathrm{vol}(R^t(\overline{D})) = \limsup_{m\to\infty} \frac{\dim_K R^t_m(\overline{D})}{m^d/d!}.$$

If $\overline{D} - t[\infty]$ is big, the lim sup is actually a limit by [BC11, Lemma 1.6] and [LM09, Theorem 2.13]. The following theorem was proved by Chen in the setting of adelic divisors [Che10, Theorem 3.8] (see also [Che11, (5.2) page 380]). We mention that it can also be seen as a consequence of Boucksom and Chen's theorem on arithmetic Okounkov bodies [BC11, Theorems 1.11 and 2.8], whose proof extends to the setting of adelic $\mathbb{R}$-divisors [Mor16, §7.3].

Theorem 3.13 (Chen)**.** *Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$. If D is big, then*

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\int_0^\infty \mathrm{vol}(R^t(\overline{D}))dt.$$

3.4. **Absolute and essential minima.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_\mathbb{R}$. The *absolute minimum* of $\overline{D}$ is

$$\mu^{\mathrm{abs}}(\overline{D}) = \inf_{x\in X(\overline{K})} h_{\overline{D}}(x).$$

Clearly we have that $\mu^{\mathrm{abs}}(\phi^*\overline{D}) = \mu^{\mathrm{abs}}(\overline{D})$ for any surjective morphism $\phi\colon X' \to X$. By definition, $\overline{D}$ is nef if and only if it is semipositive and $\mu^{\mathrm{abs}}(\overline{D}) \geq 0$. When $\overline{D}$ is semipositive, we have

$$\mu^{\mathrm{abs}}(\overline{D}) = \sup\{t \in \mathbb{R} \mid \overline{D} - t[\infty] \text{ is nef}\}.$$

The following lower bound for the height of effective cycles is a consequence of Zhang's theorem on minima [Zha95a, Theorem 5.2], [Zha95b, Theorem 1.10].

Lemma 3.14. *Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be semipositive, $r \in \{0, \dots, d\}$ and let Z be an effective r-dimensional cycle on X. Then $h_{\overline{D}}(Z) \geq (r+1)\,\mu^{\mathrm{abs}}(\overline{D}) \times (D^r \cdot Z)$. In particular, if $\overline{D}$ is nef then $h_{\overline{D}}(Z) \geq 0$.*

Proof. By linearity, we may assume that Z is a subvariety. When D is ample, the result follows from [Bal24, Corollary 2.9]. Let $\overline{A} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be semipositive with A ample. Then for any $\varepsilon > 0$, $D + \varepsilon A$ is ample and therefore

$$\begin{aligned} h_{\overline{D}+\varepsilon\overline{A}}(Z) &\geq d\,\mu^{\mathrm{abs}}(\overline{D}+\varepsilon\overline{A}) \times ((D+\varepsilon A)^r \cdot Z) \\ &\geq d\,(\mu^{\mathrm{abs}}(\overline{D}) + \varepsilon\mu^{\mathrm{abs}}(\overline{A})) \times ((D+\varepsilon A)^r \cdot Z) \end{aligned}$$

We conclude by letting ε tend to zero using multilinearity. □

The *essential minimum* of $\overline{D}$ is the quantity

$$\mu^{\mathrm{ess}}(\overline{D}) = \sup_{Y \subsetneq X} \inf_{x \in (X \setminus Y)(\overline{K})} h_{\overline{D}}(x),$$

where the supremum is over all the closed subsets $Y \subsetneq X$. It follows from the definition that $\mu^{\mathrm{ess}}(\phi^*\overline{D}) = \mu^{\mathrm{ess}}(\overline{D})$ for any normal modification $\phi\colon X' \to X$. Moreover we always have $\mu^{\mathrm{ess}}(\overline{D}) < \infty$, and $\mu^{\mathrm{ess}}(\overline{D}) \in \mathbb{R}$ as soon as $R(D) \neq \{0\}$ [BC11, Proposition 2.6].

Lemma 3.15. *Let $\overline{D}_1, \overline{D}_2 \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$.*
(1) We have $\mu^{\mathrm{ess}}(\overline{D}_1 + \overline{D}_2) \geq \mu^{\mathrm{ess}}(\overline{D}_1) + \mu^{\mathrm{ess}}(\overline{D}_2)$.
(2) If D_1 is big, then $\lim_{\lambda\to 0} \mu^{\mathrm{ess}}(\overline{D}_1 + \lambda\overline{D}_2) = \mu^{\mathrm{ess}}(\overline{D}_1)$.
(3) For every $t \in \mathbb{R}$ such that $R^t(\overline{D}_1) \neq \{0\}$, we have $\mu^{\mathrm{ess}}(\overline{D}_1) \geq t$.
(4) If D_1 is big and $\overline{D}_1 - \overline{D}_2$ is pseudo-effective, then $\mu^{\mathrm{ess}}(\overline{D}_1) \geq \mu^{\mathrm{ess}}(\overline{D}_2)$.

Proof. For the first two points, see [Bal21, Lemma 3.15]. The third one is a direct consequence of (3.4). To prove (4), let $\overline{B} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be big and let $\varepsilon > 0$. Then $\overline{D}_1 + \varepsilon\overline{B} - \overline{D}_2$ is big, and therefore $R^0(\overline{D}_1 + \varepsilon\overline{B} - \overline{D}_2) \neq 0$. By (1) and (3), we obtain

$$\mu^{\mathrm{ess}}(\overline{D}_1 + \varepsilon\overline{B}) \geq \mu^{\mathrm{ess}}(\overline{D}_1 + \varepsilon\overline{B} - \overline{D}_2) + \mu^{\mathrm{ess}}(\overline{D}_2) \geq \mu^{\mathrm{ess}}(\overline{D}_2).$$

We conclude by letting ε tend to zero, using (2). □

The following theorem gives a converse to Lemma 3.15(3). It was proved in [Bal21, Theorem 1.1] as a consequence of a theorem of Ikoma [Iko15], under the additional assumption that $\overline{D}$ is semipositive. In recent works, Qu and Yin [QY22] and Szachniewicz [Sza23] independently relaxed the latter condition.

Theorem 3.16 ([Bal21], Theorem 1.1; [QY22], Theorem 1.4; [Sza23], Theorem 3.3.7)**.** *Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor. Then*

$$\mu^{\mathrm{ess}}(\overline{D}) \leq \sup\{t \in \mathbb{R} \mid \overline{D} - t[\infty] \textit{ is pseudo-effective}\},$$

with equality if D is big. In that case, we have

$$\mu^{\mathrm{ess}}(\overline{D}) = \sup\{t \in \mathbb{R} \mid \overline{D} - t[\infty] \textit{ is big}\} = \sup\{t \in \mathbb{R} \mid R^t(\overline{D}) \neq 0\}.$$

Proof. It follows from [Sza23, Lemma 3.3.5 and Theorem 3.3.7]. □

We have the following variants of Zhang's inequality [Zha95a, Theorem 5.2].

Theorem 3.17 (Zhang)**.** *Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor with D big. Then*

$$\mu^{\mathrm{ess}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}_{\chi}(\overline{D})}{(d+1)\,\mathrm{vol}(D)}.$$

Moreover, if $\overline{D}$ is big then

$$\mu^{\mathrm{ess}}(\overline{D}) \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(R^0(\overline{D}))} \geq \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(D)}.$$

Proof. In the case where D is ample and $\overline{D}$ is semipositive, the first statement is given by Zhang's inequality [Zha95a, Theorem 5.2]. It remains true in the above generality [Bal21, Theorem 7.2]. The second one follows from Theorem 3.13 and Lemma 3.15 (3), which imply

$$(d+1)\,\mathrm{vol}(R^0(\overline{D})) \times \mu^{\mathrm{ess}}(\overline{D}) \geq (d+1)\int_0^{\mu^{\mathrm{ess}}(\overline{D})} \mathrm{vol}(R^t(\overline{D}))dt = \widehat{\mathrm{vol}}(\overline{D}).$$

□

Under mild positivity assumptions, if equality holds in Zhang's inequality then the essential and absolute minima coincide.

Theorem 3.18 ([Bal24, Theorem 6.6])**.** *Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be semipositive with D big. Then*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}$$

if and only if $\mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{abs}}(\overline{D})$.

3.5. **Extension of linear functionals on $C(X_v^{\mathrm{an}})$.** Let $v \in \mathfrak{M}_K$.

Lemma 3.19. *Let $\Lambda\colon C(X_v^{\mathrm{an}})^{G_v} \to \mathbb{R}$ be a positive linear functional.*

(1) There exists a unique G_v-invariant positive linear functional $\widetilde{\Lambda}$ on $C(X_v^{\mathrm{an}})$ extending Λ.

(2) Let $(\nu_\ell)_\ell$ be a sequence of G_v-invariant probability measures on X_v^{an} such that

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi\, d\nu_\ell = \Lambda(\varphi) \quad \textit{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v}.$$

Then $(\nu_\ell)_\ell$ converges weakly to the G_v-invariant measure on X_v^{an} defined by $\widetilde{\Lambda}$.

Proof. (1) Let $V \subset C(X_v^{\mathrm{an}})$ be the subspace of functions $\varphi \in C(X_v^{\mathrm{an}})$ for which there exists a finite extension K_v' of K_v such that φ is $\mathrm{Gal}(\overline{K}_v/K_v')$-invariant. By [Yua08, page 638, "Equivalence"] (see also [GM22, Proposition 2.11 and Theorem 2.13]), V is dense in $C(X_v^{\mathrm{an}})$ for the supremum norm. Let K_v'/K_v be a finite extension, $\varphi \in C(X_v^{\mathrm{an}})$ a $\mathrm{Gal}(\overline{K}_v/K_v')$-invariant function, and

$$\widetilde{\varphi} = \frac{1}{\#\,\mathrm{Gal}(K_v'/K_v)} \sum_{\sigma\in\mathrm{Gal}(K_v'/K_v)} \sigma^*\varphi.$$

Then $\widetilde{\varphi} \in C(X_v^{\mathrm{an}})^{G_v}$, and we let $\widetilde{\Lambda}(\varphi) = \Lambda(\widetilde{\varphi})$. This defines a positive linear functional $\widetilde{\Lambda}\colon V \to \mathbb{R}$, that extends uniquely to a G_v-invariant positive linear functional on $C(X_v^{\mathrm{an}})$ by continuity. Let $\Lambda'\colon C(X_v^{\mathrm{an}}) \to \mathbb{R}$ be another G_v-invariant positive linear functional on $C(X_v^{\mathrm{an}})$ extending Λ. Then for every φ and $\widetilde{\varphi}$ as above we have $\Lambda'(\varphi) = \Lambda(\widetilde{\varphi}) = \widetilde{\Lambda}(\varphi)$ by G_v-invariance of Λ'. Therefore $\Lambda' = \widetilde{\Lambda}$ by density of V.

(2) For every $\varphi \in V$, we can define $\widetilde{\varphi} \in C(X_v^{\mathrm{an}})^{G_v}$ as above and we have

$$\int_{X_v^{\mathrm{an}}} \varphi \, d\nu_\ell = \int_{X_v^{\mathrm{an}}} \widetilde{\varphi} \, d\nu_\ell \xrightarrow[\ell\to\infty]{} \Lambda(\widetilde{\varphi}) = \widetilde{\Lambda}(\varphi)$$

by G_v-invariance of the measures ν_ℓ, $\ell \in \mathbb{N}$. Since V is dense, the statement follows from Weyl's criterion for weak convergence [GM22, Proposition 5.11]. □

Let $\varphi \in C(X_v^{\mathrm{an}})^{G_v}$. Note that φ is a v-adic Green function on the trivial divisor $0 \in \mathrm{Div}(X)$. We define an adelic divisor $\overline{0}^{\varphi}$ over 0 by equipping it with φ at v, and with the zero function at other places. The assignment $\varphi \mapsto \overline{0}^{\varphi}$ allows us to view $C^0(X_v^{\mathrm{an}})^{G_v}$ as a linear subspace of $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Let $\overline{D} \in \widehat{\mathrm{DSP}}(X)_{\mathbb{R}}$. It follows from the definition and the Bézout formula (3.2) that

$$(\overline{D}^d \cdot \overline{0}^{\varphi}) = n_v \int_{X_v^{\mathrm{an}}} \varphi \, c_1(\overline{D}_v)^{\wedge d}. \tag{3.5}$$

4. The differentiability of the arithmetic volume function

In this section we recall Chen's differentiability theorem [Che11] and we revisit its proof in order to express the derivative in terms of arithmetic Fujita approximations. This will allow us to give a simple definition for the positive arithmetic intersection numbers that will be relevant for our results.

4.1. **Differentiability of concave functions.** We first recall the definition of differentiability of functions on real vector spaces and its relation with concavity. Our ambient will be a real vector space V endowed with the topology defined by declaring that a subset $U \subset V$ is open whenever its restriction to any finite-dimensional affine subspace $H \subset V$ is open with respect to the Euclidean topology of H.

Definition 4.1. Let $\Phi \colon U \to \mathbb{R}$ be a function on an open subset $U \subset V$. For a point $x \in U$ and a subspace $W \subset V$, we say that Φ is *differentiable at x along W* if for every $z \in W$, the one-sided derivative

$$\partial_z \Phi(x) = \lim_{\lambda \to 0^+} \frac{\Phi(x + \lambda\, z) - \Phi(x)}{\lambda}$$

exists in $\mathbb{R}$, and the map $z \in W \to \partial_z \Phi(x) \in \mathbb{R}$ is linear. When $W = V$, we simply say that Φ is differentiable at x.

A real-valued function Φ on a subset $U \subset V$ is *concave* when U is convex and

$$\Phi(\lambda x + (1-\lambda) y) \geq \lambda \Phi(x) + (1-\lambda)\Phi(y) \quad \text{for all } x, y \in U \text{ and } 0 \leq \lambda \leq 1.$$

Lemma 4.2. *Let $\Phi \colon U \to \mathbb{R}$ be a concave function on an open convex subset of V, and let $x \in U$. Then*

(1) for all $z \in V$, the one-sided derivative $\partial_z \Phi(x)$ exists in $\mathbb{R}$ and satisfies that $\partial_z \Phi(x) \leq -\partial_{-z} \Phi(x)$,

(2) for a subspace $W \subset V$, the function Φ is differentiable at x along W if and only if $\partial_z \Phi(x) = -\partial_{-z} \Phi(x)$ for all $z \in W$.

Proof. For (1), let $a > 0$ be a real number sufficiently small so that $x + \lambda z \in U$ for all $-a < \lambda < a$. Let $\iota\colon (-a, a) \to V$ be the inclusion map defined by $\iota(\lambda) = x + \lambda z$, so that $\iota^*\Phi$ is a concave function on the interval $(-a, a)$. Then the functions $r_-, r_+\colon (0, a) \to \mathbb{R}$ respectively defined by

$$r_-(\lambda) = \frac{\iota^*\Phi(-\lambda) - \iota^*\Phi(0)}{-\lambda} \quad \text{and} \quad r_+(\lambda) = \frac{\iota^*\Phi(\lambda) - \iota^*\Phi(0)}{\lambda}$$

verify that r_- is non-decreasing, r_+ is non-increasing, and $r_-(\lambda) \geq r_+(\lambda)$ for all λ. Hence both converge when $\lambda \to 0^+$ and their limits verify that

$$-\partial_{-z}\Phi(x) = \lim_{\lambda \to 0^+} r_-(\lambda) \geq \lim_{\lambda \to 0^+} r_+(\lambda) = \partial_z\Phi(x).$$

For (2), it is clear that the differentiability of Φ at x along W implies that $\partial_z\Phi(x) = -\partial_{-z}\Phi(x)$ for all $z \in W$. Conversely, let $z_1, z_2 \in W$ and suppose that this condition holds for these two vectors. For $\lambda > 0$, we deduce from the concavity of Φ that

$$\frac{\Phi(x + \lambda(z_1 + z_2)) - \Phi(x)}{\lambda} \geq \frac{\Phi(x + 2\lambda z_1) - \Phi(x)}{2\lambda} + \frac{\Phi(x + 2\lambda z_2) - \Phi(x)}{2\lambda}.$$

Letting $\lambda \to 0^+$ we get $\partial_{z_1+z_2}\Phi(x) \geq \partial_{z_1}\Phi(x) + \partial_{z_2}\Phi(x)$. Applying this inequality to $-z_1, -z_2$ together with (1) we obtain that

$$-\partial_{-z_1}\Phi(x) - \partial_{-z_2}\Phi(x) \geq -\partial_{-z_1-z_2}\Phi(x) \geq \partial_{z_1+z_2}\Phi(x) \geq \partial_{z_1}\Phi(x) + \partial_{z_2}\Phi(x).$$

By assumption both extremes in this inequalities coincide, and so $\partial_{z_1+z_2}\Phi(x) = \partial_{z_1}\Phi(x) + \partial_{z_2}\Phi(x)$. Furthermore, the one-sided derivative is positive homogeneous of degree 1 and so it is linear, as stated. □

4.2. **Differentiability of arithmetic volumes.** Let $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$.

Definition 4.3. A *semipositive approximation* of $\overline{D}$ is a pair $(\phi, \overline{Q})$ where

(1) $\phi\colon X' \to X$ is a normal modification,
(2) $\overline{Q}$ is a semipositive adelic $\mathbb{R}$-divisor on X' with big geometric $\mathbb{R}$-divisor Q,
(3) $\phi^*\overline{D} - \overline{Q}$ is pseudo-effective.

If moreover $\overline{Q}$ is nef, we say that $(\phi, \overline{Q})$ is a *nef approximation*. We denote by $\Theta(\overline{D})$ the set of nef approximations. When ϕ is the identity on X, we simply denote by $\overline{Q}$ the semipositive approximation of $\overline{D}$ corresponding to the pair $(\mathrm{Id}_X, \overline{Q})$.

If $\overline{D}$ is big, the existence of arithmetic Fujita approximations (Theorem 3.12) implies that there exists a sequence $(\phi_n, \overline{P}_n)_n$ in $\Theta(\overline{D})$ such that

$$\lim_{n\to\infty} (\overline{P}_n^{d+1}) = \widehat{\mathrm{vol}}(\overline{D}). \tag{4.1}$$

The following is a slight variant of Chen's differentiability theorem [Che11].

Theorem 4.4. *Assume that $\overline{D}$ is big, and let $(\phi_n, \overline{P}_n)_n$ be a sequence of nef approximations of $\overline{D}$ satisfying* (4.1). *Then the arithmetic volume function is differentiable at $\overline{D}$, and for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have*

$$\partial_{\overline{E}}\,\widehat{\mathrm{vol}}(\overline{D}) = (d+1) \lim_{n\to\infty} (\overline{P}_n^{d} \cdot \phi_n^*\overline{E}).$$

In particular, the limit on the right-hand side exists in $\mathbb{R}$ and does not depend on the choice of the sequence $(\phi_n, \overline{P}_n)_n$.

We need the following consequence of Yuan's arithmetic version of Siu's inequality (Theorem 3.11), which already appears in [Che11] and [Iko15].

Lemma 4.5. *Let $\overline{P}$ and $\overline{E}$ be two adelic $\mathbb{R}$-divisors with $\overline{P}$ nef. Assume that there exists a nef and big adelic $\mathbb{R}$-divisor $\overline{A}$ such that $\overline{A}-\overline{P}$ is pseudo-effective and $\overline{A}\pm\overline{E}$ are nef. There exists a constant C_d depending only on d such that*

$$\forall\,\lambda\in[0,1],\quad \widehat{\mathrm{vol}}(\overline{P}+\lambda\overline{E})\geq(\overline{P}^{d+1})+(d+1)\lambda(\overline{P}^{d}\cdot\overline{E})-C_d\lambda^2\widehat{\mathrm{vol}}(\overline{A}).$$

Proof. When all non-Archimedean Green functions are induced by a model over $\operatorname{Spec}\mathcal{O}_K$, this is given by [Iko15, Proposition 5.1]. The general case follows by continuity. □

Proof of Theorem 4.4. Consider the function $\Phi=\widehat{\mathrm{vol}}^{\frac{1}{d+1}}$ on the big cone of $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. It is positive homogeneous of degree one and super-additive by the Brunn-Minkowski inequality [Mor16, Theorem 5.3.1]. Therefore it is concave. Noting that $\widehat{\mathrm{vol}}(\overline{D})>0$ and using Lemma 4.2, it follows that the one-sided derivative $\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})$ exists for every $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, and moreover $-\partial_{-\overline{E}}\widehat{\mathrm{vol}}(\overline{D})\geq\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})$.

Let $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. We claim that

$$\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})\geq(d+1)\limsup_{n\to\infty}(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}). \tag{4.2}$$

To prove this, we first assume that $\overline{E}$ is DSP. Then there exists a nef and big $\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{A}\pm\overline{E}$ are nef (Remark 3.10). Replacing $\overline{A}$ by a sufficiently large multiple, we assume that $\overline{A}-\overline{D}$ is pseudo-effective. This implies that $\phi_n^*\overline{A}-\overline{P}_n$ is pseudo-effective for every $n\in\mathbb{N}$. By Lemma 4.5 there exists a constant C such that

$$\widehat{\mathrm{vol}}(\overline{D}+\lambda\overline{E})\geq\widehat{\mathrm{vol}}(\overline{P}_n+\lambda\phi_n^*\overline{E})\geq(\overline{P}_n^{d+1})+(d+1)\lambda(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})-C\lambda^2$$

for every $\lambda\in(0,1]$ and $n\in\mathbb{N}$. Taking the lim sup on n and using the condition (4.1) gives

$$\frac{\widehat{\mathrm{vol}}(\overline{D}+\lambda\overline{E})-\widehat{\mathrm{vol}}(\overline{D})}{\lambda}\geq(d+1)\limsup_{n\to\infty}(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})-C\lambda,$$

and we obtain (4.2) by letting λ tend to zero.

For the general case, let $\varepsilon>0$. By Lemma 3.7 there exists a DSP $\overline{E}_\varepsilon\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{E}-\overline{E}_\varepsilon$ and $\overline{E}_\varepsilon+\varepsilon[\infty]-\overline{E}$ are both pseudo-effective. The first condition implies that $\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})\geq\partial_{\overline{E}_\varepsilon}\widehat{\mathrm{vol}}(\overline{D})$, and the second one together with Lemma 3.8 gives

$$(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}_\varepsilon)\geq(\overline{P}_n^{d}\cdot\phi_n^*(\overline{E}-\varepsilon[\infty]))=(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})-\varepsilon\,\mathrm{vol}(P_n)\geq(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})-\varepsilon\,\mathrm{vol}(D)$$

for every n (note that $(P_n^d)=\mathrm{vol}(P_n)$ since P_n is nef). By the DSP case we obtain

$$\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})\geq(d+1)(\limsup_{n\to\infty}(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})-\varepsilon\,\mathrm{vol}(D))$$

and (4.2) follows by letting ε tend to zero.

Applying (4.2) to $-\overline{E}$ we obtain

$$(d+1)\liminf_{n\to\infty}(\overline{P}_n^{d}\cdot\phi_n^*\overline{E})\geq-\partial_{-\overline{E}}\widehat{\mathrm{vol}}(\overline{D})\geq\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})\geq(d+1)\limsup_{n\to\infty}(\overline{P}_n^{d}\cdot\phi_n^*\overline{E}),$$

and we conclude with Lemma 4.2. □

Corollary 4.6. *Assume that $\overline{D}$ is big. For every sequence $(\phi_n, \overline{P}_n)_n$ of nef approximations of $\overline{D}$ satisfying* (4.1)*, we have*

$$\lim_{n\to\infty}(\overline{P}_n^d\cdot\overline{D}) = \widehat{\mathrm{vol}}(\overline{D}),\quad \lim_{n\to\infty}(P_n^d) = \mathrm{vol}(R^0(\overline{D})),\ \textit{and}\ \lim_{n\to\infty}\mu^{\mathrm{ess}}(\overline{P}_n) = \mu^{\mathrm{ess}}(\overline{D}).$$

Proof. This follows from Theorem 4.4 after noting that $\partial_{\overline{D}}\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\widehat{\mathrm{vol}}(\overline{D})$ by homogeneity, and that $\partial_{[\infty]}\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\mathrm{vol}(R^0(\overline{D}))$ thanks to Chen's Theorem 3.13. Note that $(\overline{P}_n^d\cdot[\infty]) = (P_n^d)$ for every $n\in\mathbb{N}$ by (3.3). For the last claim, assume by contradiction that there exists $c>0$ such that $\mu^{\mathrm{ess}}(\overline{P}_n)\le\mu^{\mathrm{ess}}(\overline{D})-c$ for an arbitrarily large n. Let

$$C = (d+1)\int_{\mu^{\mathrm{ess}}(\overline{D})-c}^{\mu^{\mathrm{ess}}(\overline{D})}\mathrm{vol}(R^t(\overline{D}))\,dt > 0.$$

By Chen's Theorem 3.13 we have

$$\begin{aligned}\widehat{\mathrm{vol}}(\overline{P}_n) = (d+1)\int_0^{\mu^{\mathrm{ess}}(\overline{P}_n)}\mathrm{vol}(R^t(\overline{P}_n))\,dt &\le (d+1)\int_0^{\mu^{\mathrm{ess}}(\overline{P}_n)}\mathrm{vol}(R^t(\overline{D}))\,dt\\ &\le \widehat{\mathrm{vol}}(\overline{D}) - C,\end{aligned}$$

which contradicts the condition (4.1). □

Corollary 4.7. *Assume that D is big. Let $t<\mu^{\mathrm{ess}}(\overline{D})$ and $\overline{D}(t) = \overline{D}-t[\infty]$. For every $\varepsilon>0$, there exists a semipositive approximation $(\phi,\overline{Q})$ of $\overline{D}$ such that $\mu^{\mathrm{abs}}(\overline{D})\ge t$,*

$$(Q^d)\ge\mathrm{vol}(R^t(\overline{D}))-\varepsilon,\quad \textit{and}\quad (\overline{Q}^{d+1})\ge\widehat{\mathrm{vol}}(\overline{D}(t)) + (d+1)t\,\mathrm{vol}(R^t(\overline{D}))-\varepsilon.$$

Proof. By Theorem 3.16, $\overline{D}(t)$ is big. By Corollary 4.6 applied to $\overline{D}(t)$, there exists a nef approximation $(\phi,\overline{P})$ of $\overline{D}(t)$ such that

$$(\overline{P}^{d+1})\ge\widehat{\mathrm{vol}}(\overline{D}(t))-\varepsilon\quad\text{and}\quad(P^d)\ge\mathrm{vol}(R^0(\overline{D}(t)))-\varepsilon = \mathrm{vol}(R^t(\overline{D}))-\varepsilon.$$

Let $\overline{Q} = \overline{P}+t[\infty]$. By construction, $(\phi,\overline{Q})$ is a semipositive approximation of $\overline{D}$. Since $\overline{P}$ is nef we have $\mu^{\mathrm{abs}}(\overline{Q}) = \mu^{\mathrm{abs}}(\overline{P})+t\ge t$, and $(Q^d)\ge\mathrm{vol}(R^t(\overline{D}))-\varepsilon$ since $Q=P$. Moreover, by (3.3) we have

$$\begin{aligned}(\overline{Q}^{d+1}) = (\overline{P}^{d+1}) + t(d+1)(Q^d) &\ge \mathrm{vol}(\overline{D}(t)) + t(d+1)(Q^d)-\varepsilon\\ &\ge \mathrm{vol}(\overline{D}(t)) + t(d+1)\,\mathrm{vol}(R^t(\overline{D}))-\varepsilon(1+(d+1))t.\end{aligned}$$

□

Assume that $\overline{D}$ is big. For every $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we define the *positive arithmetic intersection number* $(\langle\overline{D}^d\rangle\cdot\overline{E})$ as

$$(\langle\overline{D}^d\rangle\cdot\overline{E}) = \lim_{n\to\infty}(\overline{P}_n^d\cdot\phi_n^*\overline{E}), \tag{4.3}$$

where $(\phi_n,\overline{P}_n)_n$ is any sequence of nef approximations of $\overline{D}$ satisfying (4.1). This is well-defined and does not depend on the choice of the sequence by Theorem 4.4.

Remark 4.8. Since the quantity $(d+1)(\langle\overline{D}^d\rangle\cdot\overline{E})$ coincides with the derivative $\partial_{\overline{E}}\widehat{\mathrm{vol}}(\overline{D})$, our definition coincides with the positive intersection numbers introduced by Chen [Che11] and further studied by Ikoma [Iko15]. In particular, when $\overline{E}$ is nef we have

$$(\langle\overline{D}^d\rangle\cdot\overline{E})=\sup_{(\phi,\overline{P})\in\Theta(\overline{D})}(\overline{P}^d\cdot\phi^*\overline{E}).$$

Although [Che11] and [Iko15] only deal with adelic $\mathbb{R}$-divisors whose non-Archimedean Green functions are induced by an integral model, this equality remains valid in our setting by continuity ([Che11, Proposition 3.6], [Iko15, Proposition 3.10]). We mention that more generally, positive intersection numbers of the form $(\langle\overline{D}^{d+1-i}\rangle\cdot\overline{E}^i)$ where defined in [Che11], and Ikoma [Iko15, Proposition 4.4] showed that they can be computed as limits using some specific sequences of nef approximations. It would be interesting to check whether this latter result remains true for any choice of sequence of nef approximations satisfying (4.1).

Remark 4.9. One can similarly adapt the proof of [BFJ09, Theorem A] to establish the differentiability of the geometric volume function. This shows that for any $D,E\in\mathrm{Div}(X)_\mathbb{R}$ with D big, the geometric positive intersection number $(\langle D^{d-1}\rangle\cdot E)$ introduced in [BFJ09] can be expressed as

$$(\langle D^{d-1}\rangle\cdot E)=\lim_{n\to\infty}(P_n^{d-1}\cdot\phi_n^*E)$$

where $(\phi_n,P_n)_n$ is any sequence such that P_n is a nef $\mathbb{R}$-divisor on a birational modification $\phi_n\colon X_n\to X$ with $\phi_n^*D-P_n$ pseudo-effective and $\lim_{n\to\infty}(P_n^d)=\mathrm{vol}(D)$.

