AN ARITHMETIC BERNŠTEIN-KUŠNIRENKO INEQUALITY

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Abstract. We present an upper bound for the height of the isolated zeros in the torus of a system of Laurent polynomials over an adelic field satisfying the product formula. This upper bound is expressed in terms of the mixed integrals of the local roof functions associated to the chosen height function and to the system of Laurent polynomials. We also show that this bound is close to optimal in some families of examples.

This result is an arithmetic analogue of the classical Bernštein-Kušnirenko theorem. Its proof is based on arithmetic intersection theory on toric varieties.

1. Introduction

The classical Bernštein-Kušnirenko theorem bounds the number of isolated zeros of a system of Laurent polynomials over a field, in terms of the mixed volume of their Newton polytopes [Kuš76, Ber75]. This result, initiated by Kušnirenko and put into final form by Bernštein, is also known as the BKK theorem to acknowledge Khovanskiĭ's contributions to this subject. It shows how a geometric problem (the counting of the number of solutions of a system of equations) can be translated into a combinatorial, simpler one. It is commonly used to predict when a given system of equations has a small number of solutions. As such, it is a cornerstone of polynomial equation solving and has motivated a large amount of work and results over the past 25 years, see for instance [GKZ94, Stu02, PS08] and the references therein.

When dealing with Laurent polynomials over a field with an arithmetic structure like the field of rationals, it might be also important to control the arithmetic complexity or *height* of their zero set. In this paper, we present an arithmetic version of the BKK theorem, bounding the height of the isolated zeros of a system of Laurent polynomials over such a field. It is a refinement of the arithmetic Bézout theorem that takes into account the finer monomial structure of the system.

Previous results in this direction were obtained by Maillot [Mai00] and by the second author [Som05]. Our current result improves these previous upper bounds, and generalizes them to adelic fields satisfying the product formula, and to height functions associated to arbitrary nef toric metrized divisors.

Let $\mathbb{K}$ be a field and $\overline{\mathbb{K}}$ its algebraic closure. Let $M \simeq \mathbb{Z}^n$ be a lattice and set

$$\mathbb{K}[M] \simeq \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \text{and} \quad \mathbb{T}_M = \operatorname{Spec}(\mathbb{K}[M]) \simeq \mathbb{G}_{\mathrm{m},\mathbb{K}}^n$$

for its group $\mathbb{K}$-algebra and algebraic torus over $\mathbb{K}$, respectively. For a family of Laurent polynomials $f_1, \dots, f_n \in \mathbb{K}[M]$, we denote by $Z(f_1, \dots, f_n)$ the 0-cycle of $\mathbb{T}_M$ given by the isolated solutions of the system of equations

$$f_1 = \cdots = f_n = 0$$

with their corresponding multiplicities (Definition 2.7).

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Set $M_{\mathbb{R}} = M \otimes \mathbb{R} \simeq \mathbb{R}^n$. Let vol_M be the Haar measure on $M_{\mathbb{R}}$ normalized so that M has covolume 1, and let MV_M be the corresponding mixed volume function. For $i = 1, \dots, n$, let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of f_i. The BKK theorem amounts to the upper bound

$$\deg(Z(f_1, \dots, f_n)) \leq \mathrm{MV}_M(\Delta_1, \dots, \Delta_n), \tag{1.1}$$

which is an equality when the f_i's are generic with respect to their Newton polytopes [Kuš76, Ber75], see also Theorem 2.9.

Now suppose that $\mathbb{K}$ is endowed with a set of places $\mathfrak{M}$, so that the pair $(\mathbb{K}, \mathfrak{M})$ is an adelic field (Definition 3.1). Each place $v \in \mathfrak{M}$ consists of an absolute value $|\cdot|_v$ on $\mathbb{K}$ and a weight $n_v > 0$. We assume that this set of places satisfies the *product formula*, namely, for all $\alpha \in \mathbb{K}^\times$,

$$\sum_{v \in \mathfrak{M}} n_v \log |\alpha|_v = 0.$$

The classical examples of adelic fields satisfying the product formula are the global fields, that is, number fields and function fields of regular projective curves.