Assume that $\overline{D}$ is big. The positive arithmetic intersections numbers introduced above allow us to define the linear form

$$\Omega_{\overline{D}}\colon\widehat{\mathrm{Div}}(X)_\mathbb{R}\longrightarrow\mathbb{R},\quad \overline{E}\longmapsto\frac{(\langle\overline{D}^d\rangle\cdot\overline{E})}{\mathrm{vol}(R^0(\overline{D}))}.$$

Let $(\phi_n,\overline{P}_n)_n$ be a sequence of nef approximations of $\overline{D}$ satisfying (4.1). By definition of the positive arithmetic intersection numbers and by Corollary 4.6, we have

$$\Omega_{\overline{D}}(\overline{E})=\lim_{n\to\infty}\frac{(\overline{P}_n^d\cdot\phi_n^*\overline{E})}{(P_n^d)}\quad\text{for every }\overline{E}\in\widehat{\mathrm{Div}}(X)_\mathbb{R}.\tag{4.4}$$

Remark 4.10. By (3.3) and Lemma 3.8, we have $\Omega_{\overline{D}}([\infty])=1$ and $\Omega_{\overline{D}}(\overline{E})\geq 0$ for every pseudo-effective $\overline{E}\in\widehat{\mathrm{Div}}(X)_\mathbb{R}$. Moreover, $\Omega_{\overline{D}}(\widehat{\mathrm{div}}(f))=0$ for any $f\in\mathrm{Rat}(X)_\mathbb{R}$. It follows that $\Omega_{\overline{D}}$ defines a GVF functional in the sense of [Sza23].

Let $v\in\mathfrak{M}_K$. Recall that we view $C(X_v^{\mathrm{an}})^{G_v}$ as a linear subspace of $\widehat{\mathrm{Div}}(X_v)_\mathbb{R}$ via the assignment $\varphi\mapsto\overline{0}^\varphi$. By Lemma 3.19, the restriction of $\Omega_{\overline{D}}$ to $C(X_v^{\mathrm{an}})^{G_v}$ extends uniquely to a G_v-invariant positive linear functional on $C(X_v^{\mathrm{an}})$. We denote by $\omega_{\overline{D},v}$ the unique G_v-invariant measure on X_v^{an} satisfying

$$n_v\int_{X_v^{\mathrm{an}}}\varphi\,d\omega_{\overline{D},v}=\frac{(\langle\overline{D}^d\rangle\cdot\overline{0}^\varphi)}{\mathrm{vol}(R^0(\overline{D}))}\quad\text{for every }\varphi\in C(X_v^{\mathrm{an}})^{G_v}.$$

This is a probability measure: indeed, by equality (4.4) we have

$$\int_{X_v^{\mathrm{an}}}d\omega_{\overline{D},v}=\lim_{n\to\infty}\frac{1}{(P_n^d)}\int_{X_{n,v}^{\mathrm{an}}}c_1(\overline{P}_{n,v})^{\wedge d}=1.$$

5. The variational approach to the equidistribution problem

In this section we relate the differentiability of the essential minimum function to the equidistribution problem. Recall that the essential minimum function on $\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ takes finite values on the open cone $C \subset \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ of adelic $\mathbb{R}$-divisors with big geometric $\mathbb{R}$-divisor. Moreover, it is positive homogeneous of degree one and super-additive by Lemma 3.15. Therefore it is concave on C.

Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor with big geometric $\mathbb{R}$-divisor D. We say that the essential minimum function is differentiable at $\overline{D}$ if so is its restriction to C. By Lemma 4.2, the one-sided derivative $\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$ exists in $\mathbb{R}$ for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and we have $-\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$.

5.1. **Generic small sequences and differentiability.**

Definition 5.1. We say that a sequence $(x_\ell)_\ell$ in $X(\overline{K})$ is *generic* if for every closed subset $Y \subsetneq X$, there exists $\ell_0 \in \mathbb{N}$ such that $x_\ell \notin Y(\overline{K})$ for every $\ell \geq \ell_0$. A generic sequence is called *$\overline{D}$-small* if

$$\lim_{\ell\to\infty} h_{\overline{D}}(x_\ell) = \mu^{\mathrm{ess}}(\overline{D}).$$

In the sequel, all the generic sequences considered lie in $X(\overline{K})$, unless otherwise explicitly stated. By [BPRS19, Proposition 3.2], for every such generic sequence $(x_\ell)_\ell$ we have that $\liminf_{\ell\to\infty} h_{\overline{D}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D})$, and moreover there exist generic sequences that are $\overline{D}$-small.

When one considers heights with respect to another adelic $\mathbb{R}$-divisor $\overline{E}$, the same conclusion holds if $\mu^{\mathrm{ess}}(\overline{D})$ is replaced by $\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$.

Proposition 5.2. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. For every $\overline{D}$-small generic sequence $(x_\ell)_\ell$, we have*

$$-\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}).$$

Moreover, there exists a $\overline{D}$-small generic sequence $(x_\ell)_\ell$ such that

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}).$$

Proof. Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence. Then for every $\lambda > 0$ we have that $\liminf_{\ell\to\infty} h_{\overline{D}+\lambda\overline{E}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E})$, hence

$$\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \liminf_{\ell\to\infty} \frac{h_{\overline{D}+\lambda\overline{E}}(x_\ell) - h_{\overline{D}}(x_\ell)}{\lambda} \geq \frac{\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda}.$$

Therefore $\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$, and we obtain the first statement by applying this to $-\overline{E}$. We now prove the second one. By homogeneity, after multiplying $\overline{E}$ by a positive constant we assume that that $D+E$ is big without loss of generality. For every $n \in \mathbb{N}_{>0}$, there exists a generic sequence $(x_{n,\ell})_\ell$ such that

$$\lim_{\ell\to\infty} h_{\overline{D}+\frac{1}{n}\overline{E}}(x_{n,\ell}) = \mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}). \tag{5.1}$$

Denote by $\mathcal{H} = \{H_i\}_{i\in\mathbb{N}}$ the countable set of hypersurfaces in X. Let $n \geq 2$ be an integer. By (5.1) and genericity of $(x_{n,\ell})_\ell$, there exists $\ell(n) \in \mathbb{N}$ such that the point $\xi_n = x_{n,\ell(n)} \in X(\overline{K})$ satisfies

- $\xi_n \notin \cup_{i=1}^n H_i$,

- $h_{\overline{D}+\frac{1}{n}\overline{E}}(\xi_n) < \mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}) + \frac{1}{n^2}$, and
- $h_{\overline{D}}(\xi_n) \geq \mu^{\mathrm{ess}}(\overline{D}) - 1/n^2$.

By the first point the sequence $(\xi_n)_n$ is generic. Since $D+E$ is big, this implies that there exists a real number $C \in \mathbb{R}$ such that $h_{\overline{D}+\overline{E}}(\xi_n) \geq C$ for $n \in \mathbb{N}$ sufficiently large by the lower bound in (3.4). It follows that

$$\begin{aligned} h_{\overline{D}}(\xi_n) &= \frac{n}{n-1} \times \Big(h_{\overline{D}+\frac{1}{n}\overline{E}}(\xi_n) - \frac{1}{n} h_{\overline{D}+\overline{E}}(\xi_n) \Big) \\ &\leq \frac{n}{n-1} \mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}) + \frac{1}{n(n-1)} - \frac{C}{n-1}, \end{aligned}$$

and therefore the sequence $(\xi_n)_n$ is also $\overline{D}$-small by the continuity of the essential minimum (Lemma 3.15(2)). On the other hand the conditions for ξ_n give

$$\begin{aligned} -\frac{1}{n^2} + \frac{1}{n} h_{\overline{E}}(\xi_n) \leq h_{\overline{D}}(\xi_n) + \frac{1}{n} h_{\overline{E}}(\xi_n) - \mu^{\mathrm{ess}}(\overline{D}) &= h_{\overline{D}+\frac{1}{n}\overline{E}}(\xi_n) - \mu^{\mathrm{ess}}(\overline{D}) \\ &\leq \mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}) - \mu^{\mathrm{ess}}(\overline{D}) + \frac{1}{n^2}. \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} h_{\overline{E}}(\xi_n) \leq \lim_{n\to\infty} n(\mu^{\mathrm{ess}}(\overline{D}+\frac{1}{n}\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}),$$

and we conclude that $\lim_{n\to\infty} h_{\overline{E}}(\xi_n) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ by using the first statement of the proposition. □

Proposition 5.3. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. The following conditions are equivalent:*

(1) $\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{E}) = -\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{E})$,

(2) for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$, the limit $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell)$ exists.

If they are satisfied, then $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$.

Proof. The implication (1) ⇒ (2) is given by the first part of Proposition 5.2. Conversely, assume that (2) holds. Then we have $\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$. Indeed, if the limit were different then we could construct a $\overline{D}$-small generic sequence $(x'_\ell)_\ell$ such that $(h_{\overline{E}}(x'_\ell))_\ell$ does not converge by the second part of Proposition 5.2. Moreover, applying the latter to $-\overline{E}$ gives a $\overline{D}$-small generic sequence $(x_\ell)_\ell$ satisfying

$$\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{\ell\to\infty} h_{-\overline{E}}(x_\ell) = -\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = -\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$$

□

Corollary 5.4. *The essential minimum function is differentiable at $\overline{D}$ if and only if for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ we have that*

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

5.2. **Equidistribution property and differentiability.** Let $v \in \mathfrak{M}_K$. For every $x \in X(\overline{K})$ we define a probability measure $\delta_{O(x)_v}$ on X_v^{an} by

$$\delta_{O(x)_v} = \frac{1}{\# O(x)_v} \sum_{y \in O(x)_v} \delta_y.$$

Definition 5.5. We say that $\overline{D}$ satisfies the *equidistribution property* at v if there exists a probability measure $\nu_{\overline{D},v}$ on X_v^{an} with the following property: for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$, the sequence of measures $(\delta_{O(x_\ell)_v})_\ell$ converges weakly to $\nu_{\overline{D},v}$. When this holds, we call $\nu_{\overline{D},v}$ the *(v-adic) equidistribution measure* of $\overline{D}$.

Note that for every $x \in X(\overline{K})$, $O(x)_v$ is G_v-invariant and thus $\delta_{O(x)_v}$ is a G_v-invariant measure. Therefore, if $\overline{D}$ satisfies the equidistribution property at v then the equidistribution measure $\nu_{\overline{D},v}$ is G_v-invariant as a limit of G_v-invariant measures.

For every $\varphi \in C(X_v^{\mathrm{an}})^{G_v}$ and $x \in X(\overline{K})$ we have

$$h_{\overline{0}^{\varphi}}(x) = \frac{n_v}{\# O(x)_v} \sum_{y \in O(x)_v} \varphi(y) = n_v \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x)_v}.$$

Therefore we have the following by combining Lemma 3.19 and Proposition 5.3.

Proposition 5.6. *The following conditions are equivalent:*
(1) $\overline{D}$ satisfies the equidistribution property at v,
(2) the essential minimum function is differentiable at $\overline{D}$ along $C(X_v^{\mathrm{an}})^{G_v}$.
If they are satisfied, then the equidistribution measure $\nu_{\overline{D},v}$ is the unique G_v-invariant measure on X_v^{an} such that

$$n_v \int_{X_v^{\mathrm{an}}} \varphi \, d\nu_{\overline{D},v} = \partial_{\overline{0}^{\varphi}} \mu^{\mathrm{ess}}(\overline{D}) \quad \text{for all } \varphi \in C(X_v^{\mathrm{an}})^{G_v}.$$

6. Differentiability of the essential minimum and first applications

In §6.1 we state the central result of this paper, that shows the differentiability of the essential minimum under the assumption of Theorem 1.2. The proof will be given in the next section. We explain how it implies Yuan's and Chen's equidistribution theorems [Yua08, Che11] in §6.2, and we reformulate our result using arithmetic positive intersection numbers in §6.3.

Throughout this section, $\overline{D}$ is an adelic $\mathbb{R}$-divisor on X over a big $\mathbb{R}$-divisor D.

6.1. **Main theorem.** The following result strengthens Theorem 1.2.

Theorem 6.1. *Assume that there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that*

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} = 0. \tag{6.1}$$

Then the essential minimum function is differentiable at $\overline{D}$, and for each $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have

$$\partial_{\overline{E}} \mu^{\mathrm{ess}}(\overline{D}) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E}) - d\, \mu^{\mathrm{ess}}(\overline{D}) \times (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)}.$$

In particular, the limit on the right-hand side exists in $\mathbb{R}$ and does not depend on the choice of the sequence $(\phi_n, \overline{Q}_n)_n$.

Under the assumption of the theorem, it follows from Proposition 5.3 that

$$\lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{n\to\infty} \frac{(\overline{Q}_n^d \cdot \phi_n^*\overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D}) \times (Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)}.$$

for any $\overline{D}$-small generic sequence $(x_\ell)_\ell$ and any $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. In particular, $\overline{D}$ satisfies the equidistribution property at each place $v \in \mathfrak{M}_K$ and the v-adic equidistribution measure is the weak limit $\nu_{\overline{D},v} = \lim_{n\to\infty} \nu_{n,v}$, where $\nu_{n,v}$ denotes the pushforward to X_v^{an} of the normalized Monge-Ampère measure $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$, as claimed in Theorem 1.2.

Remark 6.2. Recall that we assume X to be normal, since we did not define adelic $\mathbb{R}$-divisors on an arbitrary projective variety. However, in the case where $\overline{D} \in \widehat{\mathrm{Div}}(X)$ is an adelic divisor Theorem 6.1 remains valid on any projective variety over K. Indeed, since all the data involved are invariant under birational modifications, one can reduce to the normal case by working on the normalization.

Remark 6.3. The inradii $r(Q_n, \phi_n^* D)$ appearing in Theorem 6.1 measure the bigness of the geometric $\mathbb{R}$-divisors Q_n, $n \in \mathbb{N}$. For our purposes, these invariants are finer than the geometric volumes $\mathrm{vol}(Q_n) = (Q_n^d)$ in the sense that any sequence of semipositive approximations $(\phi_n, \overline{Q}_n)$ of $\overline{D}$ such that

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{(Q_n^d)} = 0$$

satisfies (6.1). In fact, for any ample $A \in \mathrm{Div}(X)_{\mathbb{R}}$ such that $A - D$ is pseudo-effective we have that $\phi_n^* A - Q_n$ is pseudo-effective for every n, and therefore $r(Q_n, \phi^* nD) \geq (Q_n^d) \times (d(A^d))^{-1}$ by Lemma 2.3.

By Corollary 4.7, one can always construct a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that $\mu^{\mathrm{abs}}(\overline{Q}_n)$ converges to $\mu^{\mathrm{ess}}(\overline{D})$. However, for such sequences the inradii $r(Q_n, \phi_n^* D)$ can be very "small", and the condition (6.1) is not necessarily satisfied.

6.2. **Application to classical equidistribution results.** As explained in the introduction, Yuan's Theorem 1.1 is a straightforward consequence of Theorem 6.1, which also gives the differentiability of the essential minimum in this setting. This slight improvement can already be obtained by adapting Yuan's proof, as observed by Chambert-Loir and Thuillier [CLT09, Lemma 6.1]. For completeness we state the result here, which is valid for an adelic $\mathbb{R}$-divisor with big geometric $\mathbb{R}$-divisor.

Corollary 6.4. *Assume that $\overline{D}$ is semipositive and extremal for Zhang's inequality, that is $\mu^{\mathrm{ess}}(\overline{D}) = (\overline{D}^{d+1})/((d+1)(D^d))$. Then the essential minimum function is differentiable at $\overline{D}$, and for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^d \cdot \overline{E}) - d\,\mu^{\mathrm{ess}}(\overline{D}) \times (D^{d-1}\cdot E)}{(D^d)}.$$

In particular, $\overline{D}$ satisfies the v-adic equidistribution property at each $v \in \mathfrak{M}_K$, with equidistribution measure $\nu_v(\overline{D}) = c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

Proof. Apply Theorem 6.1 to the sequence of semipositive approximations given by $\overline{Q}_n = \overline{D}$, $n \geq 1$. By Theorem 3.18 we have $\mu^{\mathrm{abs}}(\overline{D}) = \mu^{\mathrm{ess}}(\overline{D})$ and so the condition in (6.1) is verified. ☐

Similarly, Theorem 6.1 implies Chen's equidistribution theorem [Che11, Corollary 5.5].

Corollary 6.5. *Assume that $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ is big and that*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(D)}.$$

Then the essential minimum function is differentiable at $\overline{D}$, and

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{(\langle \overline{D}^{d} \rangle \cdot \overline{E}) - d\, \mu^{\mathrm{ess}}(\overline{D}) \times (\langle D^{d-1} \rangle \cdot E)}{\mathrm{vol}(D)}$$

for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. In particular, $\overline{D}$ satisfies the v-adic equidistribution property at each $v \in \mathfrak{M}_K$, with equidistribution measure $\nu_v(\overline{D}) = \omega_{\overline{D},v}$.

In the above statement, $\omega_{\overline{D},v}$ is the measure defined in §4.2 and $(\langle D^{d-1} \rangle \cdot E)$ is the geometric positive intersection number defined in [BFJ09] (see Remark 4.9).

Proof. By Chen's Theorem 3.13 and Lemma 3.15(3), we have

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{\widehat{\mathrm{vol}}(\overline{D})}{(d+1)\,\mathrm{vol}(D)} = \frac{1}{\mathrm{vol}(D)} \int_0^{\mu^{\mathrm{ess}}(\overline{D})} \mathrm{vol}(R^t(\overline{D}))\, dt.$$

Therefore $\mathrm{vol}(R^t(\overline{D})) = \mathrm{vol}(D)$ for every $t \in (0, \mu^{\mathrm{ess}}(\overline{D}))$. Letting $\overline{D}(t) = \overline{D} - t[\infty]$, we also have

$$\widehat{\mathrm{vol}}(\overline{D}(t)) = \widehat{\mathrm{vol}}(\overline{D}) - (d+1)t\,\mathrm{vol}(D)$$

by Theorem 3.13. Using Corollary 4.7, it follows that there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that

$$\lim_{n\to\infty} (\overline{Q}_n^{d+1}) = \widehat{\mathrm{vol}}(\overline{D}), \quad \lim_{n\to\infty} (Q_n^{d}) = \mathrm{vol}(D), \quad \text{and} \quad \lim_{n\to\infty} \mu^{\mathrm{abs}}(\overline{Q}_n) = \mu^{\mathrm{ess}}(\overline{D}).$$

By Remark 6.3, $(\phi_n, \overline{Q}_n)$ satisfies condition (6.1). Moreover, since $\mu^{\mathrm{ess}}(\overline{D}) > 0$, $\overline{Q}_n$ is nef for sufficiently large n and therefore $(\phi_n, \overline{Q}_n)_n$ is a sequence of nef approximations. By definition of arithmetic positive intersection numbers and by Remark 4.9, we have

$$\lim_{n\to\infty} (\overline{Q}_n^{d} \cdot \phi_n^* \overline{E}) = (\langle \overline{D}^{d} \rangle \cdot \overline{E}) \quad \text{and} \quad \lim_{n\to\infty} (Q_n^{d-1} \cdot \phi_n^* E) = (\langle D^{d-1} \rangle \cdot E).$$

Hence the result follows from Theorem 6.1. □

6.3. **Interpretation in terms of arithmetic positive intersection numbers.** Here we propose a reformulation of Theorem 6.1 that gives a more intrinsic sufficient condition for the differentiability of the essential minimum function, and shows that the derivative can be computed using limits of positive arithmetic intersection numbers defined in §4.

We define the inradius of a big adelic $\mathbb{R}$-divisor as the supremum of the geometric inradii of its nef approximations. $\overline{B}$.

Definition 6.6. Let $\overline{B} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be a big adelic $\mathbb{R}$-divisor. Let $A \in \mathrm{Div}(X)_{\mathbb{R}}$ be a big $\mathbb{R}$-divisor. The *inradius* of $\overline{B}$ with respect to A is defined as

$$\rho(\overline{B}; A) = \sup\{ r(P; \phi^* A) \mid (\phi, \overline{P}) \in \Theta(\overline{B}) \}.$$

We also set $\rho(\overline{B}) = \rho(\overline{B}; B)$ for the inradius of $\overline{B}$ with respect to its geometric $\mathbb{R}$-divisor B, which is also big.

The supremum in Definition 6.6 can be taken over the pairs $(\phi, \overline{P}) \in \Theta(\overline{B})$ with P big, and $\rho(\overline{B}; A) > 0$.

For every real number $t < \mu^{\mathrm{ess}}(\overline{D})$, we let $\overline{D}(t) = \overline{D} - t[\infty] \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Note that $\overline{D}(t)$ is big by Theorem 3.16. Recall that we defined a linear form

$$\Omega_{\overline{D}(t)} \colon \widehat{\mathrm{Div}}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad \overline{E} \longmapsto \frac{(\langle \overline{D}(t)^d \rangle \cdot \overline{E})}{\mathrm{vol}(R^t(\overline{D}))}.$$

in §4.2. In the next result and similar ones, the limits when t tends to the essential minimum are taken *from below*, so that in particular the inradii $\rho(\overline{D}(t))$ are well-defined.

Theorem 6.7. *Assume that*

$$\liminf_{t \to \mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t}{\rho(\overline{D}(t))} = 0. \tag{6.2}$$

Then the essential minimum function is differentiable at $\overline{D}$, and

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{t \to \mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \tag{6.3}$$

for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$.

We first observe that the conditions in Theorems 6.1 and 6.7 are equivalent.

Lemma 6.8. *The condition* (6.2) *holds if and only if there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ satisfying* (6.1).

Proof. Assume that (6.2) holds. Then there exist a sequence $(t_n)_n$ of real numbers in $(-\infty, \mu^{\mathrm{ess}}(\overline{D}))$ and a sequence $(\phi_n, \overline{P}_n)_n$ where $(\phi_n, \overline{P}_n) \in \Theta(\overline{D}(t_n))$ for every $n \in \mathbb{N}$, such that

$$\lim_{n \to \infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t_n}{r(P_n, \phi_n^* D)} = 0.$$

We let $n \in \mathbb{N}$ and we put $\overline{Q}_n = \overline{P}_n + t_n[\infty]$. Then $(\phi_n, \overline{Q}_n)$ is a semipositive approximation for $\overline{D}$ with $Q_n = P_n$ and $\mu^{\mathrm{abs}}(\overline{Q}_n) = \mu^{\mathrm{abs}}(\overline{P}_n) + t_n \geq t_n$. Therefore $(\phi_n, \overline{Q}_n)_n$ satisfies (6.1).

Conversely, let $(\phi_n, \overline{Q}_n)_n$ be a sequence of semipositive approximations of $\overline{D}$ satisfying (6.1). For every n we let $t_n = \mu^{\mathrm{abs}}(\overline{Q}_n)$. Then $\overline{P}_n \coloneqq \overline{Q}_n - t_n[\infty]$ is nef, since it is semipositive and $\mu^{\mathrm{abs}}(\overline{P}_n) = \mu^{\mathrm{abs}}(\overline{Q}_n) - t_n = 0$. Moreover,

$$\phi_n^* \overline{D}(t_n) - \overline{P}_n = \phi_n^* \overline{D} - t_n[\infty] - (\overline{Q}_n - t_n[\infty]) = \phi_n^* \overline{D} - \overline{Q}_n$$

is pseudo-effective since $(\phi_n, \overline{Q}_n)$ is a semipositive approximation. Therefore $(\phi_n, \overline{P}_n) \in \Theta(\overline{D}(t_n))$, and in particular

$$\rho(\overline{D}(t_n)) \geq r(P_n; \phi_n^* D) = r(Q_n; \phi_n^* D).$$

Finally, we have

$$0 \leq \liminf_{t \to \mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t}{\rho(\overline{D}(t))} \leq \liminf_{n \to \infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - t_n}{\rho(\overline{D}(t_n))} \leq \lim_{n \to \infty} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} = 0.$$

□

Lemma 6.9. *For every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we have*

$$-\partial_{-\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}). \tag{6.4}$$

In particular, if the essential minimum function is differentiable at $\overline{D}$ then the limit $\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E})$ exists and equals $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$.

Proof. Let $t < \mu^{\mathrm{ess}}(\overline{D})$ and $\lambda > 0$. By Theorem 3.16, $\overline{D} + \lambda\overline{E} - \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E})[\infty]$ is pseudo-effective. The linear form $\Omega_{\overline{D}(t)}$ takes non-negative values on pseudo-effective adelic $\mathbb{R}$-divisors and verifies $\Omega_{\overline{D}(t)}([\infty]) = 1$. Therefore

$$\Omega_{\overline{D}(t)}(\overline{D}) + \lambda\Omega_{\overline{D}(t)}(\overline{E}) \geq \mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}). \tag{6.5}$$

On the other hand

$$\Omega_{\overline{D}(t)}(\overline{D}) = \Omega_{\overline{D}(t)}(\overline{D}(t)) + t = \frac{\widehat{\mathrm{vol}}(\overline{D}(t))}{\mathrm{vol}(R^t(\overline{D}))} + t \leq (d+1)(\mu^{\mathrm{ess}}(\overline{D}) - t) + t$$

by Corollary 4.6 and Zhang's inequality (Theorem 3.17), and therefore

$$\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{D}) = \mu^{\mathrm{ess}}(\overline{D}).$$

Taking the infimum limit on $t < \mu^{\mathrm{ess}}(\overline{D})$ in (6.5) gives

$$\liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \frac{\mu^{\mathrm{ess}}(\overline{D} + \lambda\overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda},$$

and we obtain $\liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) \geq \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ by letting λ tend to zero. Applying this to $-\overline{E}$ gives (6.4), and the last assertion follows from Lemma 4.2. □

Remark 6.10. We do not know whether it can happen that

$$\limsup_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E}) > \liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \Omega_{\overline{D}(t)}(\overline{E})$$

when the essential minimum function is not differentiable at $\overline{D}$.

Proof of Theorem 6.7. If (6.2) holds, then by Lemma 6.8 and Theorem 6.1 the essential minimum function is differentiable at $\overline{D}$. The expression for the derivative in (6.3) is given by Lemma 6.9. □

Remark 6.11. Let $v \in \mathfrak{M}_K$. Recall from §4.2 that for every $t < \mu^{\mathrm{ess}}(\overline{D})$, the restriction of $\Omega_{\overline{D}(t)}$ to $C(X_v^{\mathrm{an}})^{G_v}$ defines a probability measure $\omega_{\overline{D}(t),v}$ on X_v^{an}. Lemma 6.9 together with Proposition 5.6 imply that if $\overline{D}$ satisfies the equidistribution property at v, then the equidistribution measure $\nu_{\overline{D},v}$ is the weak limit of the measures $\omega_{\overline{D}(t),v}$ as t tends to $\mu^{\mathrm{ess}}(\overline{D})$: we have

$$\int_{X_v^{\mathrm{an}}} \varphi\, d\nu_{\overline{D},v} = \lim_{t\to\mu^{\mathrm{ess}}(\overline{D})} \int_{X_v^{\mathrm{an}}} \varphi\, d\omega_{\overline{D}(t),v} \quad \text{for all } \varphi \in C(X_v^{\mathrm{an}}).$$

It would be very interesting to know whether the condition in Theorem 6.7 is actually a criterion for the differentiability of the essential minimum.

Question 6.12. If the essential minimum function is differentiable at $\overline{D}$, then does

$$\liminf_{t\to\mu^{\mathrm{ess}}(\overline{D})} \frac{\mu^{\mathrm{ess}}(\overline{D})-t}{\rho(\overline{D}(t))}=0$$

necessarily hold? More optimistically, does it hold as soon as $\overline{D}$ has the equidistribution property at every place?

The next result answers this question under an additional technical assumption, roughly saying that $\overline{D}$ has a suitable upper bound for which Zhang's inequality is an equality. As we shall see in Proposition 9.4, this assumption is always satisfied for semipositive toric adelic $\mathbb{R}$-divisors. This will allow us to give an affirmative answer to Question 6.12 in this case, see Corollary 9.12.

Proposition 6.13. *Assume that there exists a semipositive adelic $\mathbb{R}$-divisor $\overline{B}$ such that $\overline{B}-\overline{D}$ is pseudo-effective and $\mu^{\mathrm{ess}}(\overline{B})=\mu^{\mathrm{abs}}(\overline{B})=\mu^{\mathrm{ess}}(\overline{D})$. Consider the following three conditions:*

(1) $\lim_{t\to\mu^{\mathrm{ess}}(\overline{D})}(\mu^{\mathrm{ess}}(\overline{D})-t)\times\rho(\overline{D}(t))^{-1}=0,$

(2) the essential minimum function is differentiable at $\overline{D}$,

(3) $\partial_{-\overline{B}}\,\mu^{\mathrm{ess}}(\overline{D})=-\partial_{\overline{B}}\,\mu^{\mathrm{ess}}(\overline{D})$,

(4) $\overline{D}$ has the equidistribution property at every place.

Then (1) ⇔ (2) ⇔ (3)*. If moreover $\overline{B}$ is over D, then* (1) ⇔ (4)*.*

Proof. We first note that (1) ⇒ (2) ⇒ (4) by Theorem 6.7 and Proposition 5.6, and that (2) trivially implies (3). Since $\overline{B}-\overline{D}$ is pseudo-effective, so is $B-D$ and therefore B is big. For every $\lambda\geq 0$ we have

$$(1+\lambda)\mu^{\mathrm{ess}}(\overline{D})=(1+\lambda)\mu^{\mathrm{ess}}(\overline{B})=\mu^{\mathrm{ess}}(\overline{B}+\lambda\overline{B})\geq\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{B})\geq(1+\lambda)\mu^{\mathrm{ess}}(\overline{D})$$

by Lemma 3.15(4), hence $\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{B})=(1+\lambda)\mu^{\mathrm{ess}}(\overline{D})$ and $\partial_{\overline{B}}\,\mu^{\mathrm{ess}}(\overline{D})=\mu^{\mathrm{ess}}(\overline{D})$. After replacing $\overline{D}$ by $\overline{D}-\mu^{\mathrm{ess}}(\overline{D})[\infty]$, we assume that $\mu^{\mathrm{ess}}(\overline{D})=0$ without loss of generality. This implies that $\overline{B}$ is nef, since it is semipositive and $\mu^{\mathrm{abs}}(\overline{B})=0$.