Let X be toric compactification of $\mathbb{T}_M$ and $\overline{D}_0$ a nef toric metrized divisor on X, see §4 and §5 for details. This data gives a notion of height for 0-cycles of X, see [BG06, Chapter 2] or §4. The height

$$\mathrm{h}_{\overline{D}_0}(Z(f_1, \dots, f_n))$$

is a nonnegative real number, and it is our aim to bound this quantity in terms of the monomial expansion of the f_i's.

The toric Cartier divisor D_0 defines a polytope $\Delta_0 \subset M_{\mathbb{R}}$. Following [BPS14], we associate to $\overline{D}_0$ an adelic family of continuous concave functions $\vartheta_{0,v} \colon \Delta_0 \to \mathbb{R}$, $v \in \mathfrak{M}$, called the local roof functions of $\overline{D}_0$.

For $i = 1, \dots, n$, write

$$f_i = \sum_{m \in M} \alpha_{i,m} \chi^m$$

with $\alpha_{i,m} \in \mathbb{K}$. Let $N_{\mathbb{R}} = M_{\mathbb{R}}^\vee \simeq \mathbb{R}^n$ be the dual space and, for each place $v \in \mathfrak{M}$, consider the concave function $\psi_{i,v} \colon N_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\psi_{i,v}(u) = \begin{cases} -\log\Big(\displaystyle\sum_{m \in M} |\alpha_{i,m}|_v \,\mathrm{e}^{-\langle m,u\rangle}\Big) & \text{if } v \text{ is Archimedean,} \\ -\log\Big(\displaystyle\max_{m \in M} |\alpha_{i,m}|_v \,\mathrm{e}^{-\langle m,u\rangle}\Big) & \text{if } v \text{ is non-Archimedean.} \end{cases} \tag{1.2}$$

The Legendre-Fenchel dual $\vartheta_{i,v} = \psi_{i,v}^\vee$ is a continuous concave function on Δ_i.

We denote by MI_M the mixed integral of a family of $n+1$ concave functions on convex bodies of $M_{\mathbb{R}}$ (Definition 5.6). It is the polarization of $(n+1)!$ times the integral of a concave function on a convex body. It is a functional that is symmetric and linear in each variable with respect to the sup-convolution of concave functions, see [PS08, §8] for details.

The following is the main result of this paper.

Theorem 1.1. *Let $f_1, \dots, f_n \in \mathbb{K}[M]$, and let X be a proper toric variety with torus $\mathbb{T}_M$ and $\overline{D}_0$ a nef toric metrized divisor on X. Let $\Delta_0 \subset M_{\mathbb{R}}$ be the polytope of D_0 and, for $v \in \mathfrak{M}$, let $\vartheta_{0,v} \colon \Delta_0 \to \mathbb{R}$ be v-adic roof function of $\overline{D}_0$. For $i = 1, \dots, n$,* *let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of f_i and, for $v \in \mathfrak{M}$, let $\vartheta_{i,v}\colon \Delta_i \to \mathbb{R}$ be the Legendre-Fenchel dual of the concave function in* (1.2). *Then*

$$\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n)) \le \sum_{v\in\mathfrak{M}} n_v \,\mathrm{MI}(\vartheta_{0,v},\dots,\vartheta_{n,v}). \tag{1.3}$$

Using the basic properties of the mixed integral, we can bound the terms in the right-hand side of (1.3) in terms of mixed volumes. From this, we can derive the bound (Corollary 6.8)

$$\begin{aligned}\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n)) \le \mathrm{MV}_M(\Delta_1,\dots,\Delta_n)\Big(\sum_{v\in\mathfrak{M}} \max\vartheta_{0,v}\Big)&\\ +\sum_{i=1}^{n}\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n)\ell(f_i)&,\end{aligned}$$

where $\ell(f_i)$ denotes the (logarithmic) length of f_i (Definition 6.6). This bound might be compared with the one given by the arithmetic Bézout theorem (Corollary 6.9).