Assume that (3) holds. By Lemma 6.9 we have

$$0=\mu^{\mathrm{ess}}(\overline{D})=\partial_{\overline{B}}\,\mu^{\mathrm{ess}}(\overline{D})=\lim_{t\to 0}\Omega_{\overline{D}(t)}(\overline{B}). \tag{6.6}$$

Let $t<0$ and let $(\phi_n,\overline{P}_n)_n$ be a sequence in $\Theta(\overline{D}(t))$ with $\lim_{n\to\infty}(\overline{P}_n^{d+1})=\widehat{\mathrm{vol}}(\overline{D}(t))$. Then $\lim_{n\to\infty}(\overline{P}_n^{d}\cdot\overline{B})=(\langle\overline{D}(t)^d\rangle\cdot\overline{B})$ by the definition of positive intersection numbers. Moreover, $\lim_{n\to\infty}(P_n^d)=\mathrm{vol}(R^t(\overline{D}))>0$ and $\lim_{n\to\infty}\mu^{\mathrm{ess}}(\overline{P}_n)=\mu^{\mathrm{ess}}(\overline{D}(t))=-t$ by Corollary 4.6. For every n, let $\overline{P}'_n=\overline{P}_n-\mu^{\mathrm{ess}}(\overline{P}_n)[\infty]$. Then $\overline{P}'_n$ is pseudo-effective by Theorem 3.16. Moreover, by (3.3) we have

$$\frac{(\overline{P}_n^d\cdot\overline{B})}{(P_n^d)}=\frac{(\overline{P}_n^{d-1}\cdot\overline{B}\cdot\overline{P}'_n)}{(P_n^d)}+\mu^{\mathrm{ess}}(\overline{P}_n)\frac{(P_n^{d-1}\cdot B)}{(P_n^d)}.$$

The first summand is non-negative by Lemma 3.8. Therefore

$$\frac{(\overline{P}_n^d\cdot\overline{B})}{(P_n^d)}\geq\mu^{\mathrm{ess}}(\overline{P}_n)\frac{(P_n^{d-1}\cdot B)}{(P_n^d)}\geq\frac{\mu^{\mathrm{ess}}(\overline{P}_n)}{d r(P_n;B)}\geq\frac{\mu^{\mathrm{ess}}(\overline{P}_n)}{d\rho(\overline{D}(t);B)},$$

where the second inequality is given by Lemma 2.2 and the third one follows from the definition of $\rho(\overline{D}(t);B)$. Letting n tend to ∞ we obtain

$$\Omega_{\overline{D}(t)}(\overline{B}) = \frac{(\langle \overline{D}(t)^d\rangle \cdot \overline{B})}{\mathrm{vol}(R^t(\overline{D}))} = \lim_{n\to\infty} \frac{(\overline{P}_n^d \cdot \overline{B})}{(P_n^d)} \geq \lim_{n\to\infty} \frac{\mu^{\mathrm{ess}}(\overline{P}_n)}{d\rho(\overline{D}(t);B)} = \frac{-t}{d\rho(\overline{D}(t);B)}.$$

It follows from Lemma 2.1 that there exists a constant $c>0$ such that $\rho(\overline{D}(t);B) \leq c\rho(\overline{D}(t);D) = c\rho(\overline{D}(t))$ for every $t<0$. Therefore (1) follows by letting t tend to $0 = \mu^{\mathrm{ess}}(\overline{D})$ and using (6.6).

To prove the last claim, assume that $B = D$ and that $\overline{D}$ has the equidistribution property at every place. Note that $\overline{E} = \overline{B} - \overline{D}$ is an adelic divisor over $E = 0$. It follows from Example 3.2 and Proposition 5.6 that $\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = -\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$. Since $\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{D}) = (1+\lambda)\mu^{\mathrm{ess}}(\overline{D})$, we also have $\partial_{-\overline{D}}\,\mu^{\mathrm{ess}}(\overline{D}) = -\partial_{\overline{D}}\,\mu^{\mathrm{ess}}(\overline{D})$. This implies that (3) holds by the same arguments as in the proof of Lemma 4.2(2). □

7. Proof of the main theorem

In this section we prove Theorem 6.1. In §7.3 we give a slight refinement of this result in the case where $E = 0$.

7.1. **A consequence of Yuan's inequality.** The following proposition is a crucial ingredient in our proof. It is a variant of Lemma 4.5, with a precise estimate of the error term involving an inradius.

Proposition 7.1. *Let $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Assume that there exists a nef and big adelic $\mathbb{R}$-divisor $\overline{A}$ such that $\overline{A}+\overline{E}$ is pseudo-effective and $\overline{A}-\overline{E}$ is nef. Let $\overline{P}$ be a nef adelic $\mathbb{R}$-divisor with P big. For every $\lambda \geq 0$, we have*

$$\widehat{\mathrm{vol}}_\chi(\overline{P}+\lambda\overline{E}) \geq (\overline{P}^{d+1}) + (d+1)\lambda(\overline{P}^d \cdot \overline{E}) - C_d\lambda \times (\overline{P}^d \cdot \overline{A}) \max_{i\in\{1,d\}} \left(\frac{\lambda}{r(P;A)}\right)^i,$$

where C_d is a constant depending only on d.

The proof combines Yuan's Theorem 3.11 with the following technical lemma, which is a consequence of the arithmetic Hodge index theorem of Yuan and Zhang [YZ17]. The first point is due to Ikoma [Iko15, Theorem 2.7(2)], and the second one relies on similar arguments.

Lemma 7.2. *Let $\overline{P}$ and $\overline{A}$ be nef adelic $\mathbb{R}$-divisors with big geometric $\mathbb{R}$-divisors P and A. Let $i \in \{1,\ldots,d\}$.*

(1) Let $\overline{B}$ be a DSP adelic $\mathbb{R}$-divisor on X. If

$$(\overline{B}\cdot\overline{P}^{d+1-i}\cdot\overline{A}^{i-1}) = 0,$$

then $(\overline{B}^2\cdot\overline{P}^{d-i}\cdot\overline{A}^{i-1}) \leq 0$.

(2) We have

$$(P^{d+1-i}\cdot A^{i-1}) \times (\overline{P}^{d-i}\cdot\overline{A}^{i+1}) \leq 2(\overline{P}^{d+1-i}\cdot\overline{A}^{i}) \times (P^{d-i}\cdot A^i).$$

(3) We have

$$(\overline{P}^{d-i}\cdot\overline{A}^{i+1}) \leq (\overline{P}^d\cdot\overline{A}) \times \left(\frac{2}{r(P;A)}\right)^i$$

Proof. (1) This is a special case of [Iko15, Theorem 2.7 (2)]. We recall the argument for the convenience of the reader. Since P and A are nef and big, we have $(P^d) = \mathrm{vol}(P) > 0$ and $(A^d) = \mathrm{vol}(A) > 0$. This implies that $(P^{d+1-i} \cdot A^{i-1}) > 0$ by [Laz04, Theorem 1.6.1]. Let

$$\alpha = \frac{(B \cdot P^{d-i} \cdot A^{i-1})}{(P^{d+1-i} \cdot A^{i-1})}.$$

Then $((B - \alpha P) \cdot P^{d-i} \cdot A^{i-1}) = 0$. By the arithmetic Hodge index theorem [YZ17, Theorem 2.2] (which remains valid for adelic $\mathbb{R}$-divisors, see [Iko15, Theorem 2.7 (1)]), we have

$$(\overline{B}^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) + \alpha^2 (\overline{P}^{d+2-i} \cdot \overline{A}^{i-1}) = ((\overline{B} - \alpha \overline{P})^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) \leq 0,$$

and therefore $(\overline{B}^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) \leq 0$.

(2) Let

$$\alpha = \frac{(\overline{P}^{d+1-i} \cdot \overline{A}^{i})}{(P^{d+1-i} \cdot A^{i-1})}$$

and $\overline{A}(\alpha) = \overline{A} - \alpha[\infty]$. By (3.3) we have

$$(\overline{A}(\alpha) \cdot \overline{P}^{d+1-i} \cdot \overline{A}^{i-1}) = (\overline{P}^{d+1-i} \cdot \overline{A}^{i}) - \alpha (P^{d+1-i} \cdot A^{i-1}) = 0.$$

Applying (1) with $\overline{B} = \overline{A}(\alpha)$, we obtain

$$(\overline{A}(\alpha)^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) \leq 0.$$

To conclude, we observe that

$$(\overline{A}(\alpha)^2 \cdot \overline{P}^{d-i} \cdot \overline{A}^{i-1}) = (\overline{P}^{d-i} \cdot \overline{A}^{i+1}) - 2\alpha \, (P^{d-i} \cdot A^{i}).$$

(3) Note that the result holds trivially for $i = 0$. We deduce the general case by induction on i, using (2) and the fact that

$$(P^{d-i} \cdot A^{i}) \leq \frac{1}{r(P; A)} (P^{d+1-i} \cdot A^{i-1})$$

for every $i \in \{1, \ldots, d\}$. This inequality follows from (2.1) since $P - r(P, A)A$ is pseudo-effective. □

Proof of Proposition 7.1. Let $\overline{B} = \overline{A} - \overline{E}$. Then $\overline{B}$ is nef. By Yuan's Theorem 3.11, we have

$$\widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda \overline{E}) = \widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda \overline{A} - \lambda \overline{B}) \geq ((\overline{P} + \lambda \overline{A})^{d+1}) - \lambda (d+1) ((\overline{P} + \lambda \overline{A})^{d} \cdot \overline{B}).$$

After expanding the right-hand side, we find that it equals

$$(\overline{P}^{d+1}) + \lambda (d+1) (\overline{P}^{d} \cdot \overline{E}) + \sum_{i=2}^{d+1} \binom{d+1}{i} \lambda^{i} (\overline{P}^{d+1-i} \cdot \overline{A}^{i}) \\ - (d+1) \sum_{i=1}^{d} \binom{d}{i} \lambda^{i+1} (\overline{P}^{d-i} \cdot \overline{A}^{i} \cdot \overline{B}).$$

Since $\overline{P}$ and $\overline{A}$ are nef, the first sum is non-negative and therefore

$$\widehat{\mathrm{vol}}_{\chi}(\overline{P} + \lambda \overline{E}) \geq (\overline{P}^{d+1}) + \lambda (d+1) (\overline{P}^{d} \cdot \overline{E}) - (d+1) \sum_{i=1}^{d} \binom{d}{i} \lambda^{i+1} (\overline{P}^{d-i} \cdot \overline{A}^{i} \cdot \overline{B}).$$

Since $2\overline{A}-\overline{B}=\overline{A}+\overline{E}$ is pseudo-effective, we have $(\overline{P}^{d-i}\cdot\overline{A}^i\cdot\overline{B})\leq 2(\overline{P}^{d-i}\cdot\overline{A}^{i+1})$ for every $i\in\{1,\dots,d\}$ by Lemma 3.8, and

$$(\overline{P}^{d-i}\cdot\overline{A}^{i+1})\leq(\overline{P}^d\cdot\overline{A})\times\left(\frac{2}{r(P;A)}\right)^i$$

by Lemma 7.2(3). Therefore, $\widehat{\mathrm{vol}}_\chi(\overline{P}+\lambda\overline{E})$ is bounded from below by

$$(\overline{P}^{d+1})+\lambda(d+1)(\overline{P}^d\cdot\overline{E})-\lambda(\overline{P}^d\cdot\overline{A})(d+1)\sum_{i=1}^d\binom{d}{i}2^{i+1}\left(\frac{\lambda}{r(P;A)}\right)^i.$$

□

The following consequence of Proposition 7.1 plays a central role in our proof.

Corollary 7.3. *Let $\overline{E}\in\widehat{\mathrm{Div}}(X)_\mathbb{R}$. Assume that there exists a nef adelic $\mathbb{R}$-divisor $\overline{A}$ such that A is big, $\overline{A}+\overline{E}$ is pseudo-effective, and $\overline{A}-\overline{E}$ is nef. Let $\overline{P}$ be a nef adelic $\mathbb{R}$-divisor with P big. There exists a constant C_d depending only on d such that*

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\geq\frac{(\overline{P}^{d+1})}{(d+1)\,\mathrm{vol}(P+\lambda E)}+\lambda\frac{(\overline{P}^d\cdot\overline{E})}{(P^d)}-C_d\times\frac{(\overline{P}^d\cdot\overline{A})}{(P^d)}\times\frac{\lambda^2}{r(P;A)}$$

for every $\lambda\in[0,r(P;A)/2)$. In particular, if $E=0$ then

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\geq\frac{(\overline{P}^{d+1})}{(d+1)(P^d)}+\lambda\frac{(\overline{P}^d\cdot\overline{E})}{(P^d)}-C_d\times\frac{(\overline{P}^d\cdot\overline{A})}{(P^d)}\times\frac{\lambda^2}{r(P;A)}.\tag{7.1}$$

Proof. Let λ be a real number with $0\leq\lambda<r(P;A)/2$. By Lemma 2.4, we have

$$\left(1-\frac{\lambda}{r(P;A)}\right)^d(P^d)\leq\mathrm{vol}(P+\lambda E)\leq\left(1+\frac{\lambda}{r(P;A)}\right)^d(P^d).\tag{7.2}$$

In particular, $P+\lambda E$ is big. By Zhang's inequality (Theorem 3.17) we have

$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\geq\frac{\widehat{\mathrm{vol}}_\chi(\overline{P}+\lambda\overline{E})}{(d+1)\,\mathrm{vol}(P+\lambda E)}.$$

Therefore Proposition 7.1 implies that there exists a constant $C_d>0$ depending only on d such that

(7.3)
$$\mu^{\mathrm{ess}}(\overline{P}+\lambda\overline{E})\geq\frac{(\overline{P}^{d+1})}{(d+1)\,\mathrm{vol}(P+\lambda E)}+\lambda\frac{(\overline{P}^d\cdot\overline{E})}{\mathrm{vol}(P+\lambda E)}-C_d\frac{\lambda^2}{r(P,A)}\times\frac{(\overline{P}^d\cdot\overline{A})}{\mathrm{vol}(P+\lambda E)}.$$

By (7.2) we have

$$\frac{(\overline{P}^d\cdot\overline{A})}{\mathrm{vol}(P+\lambda E)}\leq\frac{(\overline{P}^d\cdot\overline{A})}{(P^d)}\times\left(1-\frac{\lambda}{r(P;A)}\right)^{-d}\leq 2^d\frac{(\overline{P}^d\cdot\overline{A})}{(P^d)}.$$

On the other hand, by Lemma 3.8 we have $|(\overline{P}^d\cdot\overline{E})|\leq(\overline{P}^d\cdot\overline{A})$ since $\overline{P}$ is nef and both $\overline{A}+\overline{E}$ and $\overline{A}-\overline{E}$ are pseudo-effective. Let $a=1$ if $(\overline{P}^d\cdot\overline{E})\geq 0$, and $a=-1$

otherwise. By (7.2), we have

$$\frac{(\overline{P}^d\cdot\overline{E})}{\mathrm{vol}(P+\lambda E)}\geq\frac{(\overline{P}^d\cdot\overline{E})}{(P^d)}\times\left(1+a\frac{\lambda}{r(P;A)}\right)^{-d}\geq\frac{(\overline{P}^d\cdot\overline{E})}{(P^d)}-C'_d\frac{\lambda}{r(P;A)}\frac{|(\overline{P}^d\cdot\overline{E})|}{(P^d)}$$
$$\geq\frac{(\overline{P}^d\cdot\overline{E})}{(P^d)}-C'_d\frac{\lambda}{r(P;A)}\frac{(\overline{P}^d\cdot\overline{A})}{(P^d)}$$

for some constant C'_d depending only on d. The result follows by combining these inequalities with (7.3). □

7.2. **Proof of Theorem 6.1.** Let $\overline{D}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with D big. To prove Theorem 6.1, we fix a sequence $(\phi_n,\overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ satisfying (6.1).

Lemma 7.4. *For every big $\mathbb{R}$-divisor A on X, we have*

$$\lim_{n\to\infty}\frac{\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n;\phi_n^*A)}=0.$$

Proof. This a straightforward consequence of (6.1) thanks to Lemma 2.1. □

For every $n\in\mathbb{N}$, we let $\widetilde{Q}_n=\overline{Q}_n-\mu^{\mathrm{abs}}(\overline{Q}_n)[\infty]$. Note that $\mu^{\mathrm{abs}}(\widetilde{Q}_n)=\mu^{\mathrm{abs}}(\overline{Q}_n)-\mu^{\mathrm{abs}}(\overline{Q}_n)=0$ and therefore $\widetilde{Q}_n$ is nef.

Lemma 7.5. *For every $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ and $n\in\mathbb{N}$, we have*

$$\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}=\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})-d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\times(Q_n^{d-1}\cdot\phi_n^*E)}{(Q_n^d)}$$

and moreover

$$\lim_{n\to\infty}\left(\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}-\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})-d\,\mu^{\mathrm{ess}}(\overline{D})\times(Q_n^{d-1}\cdot\phi_n^*E)}{(Q_n^d)}\right)=0.$$

Proof. The first equality follows from the multilinearity of arithmetic intersection products and (3.3). For $n\in\mathbb{N}$, let

$$\beta(n)=(\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}(\overline{Q}_n))\times\frac{(Q_n^{d-1}\cdot\phi_n^*E)}{(Q_n^d)}$$

The second statement just means that $\lim_{n\to\infty}\beta(n)=0$. To see this, let A be an ample divisor such that $A\pm E$ are both big. For every $n\in\mathbb{N}$ we have $|(Q_n^d\cdot\phi_n^*E)|\leq(Q_n^d\cdot\phi_n^*A)$ by (2.1). By Lemma 2.2 we have

$$|\beta(n)|\leq(\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}(\overline{Q}_n))\times\frac{(Q_n^{d-1}\cdot\phi_n^*A)}{(Q_n^d)}\leq\frac{\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n,\phi_n^*A)},$$

and we conclude with Lemma 7.4. □

Lemma 7.6. *For every nef and big $\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, we have*

$$\sup_{n\in\mathbb{N}}\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{A})}{Q_n^d}<\infty.$$

Proof. Replacing $\overline{D}$ by $\overline{D}+t[\infty]$ for t sufficiently large, we assume that $\overline{D}$ is big without loss of generality. Moreover, we assume that $\overline{D}-\overline{A}$ is pseudo-effective, after replacing $\overline{A}$ by a sufficiently small multiple if necessary. For every $n\in\mathbb{N}$ and $\lambda>0$ we have

$$\lambda(\widetilde{Q}_n^d\cdot\phi_n^*\overline{A})\leq\frac{1}{d+1}((\widetilde{Q}_n+\lambda\phi_n^*\overline{A})^{d+1}).$$

By Zhang's inequality (Theorem 3.17),

$$((\widetilde{Q}_n+\lambda\phi_n^*\overline{A})^{d+1})\leq(d+1)\mu^{\mathrm{ess}}(\widetilde{Q}_n+\lambda\phi_n^*\overline{A})\times((Q_n+\lambda\phi_n^*A)^d).$$

Note that $\mu^{\mathrm{ess}}(\widetilde{Q}_n+\lambda\phi_n^*\overline{A})=\mu^{\mathrm{ess}}(\overline{Q}_n+\lambda\phi_n^*\overline{A})-\mu^{\mathrm{abs}}(\overline{Q}_n)$. Since $\phi_n^*\overline{D}-\overline{Q}_n$ and $\overline{D}-\overline{A}$ are pseudo-effective, we have

$$\mu^{\mathrm{ess}}(\widetilde{Q}_n+\lambda\phi_n^*\overline{A})=\mu^{\mathrm{ess}}(\overline{Q}_n+\lambda\phi_n^*\overline{A})-\mu^{\mathrm{abs}}(\overline{Q}_n)\leq(1+\lambda)\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}(\overline{Q}_n).$$

Therefore

$$\lambda(\widetilde{Q}_n^d\cdot\phi_n^*\overline{A})\leq(\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}((\overline{Q}_n))(Q_n+\lambda\phi_n^*A)^d)+\lambda\mu^{\mathrm{ess}}(\overline{D})((Q_n+\lambda\phi_n^*A)^d).\tag{7.4}$$

If $\lambda\leq r(Q_n;\phi_n^*A)$, then $Q_n-\lambda\phi_n^*A$ is pseudo-effective and therefore $((Q_n+\lambda\phi_n^*A)^d)\leq 2^d(Q_n^d)$. Applying (7.4) to $\lambda=r(Q_n;\phi_n^*A)$ gives

$$\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{A})}{(Q_n^d)}\leq 2^d\left(\frac{\mu^{\mathrm{ess}}(\overline{D})-\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n;\phi_n^*A)}\right)+2^d\mu^{\mathrm{ess}}(\overline{D}).$$

This concludes the proof since the right-hand side is upper-bounded by some constant independent of n by Lemma 7.4. □

We are now ready to prove the theorem.

Proof of Theorem 6.1. Recall from §5 that for every $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, the one-sided derivative $\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$ exists and $-\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})\geq\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})$. Therefore we only need to show that

$$\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})\geq\limsup_{n\to\infty}\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}\tag{7.5}$$

for any $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Indeed, if this holds then we can apply (7.5) to $-\overline{E}$ and $\overline{E}$ to obtain

$$\liminf_{n\to\infty}\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}\geq-\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})\geq\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})\geq\limsup_{n\to\infty}\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)},$$

and Theorem 6.1 follows by Lemmas 4.2 and 7.5.

We first show (7.5) when $\overline{E}$ is DSP. In that case there exists a nef and big $\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{A}\pm\overline{E}$ are nef by Remark 3.10. By Lemma 7.6,

$$C_0:=\sup_{n\in\mathbb{N}}\frac{((\widetilde{Q}_n)^d\cdot(\phi_n^*\overline{A}))}{(Q_n^d)}$$

is a real number. Let $n\in\mathbb{N}$. Since $\widetilde{Q}_n$ is nef, we have $(\widetilde{Q}_n)^{d+1}\geq 0$. Moreover, for any $\lambda\geq 0$ such that $D+\lambda E$ is big we have

$$\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E})-\mu^{\mathrm{abs}}(\overline{Q}_n)\geq\mu^{\mathrm{ess}}(\overline{Q}_n+\lambda\phi_n^*\overline{E})-\mu^{\mathrm{abs}}(\overline{Q}_n)=\mu^{\mathrm{ess}}(\widetilde{Q}_n+\lambda\phi_n^*\overline{E})$$

by Lemma 3.15(4). Therefore Corollary 7.3 applied with $\overline{P}=\widetilde{Q}_n$ gives

$$\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E})-\mu^{\mathrm{abs}}(\overline{Q}_n)\geq\lambda\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}-C_dC_0\times\frac{\lambda^2}{r(Q_n;\phi_n^*A)},\tag{7.6}$$

for every $\lambda \in (0, r(Q_n; \phi_n^* A)/2)$, where C_d is a constant depending only on d. Let

$$\gamma(n) = \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n, \phi_n^* A)}$$

if $\mu^{\mathrm{ess}}(\overline{D}) \neq \mu^{\mathrm{abs}}(\overline{Q}_n)$ and $\gamma(n) = 1/n$ otherwise, and set $\lambda_n = \sqrt{\gamma(n)} r(Q_n, \phi_n^* A)$. By Lemma 7.4 we have $\lim_{n\to\infty} \gamma(n) = 0$, hence $\lim_{n\to\infty} \lambda_n = 0$. Applying (7.6) with $\lambda = \lambda_n$ gives

$$\frac{\mu^{\mathrm{ess}}(\overline{D} + \lambda_n \overline{E}) - \mu^{\mathrm{ess}}(\overline{D})}{\lambda_n} + \sqrt{\gamma(n)} \geq \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} - C_d C_0 \sqrt{\gamma(n)},$$

and we obtain (7.5) by letting n tend to infinity.

We now consider the general case. Let $\varepsilon > 0$. By Lemma 3.7, there exists a DSP $\overline{E}_\varepsilon \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ such that $\overline{E} - \overline{E}_\varepsilon$ and $\overline{E}_\varepsilon - \overline{E} + \varepsilon[\infty]$ are both pseudo-effective. The first condition implies that $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}_\varepsilon}\, \mu^{\mathrm{ess}}(\overline{D})$, and the second one together with Lemma 3.8 gives

$$(\widetilde{Q}_n^d \cdot \overline{E}_\varepsilon) \geq (\widetilde{Q}_n^d \cdot (\overline{E} - \varepsilon[\infty])) = (\widetilde{Q}_n^d \cdot \overline{E}) - \varepsilon(Q_n^d)$$

for every $n \in \mathbb{N}$. Therefore

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \partial_{\overline{E}_\varepsilon}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \limsup_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E}_\varepsilon)}{(Q_n^d)} \geq \limsup_{n\to\infty} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} - \varepsilon,$$

and we conclude by letting ε tend to zero. □

7.3. **A variant of Theorem 6.1.** In the course of the proof, we neglected a term of the form

$$\frac{(\widetilde{Q}_n^{d+1})}{\mathrm{vol}(Q_n + \lambda \phi_n^* E)}$$

when applying Corollary 7.3. It turns out that taking this term into account does not improve our approach in general, but it does in the case where $E = 0$ for which we can use the second part of Corollary 7.3. This leads to the following slight refinement of Theorem 6.1 in that special case.

Theorem 7.7. *Assume that there exists a sequence $(\phi_n, \overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ such that*

$$\lim_{n\to\infty} \frac{1}{r(Q_n; \phi_n^* D)} \Big(\mu^{\mathrm{ess}}(\overline{D}) - \frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)} \Big) = 0 \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} < \infty.$$

Then $\overline{D}$ satisfies the equidistribution property at every place $v \in \mathfrak{M}_K$. The v-adic equidistribution measure is the weak limit $\nu_{\overline{D},v} = \lim_{n\to\infty} \nu_{n,v}$, where $\nu_{n,v}$ denotes the pushforward to X_v^{an} of the normalized Monge-Ampère measure $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$.

We only outline the proof of this theorem, as it is almost the same as the one of Theorem 6.1. Let $(\phi_n, \overline{Q}_n)$ be a sequence satisfying the conditions of Theorem 7.7. As above, we let $\widetilde{Q}_n = \overline{Q}_n - \mu^{\mathrm{abs}}(\overline{Q}_n)[\infty]$ for every n. With this notation, the proof of Lemma 7.6 remains valid: we have

$$C_0 := \sup_{n\in\mathbb{N}} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{A})}{(Q_n^d)} < \infty.$$

By Proposition 5.6, it suffices to show that

$$-\partial_{-\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D}) = \lim_{n\to\infty}\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)} \tag{7.7}$$

for any $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ over 0. We only treat the case where $\overline{E}$ is DSP, as the general one follows as in the proof of Theorem 6.1. Let $\overline{A}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ be big and nef with $\overline{A}\pm\overline{E}$ nef. By Corollary 7.3, there exists a constant C_d such that

$$\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E})-\mu^{\mathrm{abs}}(\overline{Q}_n)\geq\frac{(\widetilde{Q}_n^d)}{(d+1)(Q_n^d)}+\frac{\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}-C_0C_d\times\frac{\lambda^2}{r(Q_n,\phi_n^*A)} \tag{7.8}$$

for every n and $\lambda\in(0,r(Q_n,\phi_n^*A)/2)$. On the other hand, since $E=0$ we have

$$(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})=(\overline{Q}_n^d\cdot\phi_n^*\overline{E})\quad\text{and}\quad(\widetilde{Q}_n^d)=(\overline{Q}_n^d)-(d+1)(Q_n^d)\times\mu^{\mathrm{abs}}(\overline{Q}_n)$$

by (3.3). Combining this with (7.8) and dividing by λ gives

$$\begin{aligned}\frac{\mu^{\mathrm{ess}}(\overline{D}+\lambda\overline{E})-\mu^{\mathrm{ess}}(\overline{D})}{\lambda}\geq\frac{1}{\lambda}\Big(\frac{(\overline{Q}_n^d)}{(d+1)(Q_n^d)}-\mu^{\mathrm{ess}}(\overline{D})\Big)+\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}\\ -C_0C_d\times\frac{\lambda}{r(Q_n,\phi_n^*A)}.\end{aligned}$$

As before, a suitable choice of $\lambda=\lambda_n$ permits to conclude that

$$\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})\geq\limsup_{n\to\infty}\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)},$$

and we obtain (7.7) by applying this to $-\overline{E}$.

Remark 7.8. Compared to Theorem 6.1, Theorem 7.7 gives more flexibility to construct the sequence $(\phi_n,\overline{Q}_n)_n$. For example, one can deduce Yuan's equidistribution theorem from Theorem 7.7 without using Theorem 3.18. However, it turns out that starting with a sequence $(\phi_n,\overline{Q}_n)_n$ satisfying the conditions of Theorem 7.7, one can modify it to construct a sequence with (6.1). This can be done with arguments similar to the ones we used in the proof of Corollary 4.7. Since we do not need this in the remainder of the text, we leave the details to the interested reader.

8. Logarithmic equidistribution

Let $\overline{D}$ be an adelic $\mathbb{R}$-divisor on X with big geometric $\mathbb{R}$-divisor D. In this section we assume that the condition of Theorem 6.1 is satisfied, namely that there exists a sequence $(\phi_n\colon X_n\to X,\overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ satisfying (6.1). By Theorem 6.1 and Lemma 7.5, the essential minimum function is differentiable at $\overline{D}$ and

$$\begin{aligned}\partial_{\overline{E}}\,\mu^{\mathrm{ess}}(\overline{D})&=\lim_{n\to\infty}\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})-d\,\mu^{\mathrm{ess}}(\overline{D})\times(Q_n^{d-1}\cdot\phi_n^*E)}{(Q_n^d)}\\&=\lim_{n\to\infty}\frac{(\overline{Q}_n^d\cdot\phi_n^*\overline{E})-d\,\mu^{\mathrm{abs}}(\overline{Q}_n)\times(Q_n^{d-1}\cdot\phi_n^*E)}{(Q_n^d)}\end{aligned} \tag{8.1}$$

for every $\overline{E}\in\widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. In particular, $\overline{D}$ satisfies the equidistribution property at every $v\in\mathfrak{M}_K$ with equidistribution measure $\nu_{\overline{D},v}=\lim_{n\to\infty}\nu_{n,v}$, where $\nu_{n,v}$ be the pushforward to X_v^{an} of the normalized v-adic Monge-Ampère measure of $\overline{Q}_n$. Following

[CLT09], we show that this equidistribution phenomenon extends to functions with logarithmic singularities along certain divisors. We closely follow [CLT09, Proof of Theorem 1.2], adapting the arguments to our setting.