The following example illustrates a typical application of these results. It concerns two height functions applied to the same 0-cycle. Our upper bounds are close to optimal for both of them and, in particular, they reflect their very different behavior on this family of Laurent polynomials.

Example 1.2. Take two integers $d,\alpha \ge 1$ and consider the system of Laurent polynomials

$$f_1 = x_1-\alpha,\quad f_2 = x_2-\alpha x_1^d\quad,\dots,\quad f_n = x_n-\alpha x_{n-1}^d\quad \in \mathbb{Q}[x_1^{\pm1},\dots,x_n^{\pm1}].$$

The 0-cycle $Y := Z(f_1,\dots,f_n)$ of $\mathbb{G}^n_{\mathrm{m},\mathbb{Q}}$ is the single point $(\alpha,\alpha^{d+1},\dots,\alpha^{d^{n-1}+\dots+d+1})$ with multiplicity 1.

Let $\mathbb{P}^n_{\mathbb{Q}}$ be the n-dimensional projective space over $\mathbb{Q}$ and $\overline{E}^{\mathrm{can}}$ the divisor of the hyperplane at infinity, equipped with the canonical metric. Its associated height function is the Weil height. We consider two toric compactifications X_1 and X_2 of $\mathbb{G}^n_{\mathrm{m}}$. These are given by compactifying the torus via the equivariant embeddings $\iota_i\colon \mathbb{G}^n_{\mathrm{m}} \hookrightarrow \mathbb{P}^n_{\mathbb{Q}}$, $i=1,2$, respectively defined, for $p=(p_1,\dots,p_n)\in \mathbb{G}^n_{\mathrm{m}}(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^{\times})^n$, by

$$\iota_1(p) = (1:p_1:\dots:p_n) \quad\text{and}\quad \iota_2(p) = (1:p_1:p_2p_1^{-d}:\dots:p_np_{n-1}^{-d}).$$

Set $\overline{D}_i = \iota_i^*\overline{E}^{\mathrm{can}}$, $i=1,2$, which are nef toric metrized divisors on X_i. By an explicit computation, we show that

$$\mathrm{h}_{\overline{D}_1}(Y) = \Big(\sum_{i=1}^{n} d^{i-1}\Big)\log(\alpha) \quad\text{and}\quad \mathrm{h}_{\overline{D}_2}(Y) = \log(\alpha).$$

On the other hand, the upper bounds given by Theorem 1.1 are

$$\mathrm{h}_{\overline{D}_1}(Y) \le \Big(\sum_{i=1}^{n} d^{i-1}\Big)\log(\alpha+1) \quad\text{and}\quad \mathrm{h}_{\overline{D}_2}(Y) \le n\log(\alpha+1),$$

see Example 7.2 for details.

To the best of our knowledge, the first arithmetic analogue of the BKK theorem was proposed by Maillot [Mai00, Corollaire 8.2.3], who considered the case of canonical toric metrics. Another result in this direction was obtained by the second author for the unmixed case and also canonical toric metrics [Som05, Théorème 0.3]. Theorem 1.1

improves and refines these previous upper bounds, and generalizes them to adelic fields satisfying the product formula and to height functions associated to arbitrary nef toric metrized divisors, see §7 for details.

The key point in the proof of Theorem 1.1 consists of the construction, for each Laurent polynomial f_i, of a nef toric metrized divisor $\overline{D}_i$ on a proper toric variety X, such that f_i corresponds to a small section of $\overline{D}_i$ (Proposition 6.2 and Lemma 6.4). The proof then proceeds by applying the constructions and results of [BPS14, BMPS16] and basic results from arithmetic intersection theory.

Trying to keep our results at a similar level of generality as those in [BPS14], we faced difficulties to define and study global heights of cycles over adelic fields. This lead us to a more detailed study of these notions. In particular, we give a new notion of adelic field extension that preserves the product formula (Proposition 3.7) and a well-defined notion of global height for cycles with respect to metrized divisors that are generated by small sections (Proposition-Definition 4.15).

As an application of Theorem 1.1, we give an upper bound for the size of the coefficients of the $\boldsymbol{u}$-resultant of the direct image under a monomial map of the solution set of a system of Laurent polynomial equations.

For the simplicity of the exposition, set $\mathbb{K}=\mathbb{Q}$ and $M=\mathbb{Z}^n$. Let $r\geq 0$, $\boldsymbol{m}_0=(m_{0,0},\dots,m_{0,r})\in(\mathbb{Z}^n)^{r+1}$ and $\boldsymbol{\alpha}_0=(\alpha_{0,0},\dots,\alpha_{0,r})\in(\mathbb{Z}\setminus\{0\})^{r+1}$, and consider the map $\varphi_{\boldsymbol{m}_0,\boldsymbol{\alpha}_0}\colon \mathbb{G}^n_{\mathrm{m},\mathbb{Q}}\to\mathbb{P}^r_{\mathbb{Q}}$ defined by

$$\varphi_{\boldsymbol{m}_0,\boldsymbol{\alpha}_0}(p)=(\alpha_{0,0}\chi^{m_{0,0}}(p):\dots:\alpha_{0,r}\chi^{m_{0,r}}(p)). \tag{1.4}$$

For a 0-cycle W of $\mathbb{P}^r_{\mathbb{Q}}$, let $\boldsymbol{u}=(u_0,\dots,u_r)$ be a set of $r+1$ variables and denote by $\mathrm{Res}(W)\in\mathbb{Z}[u_1,\dots,u_r]$ its primitive $\boldsymbol{u}$-resultant (Definition 7.4). It is well-defined up a sign. For a vector $\boldsymbol{\alpha}$ with integer entries, we denote by $\ell(\boldsymbol{\alpha})$ the logarithm of the sum of the absolute values of its entries.

Theorem 1.3. *Let $f_1,\dots,f_n\in\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1}]$, $\boldsymbol{m}_0\in(\mathbb{Z}^n)^{r+1}$ and $\boldsymbol{\alpha}_0\in(\mathbb{Z}\setminus\{0\})^{r+1}$ with $r\geq 0$. Set $\Delta_0=\mathrm{conv}(m_{0,0},\dots,m_{0,r})\subset\mathbb{R}^n$ and let φ be the monomial map associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (1.4). For $i=1,\dots,n$, let $\Delta_i\subset\mathbb{R}^n$ be the Newton polytope of f_i, and $\boldsymbol{\alpha}_i$ the vector of nonzero coefficients of f_i. Then*

$$\ell(\mathrm{Res}(\varphi_*Z(f_1,\dots,f_n)))\leq\sum_{i=0}^n \mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n)\,\ell(\boldsymbol{\alpha}_i).$$

The paper is organized as follows. In §2 we recall some preliminary material on intersection theory and on the algebraic geometry of toric varieties. In §3 we study adelic fields satisfying the product formula. In §4 we recall the notions of metrized divisors and its associated measures and heights, with an emphasis on the 0-dimensional case. In §5 we explain the notation and basic constructions of the arithmetic geometry of toric varieties. In §6 we prove Theorem 1.1, whereas in §7 we give examples illustrating the applications of our bounds, and prove Theorem 1.3.

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2. Intersection theory and toric varieties

In this section, we recall the proof of the Bernštein-Kušnirenko theorem using intersection theory on toric varieties, which is the model that we follow in our treatment of the arithmetic version of this result. Presenting this proof also allows us to introduce the basic definitions and results on the intersection of Cartier divisors with cycles, and on the algebraic geometry of toric varieties. For more details on these subjects, we refer to [Ful84, Laz04] and to [Ful93].