Definition 8.1. Let $E \in \operatorname{Div}(X)$ be an effective (Cartier) divisor on X and let $v \in \mathfrak{M}_K$. We say that a function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ has *at most logarithmic singularities* along E if it is continuous on $X_v^{\mathrm{an}} \setminus (\operatorname{Supp} E)_v^{\mathrm{an}}$ and if every $x \in X_v^{\mathrm{an}}$ has a neighborhood $U \subset X_v^{\mathrm{an}}$ with the following property: there exists an equation f_U of E_v^{an} on U and a real number c_U such that $|\varphi|_v \leq c_U \log |f_U|_v^{-1}$ on U.

We start with two preliminary lemmas.

Lemma 8.2. *For every $\overline{E} \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$ with E effective, we have*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}.$$

Proof. Let $n \in \mathbb{N}$. By Lemma 3.14 we have

$$h_{\overline{Q}_n}(\phi_n^* E) \geq d\, \mu^{\mathrm{abs}}(\overline{Q}_n) \times (Q_n^{d-1} \cdot \phi_n^* E),$$

where the left-hand side denotes the height of the $\mathbb{R}$-Weil divisor associated to $\phi_n^* E$. By the arithmetic Bézout formula (3.2) we obtain

$$\begin{aligned} \frac{(\overline{Q}_n^d \cdot \phi_n^* \overline{E}) - d\, \mu^{\mathrm{abs}}(\overline{Q}_n) \times (Q_n^{d-1} \cdot \phi_n^* E)}{(Q_n^d)} &\geq \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{n,v} \\ &\geq \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \min\{a, g_{\overline{E},v}\}\, d\nu_{n,v} \end{aligned}$$

for any $a \in \mathbb{R}$. Since E is effective, $g_{\overline{E},v}$ is bounded from below and therefore $\min\{a, g_{\overline{E},v}\} \in C(X_v^{\mathrm{an}})$ for every $v \in \mathbb{R}$. Letting n tend to infinity, (8.1) then gives

$$\begin{aligned} &\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \\ &\quad \geq \sum_{v \in \mathfrak{M}_K} n_v \liminf_{n \to \infty} \int_{X_v^{\mathrm{an}}} \min\{a, g_{\overline{E},v}\}\, d\nu_{n,v} = \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \min\{a, g_{\overline{E},v}\}\, d\nu_{\overline{D},v}. \end{aligned}$$

Letting a tend to ∞, the result follows by monotone convergence. □

Lemma 8.3. *For every $\overline{E} \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$ and $v \in \mathfrak{M}_K$, $g_{\overline{E},v}$ is integrable with respect to $\nu_{\overline{D},v}$.*

Proof. We can write $\overline{E} = \overline{E}_1 - \overline{E}_2$ with $\overline{E}_1, \overline{E}_2$ effective. By linearity, we may therefore assume that $\overline{E}$ is effective without loss of generality. Since $g_{\overline{E},w} \geq 0$ for every $w \in \mathfrak{M}_K$, we have

$$\infty > \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) \geq n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v} = n_v \int_{X_v^{\mathrm{an}}} |g_{\overline{E},v}|\, d\nu_{\overline{D},v}$$

by Lemma 8.2. □

The main result in this paragraph is the following.

Theorem 8.4. *Let* $\overline{E} = (E, (g_{\overline{E},v})_{v\in\mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X)$. *Assume that* E *is effective and that*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}. \tag{8.2}$$

Then the same equality holds for every adelic divisor $\overline{E}'$ *over* E. *Moreover, for every* $\overline{D}$*-small generic sequence* $(x_\ell)_\ell$ *and* $v \in \mathfrak{M}_K$ *we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi\, d\nu_{\overline{D},v}$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R}\cup\{\pm\infty\}$ *with at most logarithmic singularities along* E.

Proof. Let $\overline{E}'$ be another adelic divisor over E. Then there exist a finite set $S \subset \mathfrak{M}_K$ and a collection $(\varphi_v)_{v\in S}$ such that $\varphi_v \in C(X_v^{\mathrm{an}})^{G_v}$ for every $v \in S$ and $\overline{E}' - \overline{E} = \sum_{v\in S} \overline{0}^{\varphi_v}$. Since the essential minimum function is differentiable at $\overline{D}$, we have

$$\partial_{\overline{E}'}\, \mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) + \sum_{v\in S} \partial_{\overline{0}^{\varphi_v}}\, \mu^{\mathrm{ess}}(\overline{D}).$$

We let $\varphi_v = 0$ for $v \in \mathfrak{M}_K \setminus S$. Proposition 5.6 together with (8.2) then gives

$$\partial_{\overline{E}'}\, \mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} (g_{\overline{E},v} + \varphi_v)\, d\nu_{\overline{D},v} = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E}',v}\, d\nu_{\overline{D},v}.$$

To prove the second statement, it suffices to consider the case where $\varphi = g_{\overline{E},v}$ by [CLT09, Lemma 6.3]. By (8.2), Proposition 5.3 and Theorem 6.1 we have

$$\sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v} = \partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \lim_{\ell\to\infty} \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v}.$$

Therefore

$$\begin{aligned} 0 &= \lim_{\ell\to\infty} \sum_{v\in\mathfrak{M}_K} n_v \Big(\int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} - \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v} \Big) \\ &\geq \sum_{v\in\mathfrak{M}_K} n_v \Big(\liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} - \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v} \Big). \end{aligned}$$

On the other hand, since $\min\{a, g_{\overline{E},v}\} \in C(X_v)^{\mathrm{an}}$ for every v and $a \in \mathbb{R}$ we have

$$\liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} \geq \liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \min\{a, g_{\overline{E},v}\}\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \min\{a, g_{\overline{E},v}\}\, d\nu_{\overline{D},v}$$

by the equidistribution property. Letting a tend to infinity, it follows that

$$\liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} - \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v} \geq 0.$$

Since the sum of these terms over all the places v is non-positive, we obtain

$$\liminf_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}$$

for every $v \in \mathfrak{M}_K$. We conclude by observing that this equality remains true when $(x_\ell)_\ell$ is replaced by an arbitrary infinite subsequence (since the latter remains generic and $\overline{D}$-small). Therefore we have

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}$$

as desired. □

Remark 8.5. Assume that for every $v \in \mathfrak{M}_K$, the sequence of probability measures $(\nu_{n,v})_n$ is eventually constant, that is $\nu_{n,v} = \nu_{\overline{D},v}$ for every sufficiently large n. As we shall see in §10, this condition is satisfied in the case of dynamical systems and semi-abelian varieties. Then for every $\overline{E} \in \widehat{\mathrm{Div}}(X)$, the condition (8.2) is equivalent to

$$\lim_{n\to\infty} \frac{h_{\overline{Q}_n}(\phi_n^* E) - d\,\mu^{\mathrm{ess}}(\overline{D}) \times (Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} = 0.$$

Indeed, by (8.1) and the arithmetic Bézout formula (3.2) we have

$$\begin{aligned}\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) &= \lim_{n\to\infty} \Big(\frac{h_{\overline{Q}_n}(\phi_n^* E) - d\,\mu^{\mathrm{ess}}(\overline{D}) \times (Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} + \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{n,v}\Big)\\ &= \lim_{n\to\infty} \frac{h_{\overline{Q}_n}(\phi_n^* E) - d\,\mu^{\mathrm{ess}}(\overline{D}) \times (Q_n^{d-1}\cdot \phi_n^* E)}{(Q_n^d)} + \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\nu_{\overline{D},v}.\end{aligned}$$

Theorem 8.4 implies [CLT09, Theorem 1.2] as follows. Assume that $\overline{D}$ is semipositive and that

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}.$$

By Theorem 3.18 we have $\mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{abs}}(\overline{D})$, and therefore the condition of Theorem 6.1 is trivially satisfied for the constant sequence $(\phi_n, \overline{Q}_n)_n$ equal to $(\mathrm{Id}_X, \overline{D})$. By Remark 8.5, the conclusion of Theorem 8.4 is satisfied for any effective divisor $E \in \mathrm{Div}(X)_{\mathbb{R}}$ such that

$$\frac{h_{\overline{D}}(E)}{d(D^{d-1}\cdot E)} = \mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}.$$

9. Toric varieties

In this section we study the differentiability of the essential minimum function for toric adelic $\mathbb{R}$-divisors on toric varieties, and explore its consequences for the equidistribution of the Galois orbits of small generic sequences of points.

In the first and second subsections we explain the basic constructions and results from the algebraic and the Arakelov geometry of toric varieties, referring to [BPS14, BPS15, BMPS16, BPRS19] for the proofs and more information, and to Appendix A for the necessary definitions from convex analysis. As a small addition to the theory we give a formula for the positive arithmetic intersection numbers of toric adelic $\mathbb{R}$-divisors (Proposition 9.6). In the third subsection we apply these elements to prove our differentiability and equidistribution results in the toric setting (Theorems 9.8 and 9.15).

9.1. **Geometric aspects.** For $d \geq 1$ let $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m}}^d$ be a split d-dimensional torus over K. A *toric variety* is a normal variety X over K that contains $\mathbb{T}$ as an open subset and is equipped with an action of this torus extending its action onto itself by translations. An $\mathbb{R}$-divisor D on X is *toric* if it is invariant under this action. A modification $\phi\colon X' \to X$ is *toric* if X' is also a toric variety with the same torus $\mathbb{T}$ and the restriction of ϕ to this torus coincides with the identity on $\mathbb{T}$.

Toric varieties and $\mathbb{R}$-divisors can be constructed and classified with polyhedral objects like fans and $\mathbb{R}$-virtual support functions. To this end, first set

$$M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_{\mathrm{m}}) \quad \text{and} \quad N = \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{T})$$

for the lattices of characters and of co-characters of $\mathbb{T}$, respectively. They are both isomorphic to $\mathbb{Z}^d$ and are dual of each other, that is $M = N^\vee$ and $N = M^\vee$. Set $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. These spaces are also dual of each other, and for $u \in N_\mathbb{R}$ and $x \in M_\mathbb{R}$ we denote their pairing by either $\langle x, u\rangle$ or $\langle u, x\rangle$. Let $K[M]$ be the group algebra of M, and for each $m \in M$ set $\chi^m \in K[M]$ for the corresponding monomial.

A *fan* Σ on $N_\mathbb{R}$ is a polyhedral complex of strongly convex cones defined over N. To each cone $\sigma \in \Sigma$ we associate the semigroup in M defined by

$$M_\sigma = \{m \in M \mid \langle u, m\rangle \geq 0 \text{ for all } u \in \sigma\}$$

and we set $K[M_\sigma]$ for its semigroup algebra. Then $X_\sigma = \mathrm{Spec}(K[M_\sigma])$ is an affine toric variety. Moreover, for each pair $\sigma, \sigma' \in \Sigma$ we have that $X_{\sigma\cap\sigma'}$ can be naturally seen an open subset of both X_σ and $X_{\sigma'}$. The toric variety X_Σ is then defined by gluing these affine toric varieties:

$$X_\Sigma = \bigcup_{\sigma\in\Sigma} X_\sigma.$$

The affine toric variety X_0 corresponding to the zero cone is canonically isomorphic to the torus $\mathbb{T}$, and is called the *principal open subset* of X_Σ.

An $\mathbb{R}$-*virtual support function* Ψ on Σ is a real-valued function on the subset of $N_\mathbb{R}$ covered by Σ that is linear on each of the cones of this fan. For each $\sigma \in \Sigma$ choose $m_\sigma \in M_\mathbb{R}$ such that $\Psi(u) = \langle m_\sigma, u\rangle$ for all $u \in \sigma$. For each pair $\sigma, \sigma' \in \Sigma$ the $\mathbb{R}$-rational function $\chi^{-m_\sigma}/\chi^{-m_{\sigma'}}$ restricted to the overlap $X_\sigma \cap X_{\sigma'}$ is an $\mathbb{R}$-linear combination of regular functions, and so

$$D_\Psi = \{(X_\sigma, \chi^{-m_\sigma})\}_{\sigma\in\Sigma} \tag{9.1}$$

defines an $\mathbb{R}$-divisor on X_Σ that is toric and independent of the choice of these vectors.

The assignment $\Sigma \mapsto X_\Sigma$ is a bijection between the set of fans on $N_\mathbb{R}$ and that of toric varieties with torus $\mathbb{T}$. Similarly, for a fixed fan Σ the assignment $\Psi \mapsto D_\Psi$ is a bijection between the set of $\mathbb{R}$-virtual support functions on Σ and that of toric $\mathbb{R}$-divisors on X_Σ. In the sequel we revert these notations, and for a toric variety X with torus $\mathbb{T}$ and a toric $\mathbb{R}$-divisor D on it we respectively denote its fan and $\mathbb{R}$-virtual support function by Σ_X and Ψ_D.

The toric variety X is projective if and only if its fan is complete and regular, namely it covers the whole of $N_\mathbb{R}$ and can be defined as the linearity locus of a piecewise linear concave function on this vector space. We suppose this is the case from now on. We then associate to D the subset of $M_\mathbb{R}$ defined as

$$\Delta_D = \{x \in M_\mathbb{R} \mid \langle u, x\rangle \geq \Psi_D(u) \text{ for all } u \in N_\mathbb{R}\}.$$

It is a quasi-rational polytope, that is a polytope with rational slopes.

The positivity invariants and properties of a toric $\mathbb{R}$-divisor can be directly read from its $\mathbb{R}$-virtual support function and polytope. For instance,

$$\mathrm{vol}(D) = d!\,\mathrm{vol}_M(\Delta_D)$$

where vol_M denotes the Haar measure on $M_\mathbb{R}$ normalized so that M has covolume 1. In particular, if D is nef then $(D^d) = d!\,\mathrm{vol}_M(\Delta_D)$. More generally, for a family D_i, $i = 1, \dots, d$, of toric nef $\mathbb{R}$-divisors on X we have

$$(D_1 \cdots D_d) = \mathrm{MV}_M(\Delta_{D_1}, \dots, \Delta_{D_d}), \tag{9.2}$$

where MV_M denotes the mixed volume function with respect to vol_M.

The $\mathbb{R}$-divisor D is pseudo-effective if and only if $\Delta_D \neq \emptyset$, and it is big if and only if $\dim(\Delta_D) = d$. In addition, D is nef if and only if Ψ_D is concave. For another toric $\mathbb{R}$-divisor E that is nef we have that $D - E$ is pseudo-effective if and only if there is $x \in M_\mathbb{R}$ such that $x + \Delta_E \subset \Delta_D$. These properties are either contained in [BMPS16, Propositions 4.6, 4.7 and 4.9] or follow easily from these results.

This characterization of pseudo-effectivity readily implies the next proposition, showing that the inradius of a toric $\mathbb{R}$-divisor with respect to another can be computed in terms of the inradius of the associated polytopes, in the sense of convex geometry (Definition A.1). This connection was first pointed out by Teissier for toric line bundles [Tei82, §2.1 and §2.2].

Proposition 9.1. *Let D and A be toric $\mathbb{R}$-divisors on X such that D is big and A is big and nef. Then $r(D; A) = r(\Delta_D; \Delta_A)$.*

The toric modifications of X can be classified by the complete and regular refinements of the fan. Let X' be another projective toric variety with torus $\mathbb{T}$ such that $\Sigma_{X'}$ refines Σ_X. Then for each cone $\sigma' \in \Sigma_{X'}$ there is another cone $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$. The associated semigroups verify that $M_{\sigma'} \supset M_\sigma$, and this inclusion induces a morphism of affine toric varieties

$$X'_{\sigma'} = \mathrm{Spec}(K[M_{\sigma'}]) \longrightarrow X_\sigma = \mathrm{Spec}(K[M_\sigma]).$$

These morphisms glue together into a toric modification, that is a morphism $X' \to X$ whose restriction to the respective principal open subsets is the identity on $\mathbb{T}$. Indeed, every such modification of X arises in this manner [BPS14, Theorem 3.2.4].

9.2. **Arithmetic aspects.** Throughout this subsection we denote by $\mathbb{T} \simeq \mathbb{G}_\mathrm{m}^d$ a split torus and X a projective toric variety with torus $\mathbb{T}$. For each place $v \in \mathfrak{M}_K$ we denote by $\mathbb{S}_v$ the compact torus of the v-adic analytic torus $\mathbb{T}_v^\mathrm{an}$. In the Archimedean case $\mathbb{S}_v$ is isomorphic to the real torus $(S^1)^d$, whereas in the non-Archimedean case it is an analytic subgroup of $\mathbb{T}_v^\mathrm{an}$ in the sense of Berkovich. An adelic $\mathbb{R}$-divisor $\overline{D}$ on X is *toric* if its geometric $\mathbb{R}$-divisor is toric and all its v-adic Green functions are invariant under the action of $\mathbb{S}_v$.

Toric adelic $\mathbb{R}$-divisors over D can be constructed and classified with adelic families of functions on $N_\mathbb{R}$ whose behavior at infinity is governed by the $\mathbb{R}$-virtual support function Ψ_D. To this end, let $v \in \mathfrak{M}_K$ and consider the *valuation map*

$$\mathrm{val}_v \colon \mathbb{T}_v^\mathrm{an} = X_{0,v}^\mathrm{an} \longrightarrow N_\mathbb{R}.$$

Using a splitting of the torus, we can identify the dense subset $\mathbb{T}_v^\mathrm{an}(\mathbb{C}_v)$ with $(\mathbb{C}_v^\times)^d$ and the vector space $N_\mathbb{R}$ with $\mathbb{R}^d$. In these coordinates, the map writes down for a point of $\mathbb{T}_v^{\mathrm{an}}(\mathbb{C}_v)$ as

$$\mathrm{val}_v(x_1, \dots, x_n) = (-\log |x_1|_v, \dots, -\log |x_n|_v).$$

This valuation map extends to a proper continuous, $\mathbb{S}_v$-invariant and similarly noted map of topological spaces

$$\mathrm{val}_v \colon X_v^{\mathrm{an}} \longrightarrow N_{\Sigma_X},$$

where N_{Σ_X} denotes the Mumford compactification of $N_{\mathbb{R}}$ induced by the fan, see [BPS14, Section 4.1] for details.

Now let $\psi_v \colon N_{\mathbb{R}} \to \mathbb{R}$, $v \in \mathfrak{M}_K$, be a family of functions such that $\psi_v - \Psi_D$ extends to a continuous function on N_{Σ_X} for all v and such that $\psi_v = \Psi_D$ for all but a finite number of these places. Then for each v we can consider the function $g_{\psi_v} \colon X_{0,v}^{\mathrm{an}} \to \mathbb{R}$ defined as

$$g_{\psi_v}(x) = -\psi_v(\mathrm{val}_v(x)).$$

The fact that $\psi_v - \Psi_D$ admits a continuous extension to N_{Σ_X} implies that for every cone $\sigma \in \Sigma_X$ and defining vector m_σ as in (9.1) we have that $g_{\psi_v} + \log |\chi^{-m_\sigma}|_v$ extends to a continuous function on the affine chart $X_{\sigma,v}^{\mathrm{an}}$. Hence g_{ψ_v} is an $\mathbb{S}_v$-invariant v-adic Green function for D. Furthermore, the fact that $\psi_v = \Psi_D$ for all but a finite number of places implies that the pair $(D, (g_{\psi_v})_{v \in \mathfrak{M}_K})$ is a toric adelic $\mathbb{R}$-divisor [BPS14, Example 4.5.4].

This construction gives a bijection between the set of such adelic families of functions on $N_{\mathbb{R}}$ and that of toric adelic $\mathbb{R}$-divisors over D. Reverting it, for a toric adelic $\mathbb{R}$-divisor $\overline{D}$ we denote by

$$\psi_{\overline{D},v} \colon N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad v \in \mathfrak{M}_K,$$

the associated family of *metric functions*. We also associate to $\overline{D}$ its family of *local roof functions*

$$\vartheta_{\overline{D},v} \colon \Delta_D \longrightarrow \mathbb{R}, \quad v \in \mathfrak{M}_K.$$

For each $v \in \mathfrak{M}_K$, the v-adic roof function is the continuous concave function on the polytope defined as

$$\vartheta_{\overline{D},v}(x) = \inf_{u \in N_{\mathbb{R}}} \langle u, x \rangle - \psi_{\overline{D},v}(u) \quad \text{for } x \in \Delta_D. \tag{9.3}$$

We have $\vartheta_{\overline{D},v} = 0|_{\Delta_D}$, the zero function on the polytope, for all but a finite number of places. We consider then the *global roof function* $\vartheta_{\overline{D}} \colon \Delta_D \to \mathbb{R}$, defined as the weighted sum

$$\vartheta_{\overline{D}} = \sum_{v \in \mathfrak{M}_K} n_v \vartheta_{\overline{D},v}. \tag{9.4}$$

We also consider the compact convex set where this concave function is nonnegative:

$$\Gamma_{\overline{D}} = \{x \in \Delta_D \mid \vartheta_{\overline{D}}(x) \geq 0\}. \tag{9.5}$$

In analogy with the geometric case, the positivity invariants and properties of a toric adelic $\mathbb{R}$-divisor can be read from its metric and roof functions. For instance, the essential minimum of X with respect to $\overline{D}$ is the maximum of global roof function [BPS15, Theorem 1.1]:

$$\mu^{\mathrm{ess}}(\overline{D}) = \max_{x \in \Delta_D} \vartheta_{\overline{D}}(x).$$

Moreover, if $\overline{D}$ is semipositive then $\mu^{\mathrm{abs}}(\overline{D}) = \min_{x \in \Delta_D} \vartheta_{\overline{D}}(x)$ [BPS15, Remark 3.15].

The arithmetic volume and the χ-arithmetic volume of $\overline{D}$ can be respectively computed as [BMPS16, Theorem 5.6]

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)! \int_{\Gamma_D} \vartheta_{\overline{D}} \, d\mathrm{vol}_M \quad \text{and} \quad \widehat{\mathrm{vol}}_{\chi}(\overline{D}) = (d+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, d\mathrm{vol}_M \, .$$

In particular, if $\overline{D}$ is semipositive then $(\overline{D}^{d+1}) = (d+1)! \int_{\Delta_D} \vartheta_{\overline{D}} \, d\mathrm{vol}_M$. More generally, the arithmetic intersection number of a family of semipositive toric adelic $\mathbb{R}$-divisors $\overline{D}_i$, $i = 0, \dots, d$, can be computed as

$$(\overline{D}_0 \cdots \overline{D}_d) = \sum_{v \in \mathfrak{M}_K} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0,v}, \dots, \vartheta_{\overline{D}_d,v}), \tag{9.6}$$

where MI_M denotes the mixed integral function with respect to the Haar measure vol_M on $M_{\mathbb{R}}$ [BPS14, Theorem 5.2.5].

We have that $\overline{D}$ is pseudo-effective if and only if there is $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) \geq 0$, and it is big if and only if $\dim(\Delta_D) = d$ and there is $x \in \Delta_D$ such that $\vartheta_{\overline{D}}(x) > 0$. In addition, $\overline{D}$ is semipositive if and only if $\psi_{\overline{D},v}$ is concave for all $v \in \mathfrak{M}_K$. If this is the case, then $\overline{D}$ is nef if and only of $\vartheta_{\overline{D}}(x) \geq 0$ for all $x \in \Delta_D$. For another adelic $\overline{E}$ semipositive that is semipositive we have that $\overline{D} - \overline{E}$ is pseudo-effective if and only if $\Delta_E \subset \Delta_D$ and $\vartheta_{\overline{E},v}(x) \leq \vartheta_{\overline{D},v}(x)$ for all $v \in \mathfrak{M}_K$ and $x \in \Delta_E$. The listed properties can be found in [BMPS16, Sections 5 and 6].

Example 9.2. Let D be a big and nef toric $\mathbb{R}$-divisor on X, and choose a family of vectors $u_v \in N_{\mathbb{R}}$, $v \in \mathfrak{M}_K$, with $u_v = 0$ for all but a finite number of v and such that $\sum_v n_v u_v = 0$. Choose also a family of constants $c_v \in \mathbb{R}$, $v \in \mathfrak{M}_K$, with $c_v = 0$ for all but a finite number of v. We consider then the semipositive toric adelic $\mathbb{R}$-divisor $\overline{D}$ on X defined by the concave metric functions

$$\psi_v(u) = \Psi_D(u - u_v) - c_v \quad \text{for all } v \text{ and } u \in N_{\mathbb{R}}.$$

Its local roof functions are the affine functions given by $\vartheta_v(x) = \langle u_v, x \rangle + c_v$ for all v and $x \in \Delta_D$, as it follows from their definition in (9.3) and the basic properties of the Legendre-Fenchel duality of concave functions, see for instance [BPS14, Proposition 2.3.3 and Example 2.2.1]. Thus its global roof function is the constant function

$$\vartheta_{\overline{D}} = c|_{\Delta_D} \quad \text{for } c = \sum_{v \in \mathfrak{M}_K} n_v c_v \in \mathbb{R}.$$

In particular $\mu^{\mathrm{ess}}(\overline{D}) = \mu^{\mathrm{abs}}(\overline{D}) = c$, and so Zhang's lower bound is attained for $\overline{D}$.

Remark 9.3. Every semipositive toric adelic $\mathbb{R}$-divisor on X for which Zhang's lower bound for the essential minimum is attained is of this type for a suitable choice of vectors and constants [BPRS19, Proposition 5.3].

For any semipositive toric adelic $\mathbb{R}$-divisor on X it is always possible to find an upper bound as that required in Proposition 6.13. To this end we consider the notion of *balanced family of sup-gradients* for the decomposition of $\vartheta_{\overline{D}}$ into local roof functions. This is a family of vectors

$$u_v \in N_{\mathbb{R}}, \quad v \in \mathfrak{M}_K,$$

with $u_v = 0$ for all but a finite number of v and verifying $\sum_v n_v u_v = 0$, and such that there is a point $x_0 \in \Delta_D$ maximizing $\vartheta_{\overline{D}}$ with u_v lying in $\partial \vartheta_{\overline{D},v}(x_0)$ the sup-differential of the v-adic roof function at this point (Definition A.7).

Proposition 9.4. *Let $\overline{D}$ be a semipositive toric adelic $\mathbb{R}$-divisor on X with D big. Then there is toric semipositive adelic $\mathbb{R}$-divisor $\widetilde{D}$ such that $\widetilde{D}-\overline{D}$ is pseudo-effective and $\mu^{\mathrm{ess}}(\widetilde{D}) = \mu^{\mathrm{abs}}(\widetilde{D}) = \mu^{\mathrm{ess}}(\overline{D})$.*

Proof. Let $(u_v)_v$ be a balanced family of sup-gradients for $\vartheta_{\overline{D}}$, which does exist thanks to Proposition A.8. Take a point $x_0 \in \Delta_D$ maximizing $\vartheta_{\overline{D}}$. Then for each $v \in \mathfrak{M}_K$ we have

$$\vartheta_{\overline{D},v}(x) \leq \langle u_v, x\rangle + c_v \quad \text{for all } x \in \Delta_D \tag{9.7}$$

with $c_v = \vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0\rangle$. We have

$$\sum_{v\in\mathfrak{M}_K} n_v c_v = \sum_{v\in\mathfrak{M}_K} n_v \left(\vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0\rangle\right) = \vartheta_{\overline{D}}(x_0) = \mu^{\mathrm{ess}}(\overline{D}). \tag{9.8}$$

Set $\widetilde{D}$ for the semipositive toric adelic $\mathbb{R}$-divisor on X from Example 9.2 corresponding to the data $(u_v)_v$ and $(c_v)_v$. The inequality in (9.7) implies that $\widetilde{D} - \overline{D}$ is pseudo-effective, and from (9.8) we get $\mu^{\mathrm{ess}}(\widetilde{D}) = \mu^{\mathrm{abs}}(\widetilde{D}) = \mu^{\mathrm{ess}}(\overline{D})$. □

Now suppose that $\overline{D}$ is big. By the toric arithmetic Fujita approximation theorem [BMPS16, Theorem 7.2] there is a sequence of nef approximations of $\overline{D}$

$$(\phi_n \colon X_n \to X, \overline{P}_n)_n \tag{9.9}$$

such that both ϕ_n and $\overline{P}_n$ are toric and $\lim_{n\to\infty} \widehat{\mathrm{vol}}(\overline{P}_n) = \widehat{\mathrm{vol}}(\overline{D})$. It can be constructed by taking any sequence Δ_n, $n \geq 1$, of quasi-rational polytopes contained in $\Gamma_{\overline{D}}$ and arbitrarily approaching this convex body. Each Δ_n induces a toric modification $\phi_n \colon X_n \to X$ by considering a complete and regular refinement of Σ_X that is compatible with the normal fan of Δ_n, and it also induces a nef toric adelic $\mathbb{R}$-divisor $\overline{P}_n$ on X_n by considering the restrictions to Δ_n of the local roof functions of $\overline{D}$.

The next result gives a lower bound for the inradius of $\overline{D}$ in terms of the convex-geometric inradius of its associated convex body.

Proposition 9.5. *Let $\overline{D}$ be a toric adelic $\mathbb{R}$-divisor on X that is big, and A a toric $\mathbb{R}$-divisor on X that is big and nef. Then $\rho(\overline{D}; A) \geq r(\Gamma_{\overline{D}}; \Delta_A)$.*

Proof. For the sequence of toric nef approximations in (9.9) we have that

$$\rho(\overline{D}; A) \geq r(P_n; \phi_n^* A) = r(\Delta_n; \Delta_A) \quad \text{for } n \geq 1,$$

by Proposition 9.1 and the fact that the polytope of a toric $\mathbb{R}$-divisor is invariant with respect to toric modifications. Since the sequence $(\Delta_n)_n$ approaches $\Gamma_{\overline{D}}$ from the inside, we get $\rho(\overline{D}; A) \geq \sup_n r(\Delta_n; \Delta_A) = r(\Gamma_{\overline{D}}; \Delta_A)$ as stated. □

Finally, we give a convex-analytic formula for the positive arithmetic intersection numbers in the toric setting.