Let K be an infinite field and X a variety over K of dimension n. For $0 \le k \le n$, the *group of k-cycles*, denoted by $Z_k(X)$, is the free abelian group on the k-dimensional irreducible subvarieties of X. Thus, a *k-cycle* is a finite formal sum

$$Y = \sum_V m_V V$$

where the V's are k-dimensional irreducible subvarieties of X and the m_V's are integers. The *support* of Y, denoted by $|Y|$, is the union of the subvarieties V such that $m_V \neq 0$. The cycle Y is *effective* if $m_V \ge 0$ for every V. Given $Y, Y' \in Z_k(X)$, we write $Y' \le Y$ whenever $Y - Y'$ is effective.

Let Z be a subscheme of X of pure dimension k. For an irreducible component V of Z, we denote by $\mathcal{O}_{V,Z}$ the local ring of Z along V, and by $l_{\mathcal{O}_{V,Z}}(\mathcal{O}_{V,Z})$ its length as an $\mathcal{O}_{V,Z}$-module. The k-cycle associated to Z is then defined as

$$[Z] = \sum l_{\mathcal{O}_{V,Z}}(\mathcal{O}_{V,Z})\, V,$$

the sum being over the irreducible components of Z.

Let V be an irreducible subvariety of X of codimension one and f a regular function on an open subset U of X such that $U \cap V \neq \emptyset$. The *order of vanishing* of f along V is defined as

$$\mathrm{ord}_V(f) = l_{\mathcal{O}_{V,X}(U)}(\mathcal{O}_{V,X}(U)/(f)).$$

For a Cartier divisor D on X, the *order of vanishing* of D along V is defined as

$$\mathrm{ord}_V(D) = \mathrm{ord}_V(g) - \mathrm{ord}_V(h),$$

with $g, h \in \mathcal{O}_{V,X}(U)$ such that g/h is a local equation of D on an open subset U of X with $U \cap V \neq \emptyset$. This definition does not depend on the choice of U, g and h. Moreover, $\mathrm{ord}_V(D) = 0$ for all but a finite number of V's. The *support* of D, denoted by $|D|$, is the union of these subvarieties V such that $\mathrm{ord}_V(D) \neq 0$. The Weil divisor associated to D is then defined as

$$D \cdot X = \sum_V \mathrm{ord}_V(D)\, V, \tag{2.1}$$

the sum being over the irreducible components of $|D|$.

Now let W be an irreducible subvariety of X of dimension k. If $W \not\subset |D|$, then D restricts to a Cartier divisor on W. In this case, we define $D \cdot W$ as the Weil divisor of W obtained by restricting (2.1) to W. This gives a $(k-1)$-cycle of X. If $W \subset |D|$, then we set $D \cdot W = 0$, the zero element of $Z_{k-1}(X)$. We extend by linearity this intersection product to a morphism

$$Z_k(X) \longrightarrow Z_{k-1}(X), \quad Y \longmapsto D \cdot Y,$$

with the convention that $Z_{-1}(X) = 0$, the zero group.

For $0 \leq r \leq n$ and Cartier divisors D_i on X, $i = 1, \dots, r$, we define inductively the intersection product $\prod_{i=1}^r D_i \in Z_{n-r}(X)$ by

$$\prod_{i=1}^t D_i = \begin{cases} X & \text{if } t = 0, \\ D_1 \cdot \prod_{i=2}^t D_i & \text{if } 1 \leq t \leq r. \end{cases}$$

Definition 2.1. Let Y be a k-cycle of X and $D_1, \dots, D_r$ Cartier divisors on X, with $r \leq k$. We say that $D_1, \dots, D_r$ *intersect* Y *properly* if, for every subset $I \subset \{1, \dots, r\}$,

$$\dim \Big(|Y| \cap \bigcap_{i \in I} |D_i| \Big) = k - \# I.$$

If $D_1, \dots, D_r$ intersect X properly, then the cycle $\prod_{i=1}^r D_i$ does not depend on the order of the D_i's [Ful84, Corollary 2.4.2]. This conclusion does not necessarily hold if these divisors do not intersect properly.