Proposition 9.6. *Let $\overline{D}$ and $\overline{E}$ be toric adelic $\mathbb{R}$-divisors on X with $\overline{D}$ big and $\overline{E}$ semipositive. Then*

$$\mathrm{vol}(R^0(\overline{D})) = d!\,\mathrm{vol}(\Gamma_{\overline{D}}), \quad (\langle \overline{D}^d\rangle \cdot \overline{E}) = \sum_{v\in\mathfrak{M}_K} n_v\, \mathrm{MI}_M(\vartheta_{\overline{D},v}|_{\Gamma_{\overline{D}}}, \dots, \vartheta_{\overline{D},v}|_{\Gamma_{\overline{D}}}, \vartheta_{\overline{E},v}).$$

Proof. The first formula follows from Corollary 4.6 using the sequence of toric nef approximations in (9.9) and the formula for toric intersection numbers in (9.2).

For the second we use the same sequence of toric nef approximations. Applying the definition of positive arithmetic intersection numbers in (4.3) and the formula for the toric arithmetic intersection numbers in (9.6) we get

$$(\overline{P}_n^d \cdot \phi_n^* \overline{E}) = \sum_{v \in \mathfrak{M}_K} n_v \operatorname{MI}_M(\vartheta_{\overline{D},v}|_{\Delta_n}, \dots, \vartheta_{\overline{D},v|_{\Delta_n}}, \vartheta_{\overline{E},v}).$$

We conclude by taking the limit $n \to \infty$ and applying Proposition A.9. □

9.3. **Equidistribution on toric varieties.** Here we apply the constructions and results explained in §9.1, §9.2 and Appendix A to state and prove our toric differentiability and equidistribution results. In spite these preparations, we still need to assume a certain working knowledge in the Arakelov geometry of toric varieties.

Throughout this section we denote by X a projective toric variety with torus $\mathbb{T}$ and $\overline{D}$ a toric adelic $\mathbb{R}$-divisor on X with D big. For each $t \le \mu^{\mathrm{ess}}(\overline{D})$ we consider the *sup-level set* of its global roof function, that is

$$S_t(\vartheta_{\overline{D}}) = \{x \in \Delta_D \mid \vartheta_{\overline{D}}(x) \ge t\}.$$

It is a nonempty compact subset of the polytope Δ_D that is d-dimensional whenever $t < \mu^{\mathrm{ess}}(\overline{D})$. Set also $\Delta_{D,\max} = S_{\mu^{\mathrm{ess}}(\overline{D})}(\vartheta_{\overline{D}})$.

The global roof function is said to be *wide* if after fixing an arbitrary norm on $M_{\mathbb{R}}$, the width of these sup-level sets remains relatively large as the level approaches its maximum (Definition A.6). By Proposition A.3, this can be alternatively expressed as the condition that the inradius of these sup-level sets with respect to any fixed convex body remains relatively large as the level approaches the maximum. By the same result, it is also equivalent to the fact that for any maximizing point $x_0 \in \Delta_{D,\max}$ the vector $0 \in N_{\mathbb{R}}$ is a vertex of the sup-differential $\partial\vartheta_{\overline{D}}(x_0) \subset N_{\mathbb{R}}$ (Definition A.2). When this is the case, by Proposition A.8 we can associate to $\vartheta_{\overline{D}}$ a unique *balanced family of sup-gradients*

$$u_v \in N_{\mathbb{R}}, \quad v \in \mathfrak{M}_K,$$

with respect to its decomposition into local roof functions in (9.4).

We next describe the probability measures that appear as equidistribution measures in the toric setting. To this end, when v is non-Archimedean we consider the section of the v-adic valuation map

$$\zeta_v \colon N_\Sigma \longrightarrow X_v^{\mathrm{an}} \tag{9.10}$$

defined as the composition $\zeta_v = \theta_\Sigma \circ \boldsymbol{e}$ for the maps $\boldsymbol{e}$ and θ_Σ respectively defined in [BPS14, §4.1 and §4.2]. By Proposition-Definition 4.2.12 in *loc. cit.*, the section ζ_v is continuous and proper. For $u \in N_{\mathbb{R}}$ we have that $\zeta_v(u) \in X_{0,v}^{\mathrm{an}}$, and this point can be explicitly written down as the multiplicative seminorm on $K[M]$ defined as

$$|f|_{\zeta_v(u)} = \max_m |\alpha_m|_v \, e^{-\langle m,u\rangle} \quad \text{for } f = \sum_m \alpha_m \chi^m \in K[M].$$

In particular, $\zeta_v(0)$ is the Gauss point of $X_{0,v}^{\mathrm{an}} = \mathbb{T}_v^{\mathrm{an}}$.

Definition 9.7 ([BPRS19, Definition 5.1])**.** Given $v \in \mathfrak{M}_K$ and $u \in N_{\mathbb{R}}$, the probability measure $\eta_{v,u}$ on X_v^{an} is defined as follows.

(1) When v is Archimedean, we have that $\mathrm{val}_v^{-1}(u) = \mathbb{S}_v \cdot x$ for any $x \in \mathrm{val}_v^{-1}(u)$. Then $\eta_{v,u}$ is the direct image under the translation by x of the Haar probability measure of the real torus $\mathbb{S}_v \simeq (S^1)^d$.
(2) When v is non-Archimedean, $\eta_{v,u}$ is defined as the Dirac measure at the point $\zeta_v(u) \in X_v^{\mathrm{an}}$.

The following theorem is the central result of this section. It corresponds to Theorem 1.3 together with the height convergence property (1.5) in the introduction.

Theorem 9.8. *If $\vartheta_{\overline{D}}$ is wide then the essential minimum function is differentiable at $\overline{D}$, and for every $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with E toric we have*

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u_v}$$

where $u_v \in N_{\mathbb{R}}$, $v \in \mathfrak{M}_K$, is the balanced family of sup-gradients for $\vartheta_{\overline{D}}$.

In particular $\overline{D}$ satisfies the equidistribution property, and for each $v \in \mathfrak{M}_K$ we have $\nu_{\overline{D},v} = \eta_{v,u_v}$.

This is an application of Theorem 6.7, or rather of its reformulation in terms of positive arithmetic intersection numbers in Theorem 6.7. It also depends on a key lemma about the decay of the sup-level sets of concave functions on convex bodies (Proposition A.3) and on some further auxiliary constructions and results that we next explain.

Following [BPS14, Definition 4.3.3], we describe an averaging process allowing to pass from an arbitrary adelic $\mathbb{R}$-divisor on X to a toric one.

Definition 9.9. Let $\overline{E}$ be an adelic $\mathbb{R}$-divisor on X with E toric. For $v \in \mathfrak{M}_K$ we consider the function $\widehat{g}_v\colon X_v^{\mathrm{an}} \setminus E_v^{\mathrm{an}} \to \mathbb{R}$ defined as

$$\widehat{g}_v(x) = \begin{cases} \displaystyle\int_{\mathbb{S}_v} g_{\overline{E},v}(t\cdot x)\, d\,\mathrm{Haar}_v(t) & \text{if } v \text{ is Archimedean,} \\ g_{\overline{E},v}(\zeta_v(\mathrm{val}_v(x))) & \text{if } v \text{ is non-Archimedean,} \end{cases}$$

with Haar_v the Haar probability measure of the real torus $\mathbb{S}_v$ when v is Archimedean, and ζ_v the section of the v-adic valuation map in (9.10) when v is non-Archimedean. We then set $\overline{E}^{\mathrm{tor}} = (E, (\widehat{g}_v)_{v\in\mathfrak{M}_K})$, which is a toric adelic $\mathbb{R}$-divisor on X.

Using the probability measures from Definition 9.7, for every $v \in \mathfrak{M}_K$ the corresponding Green function of $\overline{E}^{\mathrm{tor}}$ can be alternatively expressed as

$$\widehat{g}_v(x) = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u} \quad \text{for } u = \mathrm{val}_v(x). \tag{9.11}$$

The relevant arithmetic intersection numbers (both standard and positive) are invariant with respect to this averaging process.

Proposition 9.10. *For any toric adelic $\mathbb{R}$-divisor $\overline{P}$ on X we have $(\overline{P}^d \cdot \overline{E}) = (\overline{P}^d \cdot \overline{E}^{\mathrm{tor}})$. In particular, if $\overline{D}$ is big then $(\langle\overline{D}^d\rangle \cdot \overline{E}) = (\langle\overline{D}^d\rangle \cdot \overline{E}^{\mathrm{tor}})$.*

Proof. With notation as in Definition 9.9, by the arithmetic Bézout formula we have

$$(\overline{P}^d \cdot \overline{E}^{\mathrm{tor}}) = h_{\overline{P}}(E) + \sum_v n_v \int_{X_v^{\mathrm{an}}} \widehat{g}_v\, c_1(\overline{P}_v)^{\wedge d}.$$

Let $v \in \mathfrak{M}_K$. When v is Archimedean we have

$$\int_{X_v^{\mathrm{an}}} \widehat{g}_v \, c_1(\overline{P}_v)^{\wedge d} = \int_{X_v^{\mathrm{an}}} \Big(\int_{\mathbb{S}_v} g_{\overline{E},v}(t\cdot x) \, d\,\mathrm{Haar}_v(t) \Big) c_1(\overline{P}_v)^{\wedge d}(x)$$
$$= \int_{\mathbb{S}_v} \Big(\int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(x) \, t_* c_1(\overline{P}_v)^{\wedge d}(x) \Big) d\,\mathrm{Haar}_v(t) = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(x) \, c_1(\overline{P}_v)^{\wedge d},$$

using Fubini's theorem, the change of variable formula and the invariance of the v-adic Monge-Ampère measure of $\overline{P}$ under the action of $\mathbb{S}_v$. On the other hand, when v is non-Archimedean we have

$$\int_{X_v^{\mathrm{an}}} \widehat{g}_v \, c_1(\overline{P}_v)^{\wedge d} = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(\zeta_v(\mathrm{val}_v(x))) \, c_1(\overline{P}_v)^{\wedge d}(x) = \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}(x) \, c_1(\overline{P}_v)^{\wedge d}(x),$$

by the characterization of the Monge-Ampère measure of semipositive toric adelic divisors in [BPS14, Theorem 4.8.11].

For the second, suppose that $\overline{D}$ is big and consider a sequence of arithmetic Fujita approximations $(\phi_n \colon X_n \to X, \overline{P}_n)$, $n \in \mathbb{N}$, as in (9.9). The averaging process commutes with toric modifications as it can be read from its definition on the principal open subset, which remains unchanged under these modifications. Hence applying the first statement, for each $n \in \mathbb{N}$ we get

$$(\overline{P}_n^d \cdot \phi_n^* \overline{E}^{\mathrm{tor}}) = (\overline{P}_n^d \cdot (\phi_n^* \overline{E})^{\mathrm{tor}}) = (\overline{P}_n^d \cdot \phi_n^* \overline{E}).$$

Then this second statement follows from the definition of these positive arithmetic intersection numbers in (4.3) by taking the limit $n \to \infty$. □

Recall that $[\infty]$ denotes the adelic $\mathbb{R}$-divisor over the zero divisor of X with v-adic Green function equal to 1 if v is Archimedean and to 0 is v is non-Archimedean. As in §6.3, for each $t < \mu^{\mathrm{ess}}(\overline{D})$ we consider the twist

$$\overline{D}(t) = \overline{D} - t[\infty].$$

We have $\vartheta_{\overline{D}(t)} = \vartheta_{\overline{D}} - t$, as it can be verified from the definitions of this twist and of the global roof function. In particular we have $\Gamma_{\overline{D}(t)} = S_t(\vartheta_{\overline{D}})$ for the convex body in (9.5).

Proof of Theorem 9.8. Set $\mu = \mu^{\mathrm{ess}}(\overline{D})$ for short and let A be a big and nef $\mathbb{R}$-divisor on X. Since $\vartheta_{\overline{D}}$ is wide we have

$$\lim_{t\to\mu} \frac{\mu - t}{r(S_t(\vartheta_{\overline{D}}); \Delta_A)} = 0. \tag{9.12}$$

For each $t < \mu$ we have $S_t(\vartheta_{\overline{D}}) = \Gamma_{\overline{D}(t)}$ and so $r(S_t(\vartheta_{\overline{D}}); \Delta_A) \le \rho(\overline{D}(t); A)$ by Proposition 9.5. Hence the limit in (9.12) gives $\lim_{t\to\mu}(\mu - t)/\rho(\overline{D}(t); A) = 0$. By Theorem 6.7 this implies that the essential minimum function is differentiable at $\overline{D}$ and that for each $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have

$$\partial_{\overline{E}} \mu^{\mathrm{ess}}(\overline{D}) = \lim_{t\to\mu} \frac{\langle \overline{D}(t)^d \cdot \overline{E} \rangle}{\mathrm{vol}(R^t(\overline{D}))}.$$

The formula for this derivative will follow from this limit together with the convex analysis interpretation of positive intersection numbers in the toric setting and the averaging process previously described.

First suppose that $\overline{E}$ is both semipositive and toric. Let $\mathfrak{S} \subset \mathfrak{M}_K$ be a finite set of places such that $\psi_{\overline{D},v} = \Psi_D$ and $u_v = 0$ for all $v \notin \mathfrak{S}$. By possibly enlarging $\mathfrak{S}$ we can also assume that $\vartheta_{\overline{E},v} = 0|_{\Delta_E}$ for all $v \notin \mathfrak{S}$.

Let $t < \mu$ and set for short $S_t = S_t(\vartheta_{\overline{D},v})$. Set $\varepsilon_v = 1$ if v is Archimedean and $\varepsilon_v = 0$ if v is non-Archimedean, and set also $\theta_{v,t} = (\vartheta_{\overline{D},v} - \varepsilon_v t)|_{S_t}$ for each v. By Proposition 9.6 we have

$$\operatorname{vol}(R^t(\overline{D})) = d! \operatorname{vol}(S_t) \quad \text{and} \quad \langle \overline{D}(t)^d \cdot \overline{E} \rangle = \sum_{v \in \mathfrak{S}} n_v \operatorname{MI}_M(\theta_{v,t}, \dots, \theta_{v,t}, \vartheta_{\overline{E},v}). \tag{9.13}$$

For $v \in \mathfrak{S}$ we have $\vartheta_{\overline{D},v}(x) \leq \langle u_v, x\rangle + c_v$ for $x \in \Delta_D$ with $c_v = \vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0\rangle$ for any $x_0 \in \Delta_{D,\max}$. Setting also $\kappa_v = \max_{x \in S_t}(\langle u_v, x\rangle + c_v - \vartheta_{\overline{D},v}(x)) \geq 0$ we have the estimates

$$\langle u_v, x\rangle + c_v - \kappa_v - \varepsilon_v t \leq \theta_{v,t}(x) \leq \langle u_v, x\rangle + c_v - \varepsilon_v t \quad \text{for } x \in S_t. \tag{9.14}$$

We have $\sum_v n_v c_v = \sum_v n_v\, (\vartheta_{\overline{D},v}(x_0) - \langle u_v, x_0\rangle) = \vartheta_{\overline{D}}(x_0) = \mu$. On the other hand, for each $v_0 \in \mathfrak{S}$ and any $x \in S_t$ we have

$$\mu - t \geq \mu - \vartheta_{\overline{D},v}(x) = \sum_{v \in \mathfrak{S}} n_v(c_v + \langle u_v, x\rangle - \vartheta_{\overline{D},v}(x)) \geq n_{v_0}(c_{v_0} + \langle u_{v_0}, x\rangle - \vartheta_{\overline{D},v_0}(x))$$

using the fact that $(u_v)_v$ is balanced together with the previous upper bound for the v-adic roof functions for $v \neq v_0$. Since this holds for every $x \in S_t$ we deduce that $0 \leq n_{v_0}\kappa_{v_0} \leq \mu - t$, and in particular that $0 \leq \sum_v n_v \kappa_v \leq \#\mathfrak{S}(\mu - t)$ where $\#\mathfrak{S}$ denote the cardinality of this finite set of places.

By the monotonicity of the mixed integral we have

$$\operatorname{MI}_M(\theta_{v,t}, \dots, \theta_{v,t}, \vartheta_{\overline{E},v}) \leq \operatorname{MI}_M((\langle u_v, x\rangle + c_v - \varepsilon_v t)|_{S_t}, \dots, (\langle u_v, x\rangle + c_v - \varepsilon_v t)|_{S_t}, \vartheta_{\overline{E},v}).$$

By Lemma A.10, the mixed integral in the right-hand side of this inequality can be computed as

$$-d! \operatorname{vol}(S_t)\, \psi_{\overline{E},v}(u_v) + d(c_v - \varepsilon_v t) \operatorname{MV}_M(S_t, \dots, S_t, \Delta_E) + \langle x_1, u_v\rangle$$

for a point $x_1 \in M_{\mathbb{R}}$ which does not depend on v, as explained in Remark A.11. Summing over all these places, it follows from the formulae in (9.13) that

$$\frac{\langle \overline{D}(t)^d \cdot \overline{E}\rangle}{\operatorname{vol}(R^t(\overline{D}))} \leq -\sum_v n_v \psi_{\overline{E},v}(u_v) + \frac{d(\mu - t) \operatorname{MV}_M(S_t, \dots, S_t, \Delta_E)}{d! \operatorname{vol}(S_t)},$$

and proceeding similarly with the lower bound in (9.14) we also obtain

$$\frac{\langle \overline{D}(t)^d \cdot \overline{E}\rangle}{\operatorname{vol}(R^t(\overline{D}))} \geq -\sum_v n_v \psi_{\overline{E},v}(u_v) - \frac{d(\#\mathfrak{S} - 1)(\mu - t) \operatorname{MV}_M(S_t, \dots, S_t, \Delta_E)}{d! \operatorname{vol}(S_t)},$$

Using the comparison between the inradius and the quotient of intersection numbers in Lemma 2.1 together with the interpretation of mixed volumes as intersection numbers of toric $\mathbb{R}$-divisors in (9.2) and the limit in (9.12) we get

$$\partial_{\overline{E}} \mu^{\mathrm{ess}}(\overline{D}) = -\sum_v n_v \psi_{\overline{E},v}(u_v). \tag{9.15}$$

By the additivity of both the derivative and the metric functions, this formula readily extends to the case when $\overline{E}$ is just DSP, and by density it extends to any toric adelic $\mathbb{R}$-divisor on X.

For an arbitrary adelic $\mathbb{R}$-divisor $\overline{E}$ with E toric we apply the averaging process in Definition 9.9. By the invariance of the positive arithmetic intersection numbers with respect to this process (Proposition 9.10) we deduce from the identities in (9.15) and (9.11) that

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \partial_{\overline{E}^{\mathrm{tor}}}\mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v\,\widehat{g}_v(x_v) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u_v}$$

where each $x_v \in X_v^{\mathrm{an}}$ is a point in the fiber $\mathrm{val}_v^{-1}(u_v)$.

Finally, by Proposition 5.6 this readily implies that $\overline{D}$ satisfies the equidistribution property with $\nu_{\overline{D},v} = \eta_{v,u_v}$ for all v. □

We can extend the range of Theorem 9.8 to compute the derivative of the essential minimum function along any adelic $\mathbb{R}$-divisor on X by reducing to the situation considered therein.

Remark 9.11. Let $\overline{E}$ be an adelic $\mathbb{R}$-divisor on X. The K-algebra $\mathcal{O}(\mathbb{T}) = K[M]$ is factorial and so we can choose $f \in \mathrm{Rat}(X)_{\mathbb{R}}^{\times}$ defining the restriction of E to the torus. Set then

$$\overline{F} = \overline{E} - \widehat{\mathrm{div}}(f)$$

Its geometric divisor F is toric because its support is contained in the boundary $X \setminus X_0$ and its v-adic Green function is related to that of $\overline{E}$ by $g_{\overline{F},v} = g_{\overline{E},v} + \log|f|_v$.

Since the essential minimum function is invariant under linear equivalence, its derivative in the direction of $\overline{E}$ coincides with that in the direction of $\overline{F}$. Hence we can apply Theorem 9.8 to compute it as

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u_v} + \sum_{v\in\mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \log|f|_v \, d\eta_{v,u_v}.$$

In the semipositive case, we can prove the following converse of Theorem 9.8. It is a direct consequence of Theorem 9.8 together with Propositions 9.4 and 6.13.

Corollary 9.12. *With notation as in Theorem 9.8, the following conditions are equivalent:*

(1) $\vartheta_{\overline{D}}$ is wide,

(2) $\overline{D}$ satisfies the equidistribution property.

When any of them is satisfied, for each $v \in \mathfrak{M}_K$ we have $\nu_{\overline{D},v} = \eta_{v,u_v}$.

Remark 9.13. A key notion in [BPRS19] is that of *monocritical* semipositive toric adelic $\mathbb{R}$-divisor. Assuming that $\overline{D}$ is semipositive, this condition is expressed as the fact that certain associated function on a space of adelic measures has a unique minimizing adelic point. Then [BPRS19, Theorem 1.1] states that $\overline{D}$ is monocritical if and only if it verifies the equidistribution property at every place, with equidistribution measures determined by this adelic point when it exists.

By Proposition 4.15 in *loc.cit.*, the condition that $\overline{D}$ is monocritical can be reformulated in more elementary terms as the fact that 0 is not a vertex of $\partial\vartheta_{\overline{D}}(x_0)$ for any $x_0 \in \Delta_{D,\max}$, with the minimizing adelic point given by the balanced family of sup-gradients for $\vartheta_{\overline{D}}$. By Proposition A.3 this condition is equivalent to the fact that the global roof function is wide, and so Corollary 9.12 recovers this toric equidistribution theorem. On the other hand, Theorem 9.8 extends the sufficient condition in this criterion to the case when $\overline{D}$ is not necessarily semipositive and strengthens its conclusion to include the differentiability of the essential minimum function. Furthermore, it adds the even more elementary interpretation of this condition in terms of the decay of the width sup-level sets of the global roof function.

For the rest of this subsection we assume that $\vartheta_{\overline{D}}$ is wide and we denote by $(u_v)_v$ the associated balanced family of sup-gradients. Then for each $f \in K[M] \setminus \{0\}$ we consider the quantity

$$m_{\overline{D}}(f) = \sum_{v \in \mathfrak{M}_K} n_v \int_{X_v^{\mathrm{an}}} \log |f|_v \, d\eta_{v,u_v} \in \mathbb{R}.$$

Taking into account the definition of the probability measures η_{v,u_v} and writing $f = \sum_{m \in M} \alpha_m \chi^m$, this can be alternatively expressed as

$$(9.16) \quad m_{\overline{D}}(f) = \sum_{v \in \mathfrak{M}_K^\infty} n_v \int_{\mathbb{S}_v} \log |f(x \cdot t_v)|_v \, d\,\mathrm{Haar}_v(x) + \sum_{v \in \mathfrak{M}_K \setminus \mathfrak{M}_K^\infty} n_v \log \max_m (e^{-\langle u_v, m\rangle} |\alpha_m|_v)$$

where as before Haar_v denotes the Haar probability measure of the real torus $\mathbb{S}_v$, and $t_v \in \mathbb{T}_v^{\mathrm{an}}(\mathbb{C}_v)$ is any point in the fiber $\mathrm{val}_v^{-1}(u_v)$. Hence the defined quantity is an extension of the classical logarithmic Gauss-Mahler measure of a Laurent polynomial, which corresponds to the case when $u_v = 0$ for all v.

Lemma 9.14. *The following properties hold:*
(1) for $f \in K[M] \setminus \{0\}$ we have $m_{\overline{D}}(f) \geq 0$,
(2) for $m \in M$ and $\alpha \in K^\times$ we have $m_{\overline{D}}(\alpha \chi^m) = 0$.
(3) for $m \in M \setminus \{0\}$ and $\gamma \in K^\times$ we have

$$m_{\overline{D}}(\chi^m - \gamma) = \sum_{v \in \mathfrak{M}_K} n_v \log \max(1, e^{-\langle u_v, m\rangle} |\gamma|_v).$$

Proof. The statement (1) might be deduced from Lemma 8.2 and the expression for the derivatives of the essential minimum in Remark 9.11. It can be alternatively deduced from standard properties of the Mahler measure, as we next explain.

Let m_0 be a vertex of the Newton polytope of f. Then for each $v \in \mathfrak{M}_K$ we have

$$\int_{X_v^{\mathrm{an}}} \log |f|_v \, d\eta_{v,u_v} \geq \log |\alpha_{m_0}|_v - \langle u_v, m_0\rangle.$$

In case v is Archimedean this follows from the fact that the Mahler measure of a Laurent polynomial is bounded below by the absolute value of any of its vertex coefficients. On the other hand, in case v is non-Archimedean this inequality is given by the explicit expression of this local term in (9.16). Hence

$$m_{\overline{D}}(f) \geq \sum_{v \in \mathfrak{M}_K} n_v \left(\log |\alpha_{m_0}|_v - \langle u_v, m_0\rangle\right) = 0$$

by the product formula of K and the fact that $(u_v)_v$ is balanced.

For (2), for each v we have $\int_{X_v^{\mathrm{an}}} \log |\alpha \chi^m|_v \, d\eta_{v,u_v} = \log |\alpha|_v - \langle u_v, m\rangle$, and so the statement follows again from the product formula and the fact that $(u_v)_v$ is balanced.

For (3), for each v we have

$$\int_{X_v^{\mathrm{an}}} \log|\chi^m-\gamma|_v\, d\eta_{v,u_v} = \log\max(1,|\chi^m(t_v)\gamma|_v) = \log\max(1, e^{-\langle u_v,m\rangle}|\gamma|_v)$$

by Jensen's formula for the Mahler measure in the Archimedean case, and the explicit expression (9.16) in the non-Archimedean case. The statement follows by considering the weighted sum of these terms. □

Combining Theorem 9.8 with the results in §8 we can strengthen the toric equidistribution property to include test functions with logarithmic singularities along effective divisors satisfying a numerical condition.

Theorem 9.15. *Let $\overline{D}$ be a toric adelic $\mathbb{R}$-divisor on X with D big and such that $\vartheta_{\overline{D}}$ is wide. Let E be an effective divisor on X whose restriction to X_0 is defined by a Laurent polynomial $f\in K[M]\setminus\{0\}$ with $m_{\overline{D}}(f)=0$. Then for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $v\in\mathfrak{M}_K$ we have*

$$\lim_{\ell\to\infty}\int_{X_v^{\mathrm{an}}}\varphi\, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}}\varphi\, d\eta_{v,u_v}$$

for any function $\varphi\colon X_v^{\mathrm{an}}\to\mathbb{R}\cup\{\pm\infty\}$ with at most logarithmic singularities along E.

In particular, this holds if each irreducible component of E is either contained in the boundary $X\setminus X_0$, or is the closure of the zero set of an irreducible binomial $\chi^m-\gamma$ with $m\in M\setminus\{0\}$ and $\gamma\in K^\times$ such that $\log|\gamma|_v=\langle u_v,m\rangle$ for all v.

Proof. By Remark 9.11 we have

$$\partial_{\overline{E}}\mu^{\mathrm{ess}}(\overline{D}) = \sum_{v\in\mathfrak{M}_K} n_v\int_{X_v^{\mathrm{an}}} g_{\overline{E},v}\, d\eta_{v,u_v} + m_{\overline{D}}(f).$$

The first statement is then a direct application of Theorem 8.4.

For the second statement, let $[E]$ denote the associated Weil divisor and consider its decomposition into irreducible components $[E]=\sum_{i\in I}k_iW_i$. For each $i\in I$ we choose a Laurent polynomial f_i defining the restriction $W_i\cap X_0$. Then $f=\prod_{i\in I}f_i^{k_i}$ defines the restriction of E to the torus. Using Lemma 9.14 we deduce that $m_{\overline{D}}(f_i)=0$ for each i and that

$$m_{\overline{D}}(f)=\sum_{i\in I}k_i\, m_{\overline{D}}(f_i)=0,$$

and so this statement follows from the first. □

It is natural to try to interpret in terms of heights the numerical condition imposed on the effective divisor by the previous theorem. To this end, first note that for every point $x\in X_0(\overline{K})$ we have $h_{\overline{D}}(x)\ge\mu^{\mathrm{ess}}(\overline{D})$ [BPS15, Lemma 3.8(1)], and so for every subvariety $V\subset X_{0,\overline{K}}$ we have

$$\mu^{\mathrm{ess}}(\overline{D}|_{\overline{V}})\ge\mu^{\mathrm{ess}}(\overline{D})$$

where $\overline{V}$ denotes its closure. Following [BPRS19, Definition 5.10], we say that V is $\overline{D}$-*special* if this lower bound is an equality.

Using the characterization of the Bogomolov property for monocritical semipositive toric adelic $\mathbb{R}$-divisors in [BPRS19, §5] we derive the following logarithmic equidistribution theorem for the semipositive case.

Corollary 9.16. *With notation as in Theorem 9.15, assume furthermore that $\overline{D}$ is semipositive. Let E be an effective divisor on X such that each of its geometric irreducible components is either contained in $X \setminus X_0$ or is the closure of a $\overline{D}$-special hypersurface of $X_{0,\overline{K}}$. Then for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $X(\overline{K})$ and $v \in \mathfrak{M}_K$ we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi \, d\eta_{v,u_v}.$$

for any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along E.

Proof. Since $\overline{D}$ is semipositive and $\vartheta_{\overline{D}}$ is wide, $\overline{D}$ is monocritical in the sense of [BPRS19], Remark 9.13. Let $W \subset X_{\overline{K}}$ be a geometric irreducible component of E such that $W_0 \coloneqq W \cap X_{0,\overline{K}}$ is $\overline{D}$-special. By the Bogomolov property for monocritical adelic $\mathbb{R}$-divisors [BPRS19, Theorem 5.12], W_0 is the translate of a subtorus. Since W_0 is a hypersurface, this implies that there exist $m \in M$ and $x_0 \in X_{0,\overline{K}}(\overline{K})$ such that

$$W_0 = Z(\chi^m - 1) \cdot x_0.$$

After extending the base field if necessary, we assume without loss of generality that $x_0 \in X_0(K)$. Note that $W_0 = Z(\chi^m - \gamma)$ for $\gamma = \chi^m(x_0)$. By [BPRS19, Proposition 5.14(1)], the fact that W_0 is $\overline{D}$-special implies that

$$u_v \in m_{\mathbb{R}}^{\perp} + \mathrm{val}_v(x_0) \quad \text{for all } v,$$

which is equivalent to the fact that $\langle u_v, m\rangle = \langle \mathrm{val}_v(x_0), m\rangle = \log|\gamma|_v$ for all v. We conclude with Theorem 9.15. □

10. Dynamical systems and semiabelian varieties

Here we study the adelic $\mathbb{R}$-divisors that arise as sums of several canonical adelic $\mathbb{R}$-divisors with different regimes with respect to an algebraic dynamical system. In this setting, Zhang's lower bound for the essential minimum might be strict, in which case Yuan's theorem cannot be applied. In spite of this situation, the essential minimum function is differentiable at such adelic $\mathbb{R}$-divisors, and for every place of K the Galois orbits of the points in a small generic sequence converge towards the normalized Monge-Ampère measure (Theorem 10.8). We also show that this equidistribution still holds for test functions with logarithmic singularities along hypersurfaces containing a dense subset of preperiodic points (Theorem 10.10).

These results apply in the context of semiabelian varieties, giving the differentiability of the essential minimum function (Theorem 10.11) and allowing to recover Kühne's semiabelian equidistribution theorem (Remark 10.12). Furthermore, they imply that this equidistribution also holds for test functions with logarithmic singularities along torsion hypersurfaces (Theorem 10.15).

10.1. **Canonical adelic $\mathbb{R}$-divisors.** Zhang's canonical metrics for dynamical systems in [Zha95b, §2] extend to the setting of adelic $\mathbb{R}$-divisors, as shown by Chen and Moriwaki [CM15]. Here we recall this notion and study some of its basic positivity properties.

Let X be a normal projective variety over K of dimension $d \geq 1$ and $\phi\colon X \to X$ a surjective endomorphism. Let $D \in \mathrm{Div}(X)_{\mathbb{R}}$ such that $\phi^*D \equiv qD$ for a real number

$q > 1$. To define the associated canonical adelic $\mathbb{R}$-divisor, choose $f \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that

$$\phi^* D = qD + \mathrm{div}(f).$$

By [CM15, §4], for every $v \in \mathfrak{M}_K$ there is a unique v-adic Green function g_v on D such that $\phi_v^{\mathrm{an},*} g_v = q\, g_v - \log |f|_v$, and moreover $\overline{D} := (D, (g_v)_{v \in \mathfrak{M}_K})$ is an adelic $\mathbb{R}$-divisor on X which verifies that

$$\phi^* \overline{D} = q\overline{D} + \widehat{\mathrm{div}}(f). \tag{10.1}$$

This adelic $\mathbb{R}$-divisor is independent of the choice of f up to a summand of the form $\widehat{\mathrm{div}}(\lambda)$ with $\lambda \in K^{\times}_{\mathbb{R}}$. Indeed, if $f' \in \mathrm{Rat}(X)_{\mathbb{R}}$ also verifies that $\phi^* D = q\, D + \mathrm{div}(f')$ then necessarily $f' = \gamma f$ with $\gamma \in K^{\times}_{\mathbb{R}}$. Hence for each $v \in \mathfrak{M}_K$ the v-adic Green function induced by f' coincides with $g_v + \log |\lambda|_v$ for $\lambda = \gamma^{1/(q-1)}$ and the adelic $\mathbb{R}$-divisor induced by f' is $\overline{D} + \widehat{\mathrm{div}}(\lambda)$.

Definition 10.1. The *canonical* adelic $\mathbb{R}$-divisor of D, denoted by $\overline{D}^{\mathrm{can}}$, is any adelic $\mathbb{R}$-divisor on X such that

$$\phi^* \overline{D}^{\mathrm{can}} \equiv q \overline{D}^{\mathrm{can}} \quad \text{on } \widehat{\mathrm{Div}}(X)_{\mathbb{R}}. \tag{10.2}$$

It is unique up to summand of the form $\widehat{\mathrm{div}}(\lambda)$ with $\lambda \in K^{\times}_{\mathbb{R}}$.

The associated height function is not affected by this indeterminacy, thanks to the product formula. By (10.2), it verifies that

$$h_{\overline{D}^{\mathrm{can}}}(\phi(x)) = q\, h_{\overline{D}^{\mathrm{can}}}(x) \quad \text{for } x \in X(\overline{K}). \tag{10.3}$$

A point $x \in X(\overline{K})$ is *preperiodic* if its orbit with respect to ϕ is finite or equivalently, if there are positive integers $j < k$ such that $\phi^{\circ j}(x) = \phi^{\circ k}(x)$. The functoriality in (10.3) implies that $h_{\overline{D}^{\mathrm{can}}}(x) = 0$ whenever x is preperiodic.

It is well-known that if D is ample then $\overline{D}^{\mathrm{can}}$ is nef and both the absolute and the essential minima vanish. In a similar vein, we give a simple condition ensuring the pseudo-effectivity of the canonical adelic $\mathbb{R}$-divisor and the vanishing of its essential minimum.

Proposition 10.2. *If $R(D) \neq \{0\}$, then $\overline{D}^{\mathrm{can}}$ is pseudo-effective and $\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = 0$.*

Proof. We prove that the essential minimum vanishes. First note that it is finite because $R(D) \neq \{0\}$. We have $\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = \mu^{\mathrm{ess}}(\phi^* \overline{D}^{\mathrm{can}})$ because the essential minimum is invariant with respect to dominant and generically finite morphisms [BPS15, Proposition 3.5(1)]. Furthermore

$$\mu^{\mathrm{ess}}(\phi^* \overline{D}^{\mathrm{can}}) = \mu^{\mathrm{ess}}(q\overline{D}^{\mathrm{can}}) = q\, \mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}),$$

by the functoriality of the canonical adelic $\mathbb{R}$-divisor and the invariance of the essential minimum under linear equivalence. Hence $\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = q\, \mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}})$, which gives $\mu^{\mathrm{ess}}(\overline{D}^{\mathrm{can}}) = 0$ as stated.

The pseudo-effectivity of $\overline{D}^{\mathrm{can}}$ then follows from Theorem 3.16. For the convenience of the reader, we give an elementary proof of this fact. Let $s = (f, eD)$ be a nonzero global section of eD for some $e \geq 1$, which exists because $R(D) \neq \{0\}$. Up to multiplying s by a nonzero scalar, we can suppose that $\|s\|_{v,\sup} \leq 1$ for every non-Archimedean place v. Then given $\varepsilon > 0$ take $k \geq 1$ such that $\log \|s\|_{v,\sup} \leq \varepsilon\, e\, q^k$ for every Archimedean place v.

The pullback $\phi^{\circ k,*}s = (\phi^{\circ k,*}f, e\,\phi^{\circ k,*}D)$ is a nonzero global section of $e\,\phi^{\circ k,*}D$ and since ϕ is surjective, it has the same v-adic sup-norms as s. Since $\phi^*\overline{D}^{\mathrm{can}} \equiv q\,\overline{D}^{\mathrm{can}}$, there is a nonzero global section s_k of eq^kD with the same v-adic sup-norms. Hence

$$\log \|s_k\|_{eq^k\overline{D}^{\mathrm{can}},v,\sup} = \log \|s\|_{\overline{D}^{\mathrm{can}},v,\sup} \le \begin{cases} \varepsilon\, e\, q^k & \text{if } v \text{ is Archimedean,} \\ 0 & \text{otherwise,} \end{cases}$$

and so $s_k \in R^{-\varepsilon}(\overline{D}^{\mathrm{can}})$. Therefore $R^{-\varepsilon}(\overline{D}^{\mathrm{can}}) \neq \{0\}$ for every $\varepsilon > 0$ and $\overline{D}^{\mathrm{can}}$ is pseudo-effective. □

We also have the next condition for its effectivity.

Proposition 10.3. *If D is effective and $\phi^*D = qD$, then $\overline{D}^{\mathrm{can}}$ is effective.*

Proof. Since D is effective, its canonical Green functions are bounded below. Furthermore, by the construction in (10.1) the equality $\phi^*D = qD$ lifts to an equality of adelic $\mathbb{R}$-divisors $\phi^*\overline{D}^{\mathrm{can}} = q\overline{D}^{\mathrm{can}}$. Hence for each $v \in \mathfrak{M}_K$ we have $\phi^*g_{\overline{D}^{\mathrm{can}},v} = q\, g_{\overline{D}^{\mathrm{can}},v}$ and so

$$\inf_{x\in X_v^{\mathrm{an}}} g_{\overline{D}^{\mathrm{can}}}(x) = q \inf_{x\in X_v^{\mathrm{an}}} g_{\overline{D}^{\mathrm{can}}}(x)$$

because ϕ is surjective. Hence this infimum is zero and so $\overline{D}^{\mathrm{can}}$ is effective. □

10.2. **Equidistribution for sums of canonical adelic $\mathbb{R}$-divisors.** Let $\phi\colon X \to X$ be a surjective endormorphism of a normal variety over K as in §10.1. This endomorphism is finite [Fak03, Lemma 5.6] and we denote by $\deg(\phi)$ its degree.

For $i = 1, \dots, s$ let $D_i \in \mathrm{Div}(X)_{\mathbb{R}}$ with $\phi^*D_i \equiv q_iD_i$ for a real number $q_i > 1$ such that $R(D_i) \neq \{0\}$ and $\overline{D}_i^{\mathrm{can}}$ is semipositive. Set

$$\overline{D} = \sum_{i=1}^{s} \overline{D}_i^{\mathrm{can}}, \tag{10.4}$$

and suppose furthermore that D is ample. Up to reordering, we can also suppose that $1 < q_1 \le q_2 \le \cdots \le q_s$.

Since $\overline{D}_i^{\mathrm{can}}$ is semipositive, we have that D_i is nef for each i and so

$$\phi^*D - D = \sum_{i=1}^{s}(q_i-1)D_i = (q_1-1)D + \sum_{i=1}^{s}(q_i-q_1)D_i$$

is ample, because it is the sum of an ample $\mathbb{R}$-divisor with a nef one. By Fakhruddin's theorem [Fak03, Theorem 5.1], this implies that the set of periodic points of ϕ is dense.

Next we study the basic properties of this adelic $\mathbb{R}$-divisor, starting with its its essential minimum, intersection numbers and Monge-Ampère measures.

Proposition 10.4. *We have $\mu^{\mathrm{ess}}(\overline{D}) = 0$.*

Proof. For every preperiodic point x we have $h_{\overline{D}}(x) = \sum_{i=1}^{s} h_{\overline{D}_i^{\mathrm{can}}}(x) = 0$. Since the set of such points is dense, we get $\mu^{\mathrm{ess}}(\overline{D}) \le 0$. On the other hand, by Proposition 10.2 we have $\mu^{\mathrm{ess}}(\overline{D}_i^{\mathrm{can}}) = 0$ for each i, and so Lemma 3.15(1) gives the reverse inequality $\mu^{\mathrm{ess}}(\overline{D}) \ge \sum_{i=1}^{s}\mu^{\mathrm{ess}}(\overline{D}_i^{\mathrm{can}}) = 0$. □

Denote by $\mathbb{N}_d^s \subset \mathbb{N}^s$ the set of s-tuples of nonnegative integers whose components sum up to d. For each $a \in \mathbb{N}_d^s$ set $q^a = \prod_{i=1}^{s} q_i^{a_i}$ and consider the subset

$$I = \{a \in \mathbb{N}_d^s \mid q^a = \deg(\phi)\}. \tag{10.5}$$

Proposition 10.5. *We have*

$$(D^d) = \sum_{a\in I} \binom{d}{a} \Big(\prod_{i=1}^{s} D_i^{a_i}\Big) > 0 \quad \textit{and} \quad c_1(\overline{D}_v)^{\wedge d} = \sum_{a\in I} \binom{d}{a} \bigwedge_{i=1}^{s} c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i}.$$

Hence if $(\prod_{i=1}^s D_i^{a_i}) > 0$ *for some* $a \in \mathbb{N}_d^s$ *then* $a \in I$. *In particular* $I \neq \emptyset$.

Proof. We have $\deg(\phi)^k (D^d) = ((\phi^{\circ k,*}D)^d) = ((\sum_{i=1}^s q_i^k D_i)^d)$ by the projection formula [Ful98, Proposition 2.3(c)]. Therefore

$$(D^d) = \sum_{a\in \mathbb{N}_d^s} \binom{d}{a} \frac{q^{ka}}{\deg(\phi)^k} \Big(\prod_{i=1}^{s} D_i^{a_i}\Big).$$

Since the right-hand side is constant with respect to k, this has to be the case for the left-hand side. Hence the terms corresponding to exponents outside I vanish, which gives the first formula. Its positivity comes from the fact that D is ample.

On the other hand, the multilinearity of the Monge-Ampère operator gives

$$c_1(\overline{D}_v)^{\wedge d} = \sum_{a\in \mathbb{N}_d^s} \binom{d}{a} \bigwedge_{i=1}^{s} c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i}.$$

Since $\overline{D}_i$ is semipositive, for each $a \in \mathbb{N}_d^s$ we have that $\bigwedge_{i=1}^s c_1(\overline{D}_{i,v}^{\mathrm{can}})^{\wedge a_i}$ is a measure of total mass $(\prod_{i=1}^s D_i^{a_i})$. Hence this measure is zero unless $a \in I$, which gives the second formula.

The last statement comes readily from the first formula and its positivity. □

The next proposition summarizes the properties of the pullback with respect to ϕ of this adelic $\mathbb{R}$-divisor.

Proposition 10.6. *We have*

(1) $r(\phi^* D; D) \geq q_1$,

(2) $\mu^{\mathrm{abs}}(\phi^*\overline{D}) = \mu^{\mathrm{abs}}(\overline{D})$,

(3) $(\phi^* D^d) = \deg(\phi) \times (D^d)$,

(4) $c_1(\phi^*\overline{D}_v)^{\wedge d} = \deg(\phi)\, c_1(\overline{D}_v)^{\wedge d}$.

Proof. For (1) we have $R(\phi^*D - q_1 D) \neq \{0\}$ because $\phi^*D - q_1 D_1 = \sum_{i=1}^s (q_i - q_1) D_i$ and $R(D_i) \neq \{0\}$ for all i. This gives $r(\phi^*D; D) \geq q_1$, as stated.

The other statements hold in greater generality: (2) is the invariance of the absolute minimum with respect to pullbacks by surjective morphisms, whereas (3) and (4) follow from the functoriality of the intersection numbers and of the Monge-Ampère operator, see for instance [BPS14, Proposition 1.4.8]. □

Remark 10.7. Let $v \in \mathfrak{M}_K$. As usual we can consider the pushforward of a measure on X_v^{an} with respect to the analytification ϕ_v^{an}, and since ϕ is finite we can also consider its pullback, as explained in [Cha06, §2.8]. In our present situation, the v-adic Monge-Ampère measure of $\overline{D}$ is invariant with respect to the dynamical system, in the sense that

$$\phi_{v,*}^{\mathrm{an}} c_1(\overline{D}_v)^{\wedge d} = c_1(\overline{D}_v)^{\wedge d} \quad \text{and} \quad \phi_v^{\mathrm{an},*} c_1(\overline{D}_v)^{\wedge d} = \deg(\phi)\, c_1(\overline{D}_v)^{\wedge d}.$$

This follows readily from Proposition 10.6(4) and the behavior of Monge-Ampère measures with respect to these operations, see for instance [Cha06, §2.8] for the non-Archimedean case.

We now state the central theorem of this section, which gives the differentiability of the essential minimum at the adelic $\mathbb{R}$-divisor in (10.4). Recall that I denote the index subset in (10.5).

Theorem 10.8. *The essential minimum function is differentiable at $\overline{D}$, and for each $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ we have*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{1}{(D^d)} \sum_{a \in I} \binom{d}{a} \Big(\overline{E} \cdot \prod_{i=1}^{s} (\overline{D}_i^{\mathrm{can}})^{a_i} \Big).$$

In particular, $\overline{D}$ satisfies the v-adic equidistribution property at every $v \in \mathfrak{M}_K$ with $\nu_{\overline{D},v} = c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

The next lemma gives the specific sequence of semipositive approximations that we will use and its properties.

Lemma 10.9. *For $n \in \mathbb{N}$ set $\overline{Q}_n = \sum_{i=1}^{s} (q_i/q_s)^n\, \overline{D}_i^{\mathrm{can}} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$. Then Q_n is ample, $\overline{Q}_n$ is semipositive and $R(D - Q_n) \neq \{0\}$. In particular, $\overline{Q}_n$ is a semipositive approximation of $\overline{D}$. Furthermore*

(1) $r(Q_n; D) \geq q_s^{-n} q_1^n$,

(2) $\mu^{\mathrm{abs}}(\overline{Q}_n) = q_s^{-n} \mu^{\mathrm{abs}}(\overline{D})$,

(3) $(Q_n^d) = q_s^{-dn} \deg(\phi)^n \times (D^d)$,

(4) $c_1(\overline{Q}_{n,v})^{\wedge d} = q_s^{-dn} \deg(\phi)^n\, c_1(\overline{D}_v^{\mathrm{can}})^{\wedge d}$ for every $v \in \mathfrak{M}_K$.

In particular, the normalized v-adic Monge-Ampère measures of $\overline{Q}_n$ and $\overline{D}^{\mathrm{can}}$ coincide.

Proof. We have that $Q_n - (q_1/q_s)^n D$ is a nonnegative linear combination of the D_i's. Since D is ample and the D_i's are nef, this implies that Q_n is ample. Clearly $\overline{Q}_n$ is semipositive because it is a nonnegative linear combination of semipositive adelic $\mathbb{R}$-divisors. Moreover $D - Q_n$ is a nonnegative linear combination of the D_i's, thus $R(D - Q_n) \neq \{0\}$ since $R(D_i) \neq \{0\}$ for all i.

The other statements follow by iteratively applying Proposition 10.6, noting that $\overline{Q}_n = q_s^{-n}\, \phi^{\circ n,*} \overline{D}$. □

Proof of Theorem 10.8. For each n let $\overline{Q}_n$ be the semipositive approximation of $\overline{D}$ given by Lemma 10.9. By this result and Proposition 10.4 we get

$$0 \leq \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; D)} \leq \frac{-q_s^{-n} \mu^{\mathrm{abs}}(\overline{D})}{q_s^{-n} q_1^n} = \frac{-\mu^{\mathrm{abs}}(\overline{D})}{q_1^n}.$$

We have $\mu^{\mathrm{abs}}(\overline{D}) > -\infty$ because D is ample, and so the above ratio converges to 0 as $n \to \infty$. Then Theorem 6.1 gives that the essential minimum function is differentiable at $\overline{D}$, and that for $\overline{E} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ the corresponding derivative is

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \lim_{n \to \infty} \frac{(\overline{Q}_n^d \cdot \overline{E})}{(Q_n^d)}. \tag{10.6}$$

By Lemma 10.9(3) we have $(Q_n^d) = q_s^{-dn} \deg(\phi)^n \times (D^d)$ and so

$$\frac{(\overline{Q}_n^d \cdot \overline{E})}{(Q_n^d)} = \frac{1}{(D^d)} \sum_{a \in \mathbb{N}_d^s} \binom{d}{a} \Big(\frac{q^a}{\deg(\phi)} \Big)^n \times \Big(\overline{E} \cdot \prod_{i=1}^{s} (\overline{D}_i^{\mathrm{can}})^{a_i} \Big).$$

We obtain the formula for the derivative letting $n \to \infty$, thanks to the existence of the limit in (10.6). The formula for the equidistribution measures follows then from their expression in terms of derivatives of the essential minimum function (Proposition 5.6) together with the formula in (3.5) and Proposition 10.5. □

We also have the following logarithmic equidistribution result for $\overline{D}$. Recall that a subvariety $Y \subset X$ is *preperiodic* if there are two positive integers $j < k$ such that $\phi^{\circ j}(Y) = \phi^{\circ k}(Y)$.

Theorem 10.10. *Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $X(\overline{K})$ and $Y \subset X$ a hypersurface such that $Y(\overline{K})$ contains a dense subset of preperiodic points. Then for every $v \in \mathfrak{M}_K$ and any function $\varphi\colon X_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along Y we have*

$$\lim_{\ell\to\infty} \int_{X_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{X_v^{\mathrm{an}}} \varphi \, \frac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}.$$

In particular, this holds when Y is preperiodic.

Proof. Consider again the sequence $(\overline{Q}_n)_n$ of semipositive approximations of $\overline{D}$ given by Lemma 10.9. By this result,

$$\lim_{n\to\infty} \frac{\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; D)} = 0 \quad \text{and} \quad \frac{c_1(\overline{Q}_{n,v})^{\wedge d}}{(Q_n^d)} = \frac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}. \tag{10.7}$$

Up to switching to linearly equivalent divisors, we can suppose that Y is not contained in the support of any of the D_i's, and so we can consider the restriction of $\overline{Q}_n$'s to Y. Since the set of preperiodic points of $Y(\overline{K})$ is dense we have $\mu^{\mathrm{ess}}(\overline{Q}_n|_Y) = 0$, and by Zhang's inequality (Theorem 3.17) we get

$$h_{\overline{Q}_n}(Y) \le d\,\mu^{\mathrm{ess}}(\overline{Q}_n|_Y) \times (Q_n^{d-1} \cdot Y) = 0.$$

On the other hand, let A be an ample divisor on X such that $A - Y$ is pseudo-effective. Since Q_n is nef we have $(Q_n^{d-1} \cdot Y) \le (Q_n^{d-1} \cdot A)$, by the inequality in (2.1). Combining this with Lemmas 3.14 and 2.2 we deduce that

$$h_{\overline{Q}_n}(Y) \ge d\,\mu^{\mathrm{abs}}(\overline{Q}_n) \times (Q_n^{d-1} \cdot A) \ge d\,\frac{\mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; A)} \times (Q_n^d).$$

These inequalities with the limit in (10.7) imply that $\lim_{n\to\infty} h_{\overline{Q}_n}(Y)/(Q_n^d) = 0$. The result then follows from Theorem 8.4 and Remark 8.5.

When Y is preperiodic, there is $j > 0$ such that $Y' := \phi^{\circ j}(Y)$ is periodic with period $k_0 > 0$. Hence the iteration $\phi^{\circ k_0}$ induces a dynamical system on Y'. Up to linear equivalence we can restrict of D to Y', and $\phi^{\circ k_0,*} D|_{Y'} - D|_{Y'}$ is ample. Hence the set of periodic point of $Y'(\overline{K})$ is dense [Fak03, Theorem 5.1] and so is that of preperiodic points of $Y(\overline{K})$. □

10.3. **Equidistribution on semiabelian varieties.** In this subsection we specialize Theorems 10.8 and 10.10 in the semiabelian setting. We first recall the basic constructions and properties that are needed to this end, referring to [Cha00, Küh22] for the proofs and more details.

Let G be a semiabelian variety over K that is the extension of an abelian variety A of dimension g by a split torus $\mathbb{G}_{\mathrm{m}}^r$, so that there is an exact sequence of commutative algebraic groups over K

$$0 \longrightarrow \mathbb{G}_{\mathrm{m}}^r \longrightarrow G \longrightarrow A \longrightarrow 0.$$

We consider the compactification $\overline{G}$ of G induced by toric compactification $(\mathbb{P}^1)^r$ of $\mathbb{G}_{\mathrm{m}}^r$. To construct it, one endows the product variety $G \times (\mathbb{P}^1)^r$ with the action of this split torus defined at the level of points by

$$t \cdot (x, y) = (t \cdot_G x, t^{-1} \cdot_{(\mathbb{P}^1)^r} y)$$

and defines $\overline{G}$ as the categorical quotient $G \times (\mathbb{P}^1)^r / \mathbb{G}_{\mathrm{m}}^r$. It is a smooth variety over K containing G as a dense open subset, and the projection $G \to A$ extends to a morphism $\pi \colon \overline{G} \to A$ allowing to consider this compactification as a $(\mathbb{P}^1)^r$-bundle over A.

For a given integer $\ell > 1$ the multiplication-by-ℓ on G extends to a morphism $[\ell]_{\overline{G}} \colon \overline{G} \to \overline{G}$ of degree ℓ^{r+2g}. If we denote by $[\ell]_A$ the multiplication-by-ℓ on A, then there is a commutative diagram

$$\begin{array}{ccc} \overline{G} & \xrightarrow{[\ell]_{\overline{G}}} & \overline{G} \\ \downarrow \pi & & \downarrow \pi \\ A & \xrightarrow{[\ell]_A} & A \end{array} \tag{10.8}$$

The boundary $\overline{G} \setminus G$ is an effective Weil divisor, and we denote by M its associated (Cartier) divisor on $\overline{G}$. It is relatively ample with respect to π and verifies that

$$[\ell]_{\overline{G}}^* M = \ell M \quad \text{on } \mathrm{Div}(\overline{G}). \tag{10.9}$$

Let N be an ample symmetric divisor on A, which therefore verifies that $[\ell]_A^* N \equiv \ell^2 N$ on $\mathrm{Div}(A)$. Then its pullback $\pi^* N$ is semiample, and by (10.8) it verifies

$$[\ell]_{\overline{G}}^* \pi^* N \equiv \ell^2 \pi^* N \quad \text{on } \mathrm{Div}(\overline{G}).$$

Furthermore, the sum $D = M + \pi^* N$ is an ample divisor on $\overline{G}$.

Let $\overline{M}^{\mathrm{can}} \in \widehat{\mathrm{Div}}(\overline{G})$ and $\overline{N}^{\mathrm{can}} \in \widehat{\mathrm{Div}}(A)$ be the canonical adelic divisors of M and N for the dynamical systems $[\ell]_{\overline{G}}$ and $[\ell]_A$, respectively. By [Cha00, Proposition 3.4] the adelic divisor $\overline{M}^{\mathrm{can}}$ does not depend of the choice of ℓ, and the same holds for $\overline{N}^{\mathrm{can}}$. By the commutativity in (10.8) we have that

$$[\ell]_{\overline{G}}^* \overline{M}^{\mathrm{can}} = \ell \overline{M}^{\mathrm{can}} \quad \text{and} \quad [\ell]_{\overline{G}}^* \pi^* \overline{N}^{\mathrm{can}} \equiv \ell^2 \pi^* \overline{N}^{\mathrm{can}} \quad \text{on } \widehat{\mathrm{Div}}(\overline{G}). \tag{10.10}$$

We have that $\overline{M}^{\mathrm{can}}$ is semipositive, as shown by Chambert-Loir in [Cha00, Proposition 3.6] relying on some specific regular models of abelian varieties constructed by Künnemann. Furthermore, Proposition 10.3 together with the equality in (10.9) implies that this adelic divisor is effective. On the other hand $\overline{N}^{\mathrm{can}}$ is nef because N is ample and so this is also the case for $\pi^* \overline{N}^{\mathrm{can}}$.

Finally set $\overline{D} = \overline{M}^{\mathrm{can}} + \pi^* \overline{N}^{\mathrm{can}} \in \widehat{\mathrm{Div}}(\overline{G})$. By (10.10), its height function verifies that

$$h_{\overline{D}}([\ell]_{\overline{G}} x) = \ell \, h_{\overline{M}^{\mathrm{can}}}(x) + \ell^2 \, h_{\overline{N}^{\mathrm{can}}}(\pi(x)) \quad \text{for } x \in \overline{G}(\overline{K}).$$

It is nonnegative on $G(\overline{K})$ and vanishes on the torsion points, and so $\mu^{\mathrm{ess}}(\overline{D}) = 0$. On the other hand, this height function might take negative values at the points in the boundary $\overline{G} \setminus G$ [Cha00, Corollaire 4.6]. In these cases we have $\mu^{\mathrm{abs}}(\overline{D}) < 0$ and so $\overline{D}$ is outside of the scope of Yuan's theorem.

The next result is a direct application of Theorem 10.8.

Theorem 10.11. *The essential minimum function is differentiable at* $\overline{D}$ *with*

$$\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D}) = \frac{((\overline{M}^{\mathrm{can}})^r \cdot (\pi^*\overline{N}^{\mathrm{can}})^g \cdot \overline{E})}{(M^r \cdot \pi^* N^g)} \quad \textit{for } \overline{E} \in \widehat{\mathrm{Div}}(\overline{G})_{\mathbb{R}}.$$

Proof. We check that the conditions in Theorem 10.8 are met in our current situation. First $R(M) \neq \{0\}$ because M is effective, and $R(\pi^*N) \neq \{0\}$ because π^*N is semiample. As explained, both $\overline{M}^{\mathrm{can}}$ and $\pi^*\overline{N}^{\mathrm{can}}$ are semipositive and D is ample. Then this theorem gives the stated differentiability for the essential minimum function.

To apply the formula therein for the derivative $\partial_{\overline{E}}\, \mu^{\mathrm{ess}}(\overline{D})$ we need to determine the elements $a \in \mathbb{N}^2_{r+g}$ which verify that $\ell^{a_1+2a_2} = \deg([\ell]_{\overline{G}}) = \ell^{r+2g}$. The only one is $a = (r, g)$, and so this formula boils down to

$$\frac{\binom{r+g}{r}((\overline{M}^{\mathrm{can}})^r \cdot (\pi^*\overline{N}^{\mathrm{can}})^g \cdot \overline{E})}{\binom{r+g}{r}(M^r \cdot \pi^*N^g)} = \frac{((\overline{M}^{\mathrm{can}})^r \cdot (\pi^*\overline{N}^{\mathrm{can}})^g \cdot \overline{E})}{(M^r \cdot \pi^*N^g)}.$$

□

Remark 10.12. This result gives Theorem 1.6 in the introduction, as it implies that $\overline{D}$ satisfies the v-adic equidistribution property at every $v \in \mathfrak{M}_K$ with

$$\nu_{\overline{D},v} = \frac{c_1(\overline{M}^{\mathrm{can}}_v)^{\wedge r} \wedge c_1(\pi^*\overline{N}^{\mathrm{can}}_v)^{\wedge g}}{(M^r \cdot \pi^*N^g)} = \frac{c_1(\overline{D}_v)^{\wedge r+g}}{(D^{r+g})}. \tag{10.11}$$

Using Theorem 10.11, the first formula for this equidistribution measure follows from its expression in terms of derivatives of the essential minimum function (Proposition 5.6), whereas the second formula follows from the first together with Proposition 10.5. Theorem 10.11 together with Proposition 5.3 also implies the height convergence property in (1.6).