Example 2.2. Let $X = \mathbb{A}_K^2$ and consider the principal Cartier divisors $D_1 = \mathrm{div}(x_1 x_2)$ and $D_2 = \mathrm{div}(x_1)$. Then

$$D_1 \cdot D_2 = 0 \quad \text{and} \quad D_2 \cdot D_1 = (0, 0).$$

Proposition 2.3. *Let X be a Cohen-Macaulay variety over K of pure dimension n and $D_1, \dots, D_n$ Cartier divisors on X. Let s_i be a global section of $\mathcal{O}(D_i)$, $i = 1, \dots, n$, and write*

$$\prod_{i=1}^n \mathrm{div}(s_i) = \sum_p m_p \, p \in Z_0(X), \tag{2.2}$$

where the sum is over the closed points p of X and $m_p \in \mathbb{Z}$. This 0-cycle is effective and, for each isolated closed point p of the intersection $\bigcap_{i=1}^n |\mathrm{div}(s_i)|$,

$$m_p = \dim_K(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)),$$

where U is a trivializing neighborhood of p, and f_i is a defining function for s_i on U, $i = 1, \dots, n$.

Proof. The fact that the cycle in (2.2) is effective follows from the hypothesis that the s_i's are global sections.

For the second statement, by possibly replacing U with a smaller open neighborhood of p, we can assume that $\mathrm{div}(s_1), \dots, \mathrm{div}(s_n)$ intersect X properly on U, and so this intersection is of dimension 0. By [Ful84, Proposition 7.1 and Example 7.1.10],

$$m_p = l_{\mathcal{O}_{p,X}(U)}(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)).$$

By [Ful84, Example A.1.1], we have the equality

$$l_{\mathcal{O}_{p,X}(U)}(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)) = \dim_K(\mathcal{O}_{p,X}(U)/(f_1, \dots, f_n)),$$

completing the proof. □

For the rest of this section, we assume that the variety X is projective. With this hypothesis, Chow's moving lemma allows to construct, given a cycle and a family of Cartier divisors, another family of linearly equivalent Cartier divisors intersecting the given cycle properly, in the sense of Definition 2.1.

Definition 2.4. Let Y be a k-cycle of X and $D_1, \dots, D_k$ Cartier divisors on X. The *degree* of Y with respect to $D_1, \dots, D_k$, denoted by $\deg_{D_1,\dots,D_k}(Y)$, is inductively defined by the rules:

(1) if $k = 0$, write $Y = \sum_p m_p\, p$ and set $\deg(Y) = \sum_p m_p\, [\mathrm{K}(p) : K]$;
(2) if $k \geq 1$, choose a rational section s_k of $\mathcal{O}(D_k)$ such that $\mathrm{div}(s_k)$ intersects Y properly, and set $\deg_{D_1,\dots,D_k}(Y) = \deg_{D_1,\dots,D_{k-1}}(\mathrm{div}(s_k) \cdot Y)$.

The degree of a cycle with respect to a family of Cartier divisors does not depend on the choice of the rational section s_k in (2), see for instance [Ful84, §2.5] or [Laz04, §1.1.C].

A Cartier divisor D on X is *nef* if $\deg_D(C) \geq 0$ for every irreducible curve C of X. By Kleiman's theorem [Laz04, §1.4.B], for a family of nef Cartier divisors $D_1, \dots, D_k$ on X and an effective k-cycle Y of X,

$$\deg_{D_1,\dots,D_k}(Y) \geq 0. \tag{2.3}$$

Proposition 2.5. *Let Y be an effective k-cycle of X and $D_1, \dots, D_k$ nef Cartier divisors on X. Let s_k be a global section of $\mathcal{O}(D_k)$. Then*

$$0 \leq \deg_{D_1,\dots,D_{k-1}}(\mathrm{div}(s_k) \cdot Y) \leq \deg_{D_1,\dots,D_k}(Y).$$

Proof. Since Y is effective and s_k is a global section, $\mathrm{div}(s_k) \cdot Y$ is also effective. Since $D_1, \dots, D_{k-1}$ are nef, by (2.3) we have that $\deg_{D_1,\dots,D_{k-1}}(\mathrm{div}(s_k) \cdot Y) \geq 0$, proving the lower bound.