Remark 10.13. When v is Archimedean, the equidistribution measure in (10.11) coincides with the Haar probability measure on the maximal compact subgroup $\mathbb{S}_v \simeq (S^1)^{r+2g}$ of G^{an}_v, see for instance [Küh22, Lemma 5.2].

When v is non-Archimedean, the description of this measure seems more complicated. For abelian varieties, they were described by Gubler in terms of convex geometry [Gub10] but the extension to the semiabelian case is still pending.

For completeness, we extend this equidistribution result to the closure of a subvariety of G with vanishing essential minimum. By the semiabelian Bogomolov conjecture, proved by David and Philippon [DP00], these subvarieties are translates of semiabelian subvarieties by torsion points, and so they do not provide examples of equidistribution phenomena beyond those already obtained. But as explained in the introduction, this extension is the centerpiece of Kühne's approach to this conjecture [Küh22, Proposition 4.1] and it is worth showing that it can also be derived from our results.

Let $Y \subset \overline{G}$ be the closure of a subvariety of G, and set $e = \dim(Y)$ and $e' = \dim(\pi(Y))$. Then Y is not contained in the support of M, and after possibly replacing the divisor $N \in \mathrm{Div}(A)$ by a linearly equivalent one, we assume without loss of generality that Y is neither contained in the support of π^*N. Hence we can consider the restriction $\overline{D}|_Y$, defined as the pullback of $\overline{D}$ with respect to the inclusion morphism $\iota\colon Y \hookrightarrow \overline{G}$.

Proposition 10.14. *With notation as above, assume that* $\mu^{\mathrm{ess}}(\overline{D}|_Y)=0$. *Then* $\overline{D}|_Y$ *satisfies the* v*-adic equidistribution property for every* $v\in\mathfrak{M}_K$ *with*

$$\nu_{\overline{D}|_Y,v}=\frac{c_1(\overline{M}_v^{\mathrm{can}})^{\wedge e-e'}\wedge c_1(\pi^*\overline{N}_v^{\mathrm{can}})^{\wedge e'}\wedge\delta_{Y_v^{\mathrm{an}}}}{(M^{e-e'}\cdot\pi^*N^{e'}\cdot Y)}.$$

Proof. For each $n\in\mathbb{N}$ consider the semipositive approximation $\overline{Q}_n=\ell^{-n}\overline{M}^{\mathrm{can}}+\pi^*\overline{N}^{\mathrm{can}}$ of $\overline{D}$ given by Lemma 10.9 in the semiabelian setting. In our current situation we have that Q_n is ample, $\overline{Q}_n$ is semipositive and $\overline{D}-\overline{Q}_n$ is effective. Let $\widetilde{Y}$ be the normalization of the subvariety Y, and denote by $\overline{D}|_{\widetilde{Y}}$ and $\overline{Q}_n|_{\widetilde{Y}}$ the adelic $\mathbb{R}$-divisors on $\widetilde{Y}$ obtained by pullback with respect to the normalization morphism. Since this morphism is birational and Y is not contained in the support of M and π^*N, we deduce that $\overline{Q}_n|_{\widetilde{Y}}$ is a semipositive approximation of $\overline{D}|_{\widetilde{Y}}$.

Its absolute minimum can be estimated as

$$0=\mu^{\mathrm{ess}}(\overline{D}|_Y)=\mu^{\mathrm{ess}}(\overline{D}|_{\widetilde{Y}})\geq\mu^{\mathrm{abs}}(\overline{Q}_n|_{\widetilde{Y}})\geq\mu^{\mathrm{abs}}(\overline{Q}_n)=\ell^{-2n}\mu^{\mathrm{abs}}(\overline{D}),$$

where the last equality comes from Lemma 10.9(2). On the other hand we have that $Q_n|_{\widetilde{Y}}-\ell^{-n}D|_{\widetilde{Y}}=(1-\ell^{-n})\pi^*N|_{\widetilde{Y}}$ is effective and so

$$r(Q_n|_{\widetilde{Y}};D|_{\widetilde{Y}})\geq\ell^{-n}.$$

Hence $\lim_{n\to\infty}\mu^{\mathrm{abs}}(\overline{Q}_n|_{\widetilde{Y}})/r(Q_n|_{\widetilde{Y}};D|_{\widetilde{Y}})=0$, and so Theorem 6.1 together with Remark 6.2 implies that $\overline{D}|_Y$ satisfies the equidistribution property for every $v\in\mathfrak{M}_K$ with

$$\nu_{\overline{D}|_Y,v}=\lim_{v\to\infty}\frac{c_1((\overline{Q}_n|_Y)_v)^{\wedge e}}{((Q_n|_Y)^e)}=\lim_{v\to\infty}\frac{c_1(\overline{Q}_{n,v})^{\wedge e}\wedge\delta_{Y_v^{\mathrm{an}}}}{(Q_n^e\cdot Y)}.$$

To compute this limit, first note that

$$(Q_n^e\cdot Y)=\sum_{j=0}^{e}\ell^{-n(e-j)}\binom{e}{j}(M^{e-j}\cdot\pi^*N^j\cdot Y).$$

For each j consider the intersection product $[M^{e-j}\cdot Y]$ in the Chow group of j-dimensional cycles of Y. By the projection formula [Ful98, Proposition 2.3(c)] we have

$$(M^{e-j}\cdot\pi^*N^j\cdot Y)=(\pi_*[M^{e-j}\cdot Y]\cdot N^j),$$

where the left intersection number is computed over Y and the right over $\pi(Y)$. Hence this quantity vanishes when $j>e'=\dim(\pi(Y))$, whereas for $j=e'$ it equals $(M^{e-e'}\cdot F)\times(N^{e'})$ for a general fiber F of the projection $Y\to\pi(Y)$, and therefore it is positive because M is relatively ample and N is ample. Hence

$$(Q_n^e\cdot Y)=\ell^{-n(e-e')}\binom{e}{e'}(M^{e-e'}\cdot\pi^*N^{e'}\cdot Y)+O(\ell^{-n(e-e'+1)}),\tag{10.12}$$

and the dominant term in this asymptotics is positive.

On the other hand, the measure $c_1(\overline{Q}_{n,v})^{\wedge e}\wedge\delta_{Y_v^{\mathrm{an}}}$ is zero whenever $j>e'$ because its total mass vanishes, and therefore

$$c_1(\overline{Q}_{n,v})^{\wedge e}\wedge\delta_{Y_v^{\mathrm{an}}}=\sum_{j=0}^{e'}\ell^{-n(e-j)}\binom{e}{j}c_1(\overline{M}_v^{\mathrm{can}})^{\wedge e-j}\wedge c_1(\pi^*\overline{N}_v^{\mathrm{can}})^{\wedge j}\wedge\delta_{Y_v^{\mathrm{an}}}.$$

The statement then follows by taking the limit for $n\to\infty$ of the ratio between this asymptotics and that in (10.12). ☐

Finally we strengthen the equidistribution property in (10.11) to include test functions with logarithmic singularities along the closure of a torsion hypersurface or an irreducible component of the boundary. Recall that a hypersurface of G is *torsion* if it is the translate of a semiabelian hypersurface of G by a torsion point.

Theorem 10.15. *Let $(x_\ell)_\ell$ be a $\overline{D}$-small generic sequence in $\overline{G}(\overline{K})$ and $Y \subset \overline{G}$ a hypersurface that is either the closure of a torsion hypersurface of G or an irreducible component of $\overline{G} \setminus G$. Then for every $v \in \mathfrak{M}_K$ and any function $\varphi \colon \overline{G}_v^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$ with at most logarithmic singularities along Y we have*

$$\lim_{\ell \to \infty} \int_{\overline{G}_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \int_{\overline{G}_v^{\mathrm{an}}} \varphi \, \frac{c_1(\overline{D}_v)^{\wedge r+g}}{(D^{r+g})}.$$

Proof. This follows from Theorem 10.10 noting that Y is a preperiodic hypersurface for the endomorphism $[\ell]_{\overline{G}}$ for any $\ell > 1$. □

11. Generalization to quasi-projective varieties

In this section we extend Theorem 1.2 to the setting of adelic divisors on quasi-projective varieties in the sense of Yuan and Zhang [YZ23] (Theorem 11.8). We first recall the construction and important facts concerning adelic divisors on quasi-projective varieties from [YZ23]. We closely follow the presentation in [BK24].

11.1. Adelic divisors on quasi-projective varieties.

11.1.1. *Geometric case.* Let k be a field, X a normal projective variety over k of dimension $d \geq 1$, $B \in \mathrm{Div}(X)$ an effective divisor on X, and $U = X \setminus \mathrm{supp}(B)$. We denote by $R(X, U)$ the category of normal modifications of X which are isomorphisms over U. Given such a modification $\pi \colon X_\pi \to X$, we write (X_π, π) or simply π for the corresponding object in $R(X, U)$. The space of *model* $\mathbb{R}$*-divisors* on U is the direct limit

$$\mathrm{Div}(U)^{\mathrm{mod}}_{\mathbb{R}} = \varinjlim_{\pi \in R(X,U)} \mathrm{Div}(X_\pi)_{\mathbb{R}}.$$

Given two model $\mathbb{R}$-divisors D, D' on U, we write $D \geq D'$ or $D' \leq D$ if there exists $(X_\pi, \pi) \in R(X, U)$ such that $D, D' \in \mathrm{Div}(X_\pi)_{\mathbb{R}}$ and $D - D'$ is effective. The B-adic norm $\|.\|$ on $\mathrm{Div}(U)^{\mathrm{mod}}_{\mathbb{R}}$ is defined by setting

$$\|D\| = \inf\{\varepsilon \in \mathbb{R}_{>0} \mid -\varepsilon B \leq D \leq \varepsilon B\}$$

for $D \in \mathrm{Div}(U)^{\mathrm{mod}}_{\mathbb{R}}$.

The space $\mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$ is the completion of $\mathrm{Div}(U)^{\mathrm{mod}}_{\mathbb{R}}$ for the B-adic topology. In [YZ23, BK24], elements of $\mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$ are called adelic divisors on U. We prefer not to use this terminology here in order to avoid confusion with the arithmetic setting. The group $\mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$ only depends on the open set U by [YZ23, Lemma 2.4.1], [BK24, Remark 3.3].

Let $D \in \mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$. By definition, D can be represented by a sequence of model $\mathbb{R}$-divisors $(D_i)_i$ on U that is Cauchy for the B-adic norm. It follows from [YZ23, Theorems 5.2.1 and 5.2.9] that the limit $\lim_{i\to\infty} \mathrm{vol}(D_i)$ exists in $\mathbb{R}$ and does not depend on the choice of $(D_i)_i$. We call this limit the *volume* of D and we denote it by $\mathrm{vol}(D)$. We say that D is *big* if $\mathrm{vol}(D) > 0$.

We say that D is *nef* if the sequence $(D_i)_i$ can be chosen such that D_i is nef for every $i \in \mathbb{N}$. We say that D is *integrable* if it is of the form $D = A_1 - A_2$ with $A_1, A_2 \in \mathrm{Div}(U)_{\mathbb{R}}^{\lim}$ nef.

The subspace of integrable adelic divisors is denoted $\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{int}}$. The intersection product defined in [YZ23, §4.1.1] extends by multilinearity to a product

$$(\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{int}})^d \longrightarrow \mathbb{R},\ (D_1, \dots, D_d) \longmapsto (D_1 \cdot \ldots \cdot D_d)$$

with the following property [BK24, Theorem 3.7]. Let $(D_1, \dots, D_d) \in (\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{int}})^d$ and let $(X_i, \pi_i)_i$ be a sequence in $R(X, U)$ such that for each $j \in \{1, \dots, j\}$, D_j is represented by a Cauchy sequence $(D_{j,i})_i$ with $D_{j,i} \in \mathrm{Div}(X_i)_{\mathbb{R}}$ for every $i \in \mathbb{N}$. Then we have

$$(D_1 \cdot \ldots \cdot D_d) = \lim_{i\to\infty} (D_{1,i} \cdot \ldots \cdot D_{d,i}).$$

Moreover, $\mathrm{vol}(D) = (D^d)$ for every nef $D \in \mathrm{Div}(U)_{\mathbb{R}}^{\lim}$.

Let $P, A \in \mathrm{Div}(U)_{\mathbb{R}}^{\lim}$ be big. The *inradius* of P with respect to A is the quantity

$$r(P; A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ is big}\}.$$

It is easy to check that

$$r(P; A) = \lim_{i\to\infty} r(P_i; A_i) \tag{11.1}$$

for any sequence $(X_i, \pi_i)_i$ in $R(X, U)$ such that P and A are represented by Cauchy sequences $(P_i)_i$ and $(A_i)_i$ with $P_i, A_i \in \mathrm{Div}(X_i)_{\mathbb{R}}$ for every $i \in \mathbb{N}$.

11.1.2. *Arithmetic case.* Let K be a number field with ring of integers $\mathcal{O}_K$. By a *projective arithmetic variety* (respectively a *quasi-projective arithmetic variety*) we mean an integral scheme $\mathcal{X}$ which is flat and projective (respectively quasi-projective) over $\operatorname{Spec} \mathcal{O}_K$. Let $\mathcal{X}$ be a normal projective arithmetic variety. Then its generic fiber $X = \mathcal{X} \times_{\mathcal{O}_K} \operatorname{Spec} K$ is also normal. We let $d+1$ be the Krull dimension of $\mathcal{X}$, so that X has dimension d.

Definition 11.1. An *arithmetic* $\mathbb{R}$*-divisor* on $\mathcal{X}$ is a pair $\overline{\mathcal{D}} = (\mathcal{D}, (g_v)_{v\in\mathfrak{M}_K^\infty})$, where $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_{\mathbb{R}} = \mathrm{Div}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ is an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$ and for each $v \in \mathfrak{M}_K^\infty$, g_v is a continuous v-adic Green function on $\mathcal{D}_{|X}$. The space of arithmetic $\mathbb{R}$-divisors on $\mathcal{X}$ is denoted $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{R}}$.

We let $[\infty] \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{R}}$ be the arithmetic divisor on $\mathcal{X}$ given by $\mathcal{D} = 0$ and $g_v = 1$ for every $v \in \mathfrak{M}_K^\infty$.

To any $\overline{\mathcal{D}} = (\mathcal{D}, (g_v)_{v\in\mathfrak{M}_K^\infty}) \in \widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{R}}$ we can associate an adelic $\mathbb{R}$-divisor $\overline{\mathcal{D}}^{\mathrm{ad}} = (D, (g_v)_{v\in\mathfrak{M}_K}) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ by setting $D = \mathcal{D}_{|X}$ and $g_v = g_{\mathcal{D},v}$ for $v \in \mathfrak{M}_K \setminus \mathfrak{M}_K^\infty$ (see §3.1). We denote by $h_{\overline{\mathcal{D}}} = \mathrm{h}_{\overline{\mathcal{D}}^{\mathrm{ad}}} \colon X(\overline{K}) \to \mathbb{R}$ the associated height function.

We say that $\overline{\mathcal{D}}$ is *effective* (respectively *strictly effective*) if $\mathcal{D}$ is effective and if $g_v \geq 0$ (respectively $g_v > 0$) on $X_v^{\mathrm{an}} \setminus \mathrm{supp}(D)_v^{\mathrm{an}}$ for every $v \in \mathfrak{M}_K^\infty$. Let $\overline{\mathcal{B}} = (\mathcal{B}, (g_{\mathcal{B},v})_v)$ be a strictly effective arithmetic $\mathbb{R}$-divisor on $\mathcal{X}$ and $\mathcal{U} = \mathcal{X} \setminus \mathrm{supp}(\mathcal{B})$. We also let $B = \mathcal{B}_{|X} \in \mathrm{Div}(X)_{\mathbb{R}}$ and $U = X \setminus \mathrm{supp}(B)$. We denote by $R(\mathcal{X}, \mathcal{U})$ the category of normal modifications of $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}_K$ which are isomorphisms over $\mathcal{U}$. By definition, the objects of $R(\mathcal{X}, \mathcal{U})$ are pairs $(\pi, \mathcal{X}_\pi)$ (also denoted π for short) where $\mathcal{X}_\pi$ is a normal projective arithmetic variety and $\pi \colon \mathcal{X}_\pi \to \mathcal{X}$ is a morphism that induces an isomorphism over $\mathcal{U}$. The space of *model arithmetic* $\mathbb{R}$*-divisors* on $\mathcal{U}$ is defined as the direct limit

$$\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}} = \varinjlim_{\pi \in R(\mathcal{X},\mathcal{U})} \widehat{\mathrm{Div}}(\mathcal{X}_\pi)_{\mathbb{R}}.$$

For every $\overline{\mathcal{D}}_1, \overline{\mathcal{D}}_2 \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$, we write $\overline{\mathcal{D}}_1 \geq \overline{\mathcal{D}}_2$ or $\overline{\mathcal{D}}_1 \leq \overline{\mathcal{D}}_2$ if $\overline{\mathcal{D}}_1 - \overline{\mathcal{D}}_2$ is effective on a model where $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ are defined. The $\overline{\mathcal{B}}$-adic norm is defined by letting

$$\|\overline{\mathcal{D}}\| = \inf\{\varepsilon \in \mathbb{R}_{>0} \mid -\varepsilon\overline{\mathcal{B}} \leq \overline{\mathcal{D}} \leq \varepsilon\overline{\mathcal{B}}\}$$

for $\overline{\mathcal{D}} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$.

Definition 11.2. The space $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ of *adelic* $\mathbb{R}$*-divisors* on $\mathcal{U}$ is the completion of $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ with respect to the $\overline{\mathcal{B}}$-adic topology.

As in the geometric case, $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ depends only on $\mathcal{U}$ [BK24, Proposition 3.36]. Let $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$. Then $\overline{D}$ can be represented by a sequence $(\overline{\mathcal{D}}_i)_i$ in $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ which is Cauchy for the $\overline{\mathcal{B}}$-adic norm. For every $i \in \mathbb{N}$, let $(\pi_i, \mathcal{X}_i) \in R(\mathcal{X},\mathcal{U})$ such that $\overline{\mathcal{D}}_i \in \widehat{\mathrm{Div}}(\mathcal{X}_i)_{\mathbb{R}}$ and let $X_i = \mathcal{X}_i \times_{\mathcal{O}_K} \mathrm{Spec}\, K$, $\overline{D}_i = \overline{\mathcal{D}}_i^{\mathrm{ad}} \in \widehat{\mathrm{Div}}(X_i)_{\mathbb{R}}$, and $D_i = \mathcal{D}_{i|X_i}$. Note that $(D_i)_i$ is a sequence in $\mathrm{Div}(U)^{\mathrm{mod}}_{\mathbb{R}}$ which is Cauchy for the B-adic topology in the sense of §11.1.1, hence defines an object $D \in \mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$. In the sequel, given an adelic $\mathbb{R}$-divisor $\overline{A} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ we always use the same letter A to denote the corresponding object in $\mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$.

We say that $\overline{D}$ is *pseudo-effective* (respectively *semipositive*, respectively *nef*) if the sequence $(\overline{\mathcal{D}}_i)_i$ can be chosen such that $\overline{D}_i = \overline{\mathcal{D}}_i^{\mathrm{ad}}$ is pseudo-effective (respectively semipositive, respectively nef) for every i. We say that $\overline{D}$ is *integrable* if it is the difference of two nef adelic $\mathbb{R}$-divisors on $\mathcal{U}$. We denote by $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{int}}_{\mathbb{R}} \subset \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ the subspace of integrable adelic $\mathbb{R}$-divisors on $\mathcal{U}$.

Remark 11.3. Our definition of nef adelic $\mathbb{R}$-divisors on $\mathcal{U}$ coincides with the one in [BK24], but it differs slightly from the one in [YZ23] (where nef adelic divisors in the above sense are called strongly nef). However, the two definitions coincide after possibly shrinking $\mathcal{U}$ [BK24, Remarks 3.5, 3.16 and 3.39].

The fact that $(\overline{\mathcal{D}}_i)_i$ is a Cauchy sequence readily implies that $\lim_{i\to\infty} h_{\overline{\mathcal{D}}_i}(x)$ exists for every $x \in U(\overline{K})$. Moreover this limit does not depend on the choice of $(\overline{\mathcal{D}}_i)_i$. Therefore we can define a height function $h_{\overline{D}} \colon U(\overline{K}) \to \mathbb{R}$ associated to $\overline{D}$ by setting

$$h_{\overline{D}}(x) = \lim_{i\to\infty} h_{\overline{\mathcal{D}}_i}(x) \quad \text{for every } x \in U(\overline{K}).$$

As in the geometric case, the arithmetic intersection product defined in [YZ23, §4.1.1] extends to a multilinear map

$$(\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{int}}_{\mathbb{R}})^{d+1} \longrightarrow \mathbb{R},\ (\overline{D}_0, \dots, \overline{D}_d) \longmapsto (\overline{D}_0 \cdot \ldots \cdot \overline{D}_d)$$

and

$$(\overline{D}_0 \cdot \ldots \cdot \overline{D}_d) = \lim_{i\to\infty} (\overline{\mathcal{D}}_{0,i}^{\mathrm{ad}} \cdot \ldots \cdot \overline{D}_{d,i}^{\mathrm{ad}})$$

if for each $j \in \{0, \dots, d\}$, $\overline{D}_j$ is represented by a Cauchy sequence $(\overline{\mathcal{D}}_{j,i})_i$ in $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ [BK24, Theorem 3.37]. Here the intersection products on the right-hand side are the arithmetic intersection products defined in §3.2 taken on common models. For any dominant morphism $\phi\colon \mathcal{U}' \to \mathcal{U}$ of normal quasi-projective arithmetic varieties, there is a pull-back map $\phi^*\colon \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}} \to \widehat{\mathrm{Div}}(\mathcal{U}')^{\mathrm{adel}}_{\mathbb{R}}$ and we have

$$(\overline{D}_0 \cdot \ldots \cdot \overline{D}_d) = (\phi^*\overline{D}_0 \cdot \ldots \cdot \phi^*\overline{D}_d)$$

by [BK24, Proposition 3.46].

For any $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{int}}_{\mathbb{R}}$ and $v \in \mathfrak{M}_K$, we denote by $c_1(\overline{D}_v)^{\wedge d}$ the signed measure on U_v^{an} defined in [YZ23, §3.6.6] (extended to adelic $\mathbb{R}$-divisors by multilinearity). Let $(\overline{\mathcal{D}}_i)_i$ be a Cauchy sequence representing $\overline{D}$. For each $i \in \mathbb{N}$ let $(\pi_i, \mathcal{X}_i) \in R(\mathcal{X},\mathcal{U})$ such that $\overline{\mathcal{D}}_i \in \widehat{\mathrm{Div}}(\mathcal{X}_i)_{\mathbb{R}}$, $X_i = \mathcal{X}_i \times_{\mathcal{O}_K} \operatorname{Spec} K$, and $\overline{D}_i = \overline{\mathcal{D}}_i^{\mathrm{ad}} \in \widehat{\mathrm{Div}}(X_i)_{\mathbb{R}}$. By construction [YZ23, §3.6.6], we have

$$\int_{U_v^{\mathrm{an}}} \varphi\, c_1(\overline{D}_v)^{\wedge d} = \lim_{i\to\infty} \int_{X_{i,v}^{\mathrm{an}}} \varphi\, c_1(\overline{D}_{i,v})^{\wedge d} \tag{11.2}$$

for every real-valued continuous function $\varphi\colon U_v^{\mathrm{an}} \to \mathbb{R}$ with compact support. Here we view φ as a function on $X_{i,v}^{\mathrm{an}}$ via the open immersion $U_v^{\mathrm{an}} \to X_{i,v}^{\mathrm{an}}$. The measure $c_1(\overline{D}_v)^{\wedge d}$ has total mass (D^d) [YZ23, Lemma 5.4.4].

11.2. **Essential and absolute minima of an adelic $\mathbb{R}$-divisor.** Let $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ with big $D \in \mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$. The *essential minimum* of $\overline{D}$ is defined as

$$\mu^{\mathrm{ess}}(\overline{D}) = \sup_{V\subset U} \inf_{x\in V(\overline{K})} h_{\overline{D}}(x),$$

where the supremum is over all the Zariski-open subschemes of U.

Lemma 11.4. *Let $(\overline{\mathcal{D}}_i)_i$ be a Cauchy sequence in $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ representing $\overline{D}$. Then*

$$\mu^{\mathrm{ess}}(\overline{D}) = \lim_{i\to\infty} \mu^{\mathrm{ess}}(\overline{\mathcal{D}}_i^{\mathrm{ad}}).$$

Proof. Note that for any $N \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\mu^{\mathrm{ess}}(N\overline{D} + t[\infty]) = N\mu^{\mathrm{ess}}(\overline{D}) + t \quad \text{and} \quad \mu^{\mathrm{ess}}(N\overline{\mathcal{D}}_i^{\mathrm{ad}} + t[\infty]) = N\mu^{\mathrm{ess}}(\overline{\mathcal{D}}_i^{\mathrm{ad}}) + t$$

for every $i \in \mathbb{N}$. Therefore we can replace $\overline{D}$ by $N\overline{D} + t[\infty]$ without loss of generality. Taking N and t sufficiently large, we can therefore assume that $\overline{\mathcal{D}}_i^{\mathrm{ad}} - \overline{\mathcal{B}}^{\mathrm{ad}}$ is big for every $i \in \mathbb{N}$. Since $(\overline{\mathcal{D}}_i)_i$ is a Cauchy sequence, there exists a sequence of positive real numbers $(\varepsilon_i)_i$ converging to zero such that

$$\overline{\mathcal{D}}_i - \varepsilon_i\overline{\mathcal{B}} \le \overline{\mathcal{D}}_j \le \overline{\mathcal{D}}_i + \varepsilon_i\overline{\mathcal{B}} \tag{11.3}$$

for every integers i, j with $j \ge i \ge 0$. Let $i \in \mathbb{N}$. Since for every $j \ge i$ the support of $\pm(\overline{\mathcal{D}}_j - \overline{\mathcal{D}}_i) + \varepsilon_i\overline{\mathcal{B}}$ does not intersect U, it follows from (11.3) that

$$h_{\overline{\mathcal{D}}_i}(x) - \varepsilon_i h_{\overline{\mathcal{B}}}(x) \le h_{\overline{D}}(x) \le h_{\overline{\mathcal{D}}_i}(x) + \varepsilon_i h_{\overline{\mathcal{B}}}(x)$$

for every $x \in U(\overline{K})$. Since $\overline{\mathcal{D}}_i^{\mathrm{ad}} - \overline{\mathcal{B}}^{\mathrm{ad}}$ is big, there exists a dense open subset $U_i \subseteq U$ such that $h_{\overline{\mathcal{B}}}(x) \le h_{\overline{\mathcal{D}}_i}(x)$ for every $x \in U_i(\overline{K})$, and therefore

$$(1 - \varepsilon_i)h_{\overline{\mathcal{D}}_i}(x) \le h_{\overline{D}}(x) \le (1 + \varepsilon_i)h_{\overline{\mathcal{D}}_i}(x)$$

for every $x \in U_i(\overline{K})$. Since U_i is dense, this implies that $|\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{ess}}(\overline{\mathcal{D}}_i^{\mathrm{ad}})| \le \varepsilon_i$ and the lemma follows. ☐

Lemma 11.5. *For every $\overline{E} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ such that $\overline{D} - \overline{E}$ is pseudo-effective, we have $\mu^{\mathrm{ess}}(\overline{D}) \ge \mu^{\mathrm{ess}}(\overline{E})$.*

Proof. This follows from the analogous result for adelic $\mathbb{R}$-divisors on projective varieties (Lemma 3.15 (4)) thanks to Lemma 11.4. □

If $\overline{D}$ is semipositive we define its *absolute minimum* as

$$\mu^{\mathrm{abs}}(\overline{D}) = \sup\{\lambda \in \mathbb{R} \mid \overline{D} - \lambda[\infty] \text{ is nef}\}.$$

11.3. **The key inequality.**

Proposition 11.6. *Let $\overline{P}, \overline{E} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ with $\overline{P}$ nef and P big. Assume that there exists a nef $\overline{A} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ such that A is big and $\overline{A} \pm \overline{E}$ are nef. There exists a constant C_d depending only on d such that for every $\lambda \in [0, r(P; A)/2)$ we have*

$$\mu^{\mathrm{ess}}(\overline{P} + \lambda\overline{E}) \geq \frac{(\overline{P}^{d+1})}{(d+1)\,\mathrm{vol}(P + \lambda E)} + \lambda\frac{(\overline{P}^{d} \cdot \overline{E})}{(P^d)} - C_d \times \frac{(\overline{P}^{d} \cdot \overline{A})}{(P^d)} \times \frac{\lambda^2}{r(P;A)}.$$

In particular, if $E = 0$ in $\mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$ then

$$\mu^{\mathrm{ess}}(\overline{P} + \lambda\overline{E}) \geq \frac{(\overline{P}^{d+1})}{(d+1)(P^d)} + \lambda\frac{(\overline{P}^{d} \cdot \overline{E})}{(P^d)} - C_d \times \frac{(\overline{P}^{d} \cdot \overline{A})}{(P^d)} \times \frac{\lambda^2}{r(P;A)}.$$

Proof. Let $(\overline{\mathcal{P}}_i)_i, (\overline{\mathcal{E}}_i)_i$ and $(\overline{\mathcal{A}}_i)_i$ be Cauchy sequences in $\widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{mod}}_{\mathbb{R}}$ representing $\overline{P}, \overline{E}$ and $\overline{A}$ respectively and such that for every $i \in \mathbb{N}$, $\overline{\mathcal{P}}_i, \overline{\mathcal{E}}_i$ and $\overline{\mathcal{A}}_i$ are defined on the same projective arithmetic variety $\mathcal{X}_i$. For every $i \in \mathbb{N}$, we denote by $\overline{P}_i = \overline{\mathcal{P}}_i^{\mathrm{ad}}, \overline{E}_i = \overline{\mathcal{E}}_i^{\mathrm{ad}}$ and $\overline{A}_i = \overline{\mathcal{A}}_i^{\mathrm{ad}}$ the associated adelic $\mathbb{R}$-divisors on $X_i = \mathcal{X}_i \times_{\mathcal{O}_K} \mathrm{Spec}\, K$.

Let $\lambda \in [0, r(P; A)/2)$. Then for sufficiently large $i \in \mathbb{N}$ we have $\lambda \in [0, r(P_i; A_i)/2)$ by (11.1). By Corollary 7.3, there is a constant C_d depending only on d such that

$$\mu^{\mathrm{ess}}(\overline{P}_i + \lambda\overline{E}_i) \geq \frac{(\overline{P}_i^{d+1})}{(d+1)\,\mathrm{vol}(P_i + \lambda E_i)} + \lambda\frac{(\overline{P}_i^{d} \cdot \overline{E}_i)}{(P_i^d)} - C_d \times \frac{(\overline{P}_i^{d} \cdot \overline{A}_i)}{(P_i^d)} \times \frac{\lambda^2}{r(P_i;A_i)}.$$

We conclude by letting i tend to infinity. □

11.4. **Equidistribution on quasi-projective varieties.** We keep the notation of §11.1.2. Let $\overline{D} \in \widehat{\mathrm{Div}}(\mathcal{U})^{\mathrm{adel}}_{\mathbb{R}}$ be an adelic $\mathbb{R}$-divisor on $\mathcal{U}$ with $D \in \mathrm{Div}(U)^{\mathrm{lim}}_{\mathbb{R}}$ big.

Definition 11.7. A *semipositive approximation* of $\overline{D}$ is a pair $(\phi\colon \mathcal{U}' \to \mathcal{U}, \overline{Q})$ where

- $\phi\colon \mathcal{U}' \to \mathcal{U}$ is a birational morphism of normal quasi-projective arithmetic varieties,
- $\overline{Q} \in \widehat{\mathrm{Div}}(\mathcal{U}')^{\mathrm{adel}}_{\mathbb{R}}$ is a semipositive adelic $\mathbb{R}$-divisor on $\mathcal{U}'$ with Q big,
- $\phi^*\overline{D} - \overline{Q}$ is pseudo-effective.

A sequence $(x_\ell)_\ell$ in $U(\overline{K})$ is called *generic* if it is generic as a sequence in $X(\overline{K})$. In the sequel, all the generic sequences we consider lie in $U(\overline{K})$. A generic sequence $(x_\ell)_\ell$ is called $\overline{D}$*-small* if $\lim_{\ell\to\infty} h_{\overline{D}}(x_\ell) = \mu^{\mathrm{ess}}(\overline{D})$.

For every $x \in U(\overline{K})$, the orbit $O(x) \subset X(\overline{K})$ of x lies in $U(\overline{K})$. It follows that for any $v \in \mathfrak{M}_K$, $O(x)_v = \iota_v(O(x)) \subset U_v^{\mathrm{an}}$, and in particular $\delta_{O(x)_v}$ defines a probability measure on U_v^{an}.

The following is a quasi-projective analogue of Theorem 7.7. Similar arguments would allow to extend Theorem 6.1 to the quasi-projective setting, but we focus on this statement as it facilitates the comparison with [YZ23, Theorem 5.4.3].

Theorem 11.8. *Assume that there exists a sequence* $(\phi_n \colon \mathcal{U}_n \to \mathcal{U}, \overline{Q}_n)_n$ *of semipositive approximations of* $\overline{D}$ *such that*

$$\lim_{n\to\infty} \frac{1}{r(Q_n; \phi_n^* D)} \Big(\mu^{\mathrm{ess}}(\overline{D}) - \frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)} \Big) = 0 \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \frac{\mu^{\mathrm{ess}}(\overline{D}) - \mu^{\mathrm{abs}}(\overline{Q}_n)}{r(Q_n; \phi_n^* D)} < \infty.$$

Let $v \in \mathfrak{M}_K$*, and for each* $n \geq 1$ *let* $\nu_{n,v}$ *be the probability measure on* U_v^{an} *given by the pushforward of the normalized* v*-adic Monge-Ampère measure* $c_1(\overline{Q}_{n,v})^{\wedge d}/(Q_n^d)$ *on* $U_{n,v}^{\mathrm{an}}$*. Then*

(1) the sequence $(\nu_{n,v})_n$ *converges weakly to a probability measure* ν_v *on* U_v^{an}*,*

(2) for every $\overline{D}$*-small generic sequence* $(x_\ell)_\ell$ *in* $U(\overline{K})$*, the sequence of probability measures* $(\delta_{O(x_\ell)_v})_\ell$ *on* U_v^{an} *converges weakly to* ν_v*.*

The proof adapts the approach we used for Theorem 7.7, the central result being Proposition 11.6.

Proof. Let $(x_\ell)_\ell$ be a generic $\overline{D}$-small sequence in $U(\overline{K})$ and $v \in \mathfrak{M}_K$. Let $\varphi \colon U_v^{\mathrm{an}} \to \mathbb{R}$ be a continuous function with compact support. We need to show that

$$\lim_{\ell\to\infty} \int_{U_v^{\mathrm{an}}} \varphi \, d\delta_{O(x_\ell)_v} = \lim_{n\to\infty} \int_{U_v^{\mathrm{an}}} \varphi \, d\nu_{n,v}. \tag{11.4}$$

Let $\varepsilon > 0$. We view φ as an element of $C(X_v^{\mathrm{an}})$. By [GM22, Proposition 2.11 and Theorem 2.13], there exists a function $\varphi_\varepsilon \in C(X_v^{\mathrm{an}})$ such that $|\varphi_\varepsilon - \varphi| < \varepsilon$ on X_v^{an} and, after possibly extending the base field K, φ_ε is G_v-invariant and the corresponding adelic divisor $\overline{0}^{\varphi_\varepsilon}$ defined in §3.5 is DSP. After shrinking $\mathcal{U}$ if necessary, we view $\overline{E}$ as an element of $\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{int}}$.

For every $n \in \mathbb{N}$, we let $\widetilde{Q}_n = \overline{Q}_n - \mu^{\mathrm{abs}}(\overline{Q}_n)[\infty]$. Then $\widetilde{Q}_n$ is nef and

$$C_0 = \sup_{n\in\mathbb{N}} \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{A})}{(Q_n^d)} < \infty.$$

We omit the proof, as it is identical to the one of Lemma 7.6 (note that Zhang's inequality remains valid in the quasi-projective setting: this is [YZ23, Theorem 5.3.3], and alternatively it follows from Proposition 11.6 applied with $\lambda = 0$).

To conclude, we adapt the arguments in [YZ23, Proof of Theorem 5.4.3]. Let $n \in \mathbb{N}$ and $\lambda \in (0, r(Q_n; \phi_n^* A)/2)$. By Proposition 11.6 applied with $\overline{P} = \widetilde{Q}_n$, there exists a constant C_d depending only on d such that

$$\mu^{\mathrm{ess}}(\widetilde{Q}_n + \lambda \phi_n^* \overline{E}) \geq \frac{(\widetilde{Q}_n^{d+1})}{(d+1)(Q_n^d)} + \lambda \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} - C_d C_0 \times \frac{\lambda^2}{r(Q_n; \phi_n^* A)}.$$

Since $(\widetilde{Q}_n^{d+1}) = (\overline{Q}_n^{d+1}) - (d+1)\mu^{\mathrm{abs}}(\overline{Q}_n)(Q_n^d)$ and

$$\mu^{\mathrm{ess}}(\widetilde{Q}_n + \lambda \phi_n^* \overline{E}) = \mu^{\mathrm{ess}}(\overline{Q}_n + \lambda \phi_n^* \overline{E}) - \mu^{\mathrm{abs}}(\overline{Q}_n),$$

we obtain

$$\mu^{\mathrm{ess}}(\overline{Q}_n + \lambda \phi_n^* \overline{E}) \geq \frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)} + \lambda \frac{(\widetilde{Q}_n^d \cdot \phi_n^* \overline{E})}{(Q_n^d)} - C_d C_0 \times \frac{\lambda^2}{r(Q_n; \phi_n^* A)}.$$

On the other hand we have

$$\mu^{\mathrm{ess}}(\overline{D}) + \lambda \liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) = \liminf_{\ell\to\infty} h_{\overline{D}+\lambda\overline{E}}(x_\ell) \geq \mu^{\mathrm{ess}}(\overline{D} + \lambda \overline{E}) \geq \mu^{\mathrm{ess}}(\overline{Q}_n + \lambda \phi_n^* \overline{E}),$$

where the last inequality is given by Lemma 11.5. Therefore

$$\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \frac{1}{\lambda}\Big(\frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)} - \mu^{\mathrm{ess}}(\overline{D})\Big) + \frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)} - C_dC_0\times\frac{\lambda}{r(Q_n;\phi_n^*A)}.$$

It follows from our assumptions and Lemma 2.1 that

$$\lim_{n\to\infty}\frac{1}{r(Q_n;\phi_n^*A)}\Big(\mu^{\mathrm{ess}}(\overline{D}) - \frac{(\overline{Q}_n^{d+1})}{(d+1)(Q_n^d)}\Big) = 0.$$

Therefore, applying the above inequality to a suitable choice of $\lambda=\lambda_n$ and taking the supremum limit on $n\in\mathbb{N}$ gives

$$\liminf_{\ell\to\infty} h_{\overline{E}}(x_\ell) \geq \limsup_{n\to\infty}\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)}. \tag{11.5}$$

To conclude, we adapt the arguments in [YZ23, Proof of Theorem 5.4.3]. Let $n\in\mathbb{N}$ and let $(\overline{\mathcal{Q}}_{n,i})_i$ be a Cauchy sequence defining $\overline{Q}_n\in\widehat{\mathrm{Div}}(\mathcal{U}_n)^{\mathrm{adel}}$. For every $i\in\mathbb{N}$, let $\mathcal{X}_{n,i}$ be a projective arithmetic variety on which $\overline{\mathcal{Q}}_{n,i}$ is defined, $X_{n,i,v}$ the generic fiber of $\mathcal{X}_{n,i}$, and $\overline{Q}_{n,i}=(\overline{\mathcal{Q}}_{n,i})^{\mathrm{ad}}$. By abuse of notation, in the integrals below we write φ_ε and φ to denote their pullbacks to $X_{n,i,v}^{\mathrm{an}}$ and $U_{n,v}^{\mathrm{an}}$. Then

$$\begin{aligned}\frac{(\widetilde{Q}_n^d\cdot\phi_n^*\overline{E})}{(Q_n^d)} = \frac{n_v}{(Q_n^d)}\lim_{i\to\infty}\int_{X_{n,i,v}^{\mathrm{an}}}\varphi_\varepsilon\, c_1(\overline{Q}_{n,i,v})^{\wedge d} &\geq \frac{n_v}{(Q_n^d)}\lim_{i\to\infty}\int_{X_{n,i,v}^{\mathrm{an}}}\varphi\, c_1(\overline{Q}_{n,i,v})^{\wedge d} - n_v\varepsilon\\ &= \frac{n_v}{(Q_n^d)}\int_{U_{n,v}^{\mathrm{an}}}\varphi\, c_1(\overline{Q}_{n,v})^{\wedge d} - n_v\varepsilon\\ &= n_v\int_{U_v^{\mathrm{an}}}\varphi\, d\nu_{n,v} - n_v\varepsilon,\end{aligned}$$

where the penultimate equality follows from (11.2). Moreover, we have

$$n_v\varepsilon + n_v\int_{U_v^{\mathrm{an}}}\varphi\, d\delta_{O(x_\ell)_v} \geq n_v\int_{X_v^{\mathrm{an}}}\varphi_\varepsilon\, d\delta_{O(x_\ell)_v} = h_{\overline{E}}(x_\ell)$$

for every $\ell\in\mathbb{N}$. Combining this with (11.5) and letting ε tend to zero gives

$$\liminf_{\ell\to\infty}\int_{U_v^{\mathrm{an}}}\varphi\, d\delta_{O(x_\ell)_v} \geq \limsup_{n\to\infty}\int_{U_{n,v}^{\mathrm{an}}}\varphi\, d\nu_{n,v},$$

and we obtain (11.4) by applying this to $-\varphi$. □

As in the projective case, the only reason why we assume $\mathcal{U}$ to be normal is that we work with adelic $\mathbb{R}$-divisors. When working with adelic divisors (or equivalently with adelic line bundles) in the sense of [YZ23], one can drop this assumption by passing to a normalization (see Remark 6.2).

Corollary 11.9 ([YZ23, Theorem 5.4.3])**.** *Let $\overline{D}\in\widehat{\mathrm{Div}}(\mathcal{U})_{\mathbb{R}}^{\mathrm{adel}}$ be semipositive with big $D\in\mathrm{Div}(U)_{\mathbb{R}}^{\mathrm{lim}}$. Assume that $\mu^{\mathrm{abs}}(\overline{D})>-\infty$ and*

$$\mu^{\mathrm{ess}}(\overline{D}) = \frac{(\overline{D}^{d+1})}{(d+1)(D^d)}.$$

Then for every $v\in\mathfrak{M}_K$ and for every $\overline{D}$-small generic sequence $(x_\ell)_\ell$ in $U(\overline{K})$, the sequence of probability measures $(\delta_{O(x_\ell)_v})_\ell$ on U_v^{an} converges weakly to $c_1(\overline{D}_v)^{\wedge d}/(D^d)$.

Proof. Apply Theorem 11.8 to the constant sequence $(\phi_n, \overline{Q}_n)$ equal to $(\mathrm{Id}_{\mathcal{U}}, \overline{D})$. □

In [YZ23, Theorem 5.4.3], Yuan and Zhang assume that $\overline{D}$ is nef while here we ask that $\overline{D}$ is semipositive and $\mu^{\mathrm{abs}}(\overline{D}) > -\infty$. This is not more general, since the statement is invariant under translation by $[\infty]$. Recently, Biswas [Bis23] proved the differentiability of the arithmetic volume function and deduced a quasi-projective analogue of Chen's equidistribution theorem [Che11, Corollary 5.5]. It would be interesting to check whether such a statement follows from Theorem 11.8, by adapting the arguments we used in the proof of Corollary 6.5.

Appendix A. Auxiliary results on convex analysis

In this section we recall the basic constructions and properties from convex analysis that are used in our study of toric varieties in §9. We also establish some auxiliary results, most notably Proposition A.3 concerning rate of the decay of the sup-level sets of a concave function as the level approaches the maximum value.

Fix an integer $d \geq 1$. Let $C \subset \mathbb{R}^d$ be a *convex body*, that is a compact convex subset with nonempty interior.

Definition A.1. For a functional $u \in (\mathbb{R}^d)^\vee$ we denote by $w(C, u)$ the length of the interval $u(C) \subset \mathbb{R}$. The *width* of C is then defined as

$$w(C) = \inf_{u \in S^{d-1}} w(C, u),$$

where S^{d-1} denotes the unit sphere of $(\mathbb{R}^d)^\vee \simeq \mathbb{R}^d$.

For another convex body $B \subset \mathbb{R}^d$, the *inradius* of C with respect to B is defined as

$$r(C; B) = \sup\{\lambda \in \mathbb{R}_{>0} \mid \exists\, x \in \mathbb{R}^d \text{ such that } \lambda B + x \subset C\}. \tag{A.1}$$

When B is the unit ball of $\mathbb{R}^d$, it coincides with the classical inradius from Euclidean geometry.

The inradius and the width can be compared up to scalar factors: there are explicit constants $c_1, c_2 > 0$ depending only on d and B such that

$$c_1\, w(C) \leq r(C; B) \leq c_2\, w(C). \tag{A.2}$$

The first inequality comes from [BF87, page 86, inequality (9)], whereas the second is clear from the definitions.

Given convex bodies $B, B' \subset \mathbb{R}^d$, the corresponding inradii compares as

$$r(B', B)\, r(C; B') \leq r(C; B) \leq \frac{1}{r(B; B')}\, r(C; B').$$

Let $f \colon C \to \mathbb{R}$ be a concave function and set $\mu = \sup_{x \in C} f(x)$. For each $t \leq \mu$ we consider the *sup-level set*

$$S_t(f) = \{x \in C \mid f(x) \geq t\}. \tag{A.3}$$

It is a nonempty compact convex subset of C, that is a convex body whenever $t < \mu$. Set also $C_{\max} = S_\mu(f)$.

Definition A.2. For $x_0 \in C$, the *sup-differential* of f at x_0 is the closed convex subset of $(\mathbb{R}^d)^\vee$ defined as

$$\partial f(x_0) = \{u \in (\mathbb{R}^d)^\vee \mid \langle u, x - x_0 \rangle \geq f(x) - f(x_0) \text{ for all } x \in C\}.$$

Its elements are called the *sup-gradients* of f at x_0.

A point $x_0 \in C$ lies in $C_{\max}$ if and only if $0 \in \partial f(x_0)$. When this is the case, the vector 0 is a vertex of the sup-differential $\partial f(x_0)$ if and only if there is no $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ such that both u and $-u$ belong to this convex subset. Equivalently, 0 is *not* a vertex of $\partial f(x_0)$ if and only if there is $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ such that

$$f(x) \leq \mu - |\langle u, x - x_0\rangle| \quad \text{for all } x \in C. \tag{A.4}$$

This condition does not depend on the choice of the point $x_0 \in C_{\max}$: indeed, if (A.4) holds then $\langle u, x_1 - x_0\rangle = 0$ for any $x_1 \in C_{\max}$ and so this inequality also holds when x_0 is replaced by x_1.

The next proposition is a rigidity result that allows to determine when the concave function f admits a "Canadian tent" upper bound like that in (A.4) in terms of the rate of decay of the inradius or the width of its sup-level sets as the level approaches its maximum.

Proposition A.3. *The following conditions are equivalent:*

(1) for any convex body $B \subset \mathbb{R}^d$ *we have* $\lim_{t\to\mu} \dfrac{\mu - t}{r(S_t(f); B)} = 0$,

(2) $\lim_{t\to\mu} \dfrac{\mu - t}{w(S_t(f))} = 0$,

(3) for all $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ *we have* $\lim_{t\to\mu} \dfrac{\mu - t}{w(S_t(f), u)} = 0$,

(4) for any $x_0 \in C_{\max}$ *we have that* 0 *is a vertex of* $\partial f(x_0)$.

Its proof relies on the next two lemmas.

Lemma A.4. *With the previous notation, we have that*

(1) for all $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$, *the function* $t \in (-\infty, \mu) \mapsto \dfrac{\mu - t}{w(S_t(f), u)}$ *is decreasing,*

(2) the function $t \in (-\infty, \mu) \mapsto \dfrac{\mu - t}{w(S_t(f))}$ *is decreasing.*

Proof. First consider the case $d = 1$. Then take $t' < t < \mu$, choose $x_0 \in C_{\max}$ and consider the affine map

$$\iota \colon \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{t - t'}{\mu - t'}\, x_0 + \frac{\mu - t}{\mu - t'}\, x.$$

It follows from the concavity of f that $\iota(S_{t'}(f)) \subset S_t(f)$. Denoting by ℓ the Lebesgue measure on $\mathbb{R}$, this gives

$$\frac{\mu - t}{\mu - t'}\, \ell(S_{t'}(f)) \leq \ell(S_t(f)). \tag{A.5}$$

Now let $d \geq 1$. Take $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ and consider the direct image of f with respect to u, which is the concave function $u_* f \colon u(C) \to \mathbb{R}$ defined by

$$u_* f(y) = \sup\{f(x) \mid x \in C, \langle u, x\rangle = y\} \quad \text{for } y \in u(C).$$

Clearly $\sup_{y\in u(C)} u_* f(y) = \mu$ and $S_t(u_* f) = u(S_t(f))$ for all $t \leq \mu$.

Take again $t' < t < \mu$. The inequality in (A.5) gives

$$\frac{\mu - t'}{w(S_{t'}(f), u)} = \frac{\mu - t'}{\ell(S_{t'}(u_* f))} \geq \frac{\mu - t}{\ell(S_t(u_* f))} = \frac{\mu - t}{w(S_t(f), u)}, \tag{A.6}$$

proving (1). Then (2) follows by choosing $u \in S^{d-1}$ such that $w(S_t(f), u) = w(S_t(f))$ and applying (A.6) to show that

$$\frac{\mu - t'}{w(S_{t'}(f))} \geq \frac{\mu - t'}{w(S_{t'}(f), u)} \geq \frac{\mu - t}{w(S_t(f), u)} = \frac{\mu - t'}{w(S_t(f))}.$$

□

Lemma A.5. *Let $t < \mu$ and $x_0 \in C_{\max}$. Then for all $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ we have*

$$\mu - \frac{\mu - t}{w(S_t(f), u)} |\langle u, x - x_0 \rangle| \geq f(x) \quad \textit{for all } x \in C \setminus S_t(f).$$

Proof. Let $x \in C \setminus S_t(f)$ and set $t' = f(x) < t$. Since both x and x_0 lie in $S_{t'}(f)$ we have $|\langle u, x - x_0 \rangle| \leq w(S_{t'}(f), u)$. Combining this with Lemma A.4(1) we get

$$\mu - f(x) \geq \frac{\mu - t'}{w(S_{t'}(f), u)} |\langle u, x - x_0 \rangle| \geq \frac{\mu - t}{w(S_t(f), u)} |\langle u, x - x_0 \rangle|,$$

which gives the statement. □

Proof of Proposition A.3. The equivalence between (1) and (2) follows from the inequalities in (A.2), and clearly (2) implies (3).

Now assume (3). If (4) does not hold, then there is $x_0 \in C_{\max}$ such that 0 is not a vertex of $\partial f(x_0)$. Let $u \in (\mathbb{R}^d)^\vee \setminus \{0\}$ such that $\mu - |\langle u, x - x_0 \rangle| \geq f(x)$ for all $x \in C$ as in (A.4). For each $t < \mu$, we can choose $y \in S_t(u_* f) = u(S_t(f))$ such that

$$|y - \langle u, x_0 \rangle| \geq \frac{1}{2} w(S_t(f), u).$$

Taking $x \in S_t(f)$ such that $\langle u, x \rangle = y$ we have $|\langle u, x - x_0 \rangle| \geq w(S_t(f), u)/2$ and so

$$\frac{\mu - t}{w(S_t(f), u)} \geq \frac{\mu - f(x)}{w(S_t(f), u)} \geq \frac{|\langle u, x - x_0 \rangle|}{w(S_t(f), u)} \geq \frac{1}{2},$$

which contradicts (3) and thus implies (4).

Finally we show that (4) implies (2). To this end, suppose that (2) does not hold. By Lemma A.4, this implies that there is $c > 0$ such that

$$\frac{\mu - t}{w(S_t(f))} \geq c \quad \text{for all } t < \mu.$$

In particular $\dim(C_{\max}) < d$ because otherwise $w(S_t(f)) \geq w(C_{\max}) > 0$ for all $t < \mu$.

Take sequences $(t_k)_{k \geq 1}$ in $(-\infty, \mu)$ and $(u_k)_{k \geq 1}$ in S^{d-1} with $\lim_{k \to \infty} t_k = \mu$ such that $w(S_{t_k}(f)) = w(S_{t_k}(f), u_k)$ for all k. By the compacity of S^{d-1}, we can assume that $\lim_{k \to \infty} u_k = u$ for a point $u \in S^{d-1}$. Take also $x_0 \in C_{\max}$. By Lemma A.5 we have

$$\mu - c\,|\langle u_k, x - x_0 \rangle| \geq f(x) \quad \text{for all } x \in C \setminus S_{t_k}(f).$$

Now let $x \in C \setminus C_{\max}$. Then $x \notin S_{t_k}(f)$ for $k \gg 1$ and so

$$\mu - c\,|\langle u, x - x_0 \rangle| = \lim_{k \to \infty} \mu - c\,|\langle u_k, x - x_0 \rangle| \geq f(x).$$

Since $\dim(C_{\max}) < d$, this inequality extends to $x \in C_{\max}$ by continuity. Therefore 0 is not a vertex of $\partial f(x_0)$ and so (4) does not hold. □

Definition A.6. The concave function f is said to be *wide (at its maximum)* if it verifies any of the equivalent conditions in Proposition A.3.

Let I be a finite set, and for each $i \in I$ let $n_i > 0$ be a real number and $f_i\colon C \to \mathbb{R}$ a concave function such that

$$f = \sum_{i \in I} n_i f_i. \tag{A.7}$$

Definition A.7. A *balanced family of sup-gradients* for the decomposition in (A.7) is a family of vectors $u_i \in (\mathbb{R}^n)^\vee$, $i \in I$, such that there is $x_0 \in C_{\max}$ with $u_i \in \partial f_i(x_0)$ for all i and $\sum_{i\in I} n_i u_i = 0$.

Proposition A.8. *The decomposition $f = \sum_{i\in I} n_i f_i$ admits a balanced family of sup-gradients. If f is wide, then this family is unique.*

Proof. For the first statement, we can derive from the decomposition of f in (A.7) the decomposition of its sup-differential at any point $x_0 \in C$ as

$$\partial f(x_0) = \sum_{i\in I} n_i \, \partial f_i(x_0), \tag{A.8}$$

see for instance [BPS14, Proposition 2.3.9]. If $x_0 \in C_{\max}$ then $0 \in \partial f(x_0)$, and we obtain a balanced family of sup-gradients by considering any decomposition of this zero vector according to (A.8).

If f is wide then 0 is a vertex of $0 \in \partial f(x_0)$. The second statement is then given by [BPRS19, Proposition 3.15]. □

We denote by $\mathrm{MV}(C_1, \dots, C_d)$ the mixed volume of a family of d convex bodies of $\mathbb{R}^d$, and by $\mathrm{MI}(f_0, \dots, f_d)$ the mixed integral of a family of concave functions on convex bodies of $\mathbb{R}^d$, both with respect to the Lebesgue measure of $\mathbb{R}^d$ [BPS14, Definitions 2.7.14 and 2.7.16].

Proposition A.9. *For $i = 0, \dots, d$ let $f_i\colon C_i \to \mathbb{R}$ be a continuous concave function on a convex body, and $(C_{i,n})_n$ a sequence of convex bodies approaching C_i uniformly. Then*

$$\lim_{n\to\infty} \mathrm{MI}(f_0|_{C_{0,n}}, \dots, f_d|_{C_{d,n}}) = \mathrm{MI}(f_0, \dots, f_d).$$

Proof. For the sequences of convex bodies $(C_{i,n})_n$, $i = 0, \dots, d$, approaching C_i uniformly, the same holds for the corresponding Minkowski sums. Furthermore, for the sequences of concave functions $(f_i|_{C_{i,n}})_n$, $i = 0, \dots, d$, their sup-convolutions approach the corresponding sup-convolutions of the concave functions f_i, $i = 0, \dots, d$, restricted to the domain of the approximant.

The statement then follows from the continuity of the integral of a continuous concave function on a convex body with respect to the restriction of its domain to a sequence of convex bodies approaching it. □

Finally, the next lemma gives an explicit expression for the mixed integral when all but one of the involved concave functions are equal to an affine one.

Lemma A.10. *Let $C \subset \mathbb{R}^d$ be a convex body and $\ell\colon \mathbb{R}^d \to \mathbb{R}$ an affine function with linear part $u \in (\mathbb{R}^d)^\vee$ and constant $c \in \mathbb{R}$. Then for any concave function on a convex body $g\colon B \to \mathbb{R}$ we have*

$$\mathrm{MI}(\ell|_C, \dots, \ell|_C, g) = \mathrm{MI}(u|_C, \dots, u|_C, u|_B) + c\, d\, \mathrm{MV}(C, \dots, C, B) - d!\, \mathrm{vol}(C)\, g^\vee(u),$$

where $g^\vee$ denotes the Legendre-Fenchel dual of g.

Proof. The proof is based on the results in [Gua18, §1.3] and we will freely use the notation therein. This requires that both C and B are polytopes. Nevertheless, the general case follows from this one by the continuity of the formula with respect to uniform approximation.

First, by Corollary 1.10 in *loc. cit.* we have

$$\mathrm{MI}(\ell|_C, \dots, \ell|_C, g) = \mathrm{MI}(u|_C, \dots, u|_C, g) + c\, d\, \mathrm{MV}(C, \dots, C, B).$$

By Proposition 1.5 in *loc. cit.*, we can compute the mixed real Monge-Ampère measure of $u|_C$ as

$$\mathrm{M}\mathcal{M}(u|_C, \dots, u|_C) = d!\, \mathrm{vol}(C)\, \delta_u,$$

where δ_u is the Dirac measure on u. Then applying the recursive formula in Theorem 1.6 in *loc. cit.* to g and to $u|_B$ we get

$$\mathrm{MI}(u|_C, \dots, u|_C, g) - \mathrm{MI}(u|_C, \dots, u|_C, u|_B) = d!\, \mathrm{vol}(C)\, ((u|_B)^\vee(u) - g^\vee(u)).$$

In fact, the first part of this recursive formula depends only on the domain of these concave functions, and so they cancel each other. Finally, from the definition of the Legendre-Fenchel duality we get $(u|_B)^\vee(u) = \inf_{x \in B}(\langle u, x\rangle - \langle u, x\rangle) = 0$. □

Remark A.11. Given two convex bodies $C_1, C_2 \subset \mathbb{R}^d$, for each $u \in (\mathbb{R}^d)^\vee$ the sup-convolution of the restrictions $u|_{C_1}$ and $u|_{C_2}$ coincides with the restriction $u|_{C_1+C_2}$, that is

$$u|_{C_1} \boxplus u|_{C_2} = u|_{C_1+C_2}.$$

Hence the mixed integral $\mathrm{MI}(u|_C, \dots, u|_C, u|_B)$ can be written as an alternating sum of integrals of the linear function u on different Minkowski sums of C and B. In particular, the map $u \mapsto \mathrm{MI}(u|_C, \dots, u|_C, u|_B)$ is linear.

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