For the upper bound, we reduce without loss of generality to the case when $Y = V$ is an irreducible subvariety of dimension k. If $V \subset |\mathrm{div}(s_k)|$, then $\mathrm{div}(s_k) \cdot Y = 0 \in Z_{k-1}(X)$. Hence $\deg(\mathrm{div}(s_k) \cdot Y) = 0$ and the bound follows from the nefness of the D_i's. Otherwise, from the definition of the degree,

$$\deg_{D_1,\dots,D_{k-1}}(\mathrm{div}(s_k) \cdot V) = \deg_{D_1,\dots,D_k}(V),$$

which completes the proof. □

Corollary 2.6. *Let $D_1, \dots, D_n$ be nef Cartier divisors on X and, for $i = 1, \dots, n$, let s_i be a global section of $\mathcal{O}(D_i)$. Then*

$$0 \leq \deg\left(\prod_{i=1}^{n} \mathrm{div}(s_i)\right) \leq \deg_{D_1,\dots,D_n}(X).$$

We now turn to toric varieties. Let $M \simeq \mathbb{Z}^n$ be a lattice and set

$$K[M] \simeq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \text{and} \quad \mathbb{T} = \mathrm{Spec}(K[M]) \simeq \mathbb{G}_{\mathrm{m},K}^n \tag{2.4}$$

for its group K-algebra and algebraic torus over K, respectively. The elements of M correspond to the characters of $\mathbb{T}$ and, given $m \in M$, we denote by $\chi^m \in \mathrm{Hom}(\mathbb{T}, \mathbb{G}_{\mathrm{m},K})$ the corresponding character. Set also $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Let $N = M^{\vee} \simeq \mathbb{Z}^n$ be the dual lattice and set $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Given a complete fan Σ in $N_{\mathbb{R}}$, we denote by X_{Σ} the associated toric variety with torus $\mathbb{T}$. It is a proper normal variety over K containing $\mathbb{T}$ as a dense open subset. When the fan Σ is *regular*, in the sense that it is induced by a piecewise linear concave function on $N_{\mathbb{R}}$, the toric variety X_{Σ} is projective.

Set $X = X_{\Sigma}$ for short. Let D be a toric Cartier divisor on X, and denote by Ψ_D its associated virtual support function on Σ. This is a piecewise linear function

$\Psi_D\colon N_{\mathbb{R}} \to \mathbb{R}$ satisfying that, for each cone $\sigma \in \Sigma$, there exists $m \in M$ such that, for all $u \in \sigma$,

$$\Psi_D(u) = \langle m, u\rangle.$$

The condition that Ψ_D is concave is both equivalent to the conditions that D is nef and that the line bundle $\mathcal{O}(D)$ is globally generated. This line bundle $\mathcal{O}(D)$ is a subsheaf of the sheaf of rational functions of X. For each $m \in M$, the character χ^m is a rational function of X, and so it induces a rational section of $\mathcal{O}(D)$ that is regular and nowhere vanishing on $\mathbb{T}$. The rational section corresponding to the point $m = 0$ is called the *distinguished rational section* of $\mathcal{O}(D)$ and denoted by s_D.

The toric Cartier divisor D also determines the lattice polytope of $M_{\mathbb{R}}$ given by

$$\Delta_D = \{x \in M_{\mathbb{R}} \mid \ \langle x, u\rangle \geq \Psi_D(u) \text{ for every } u \in N_{\mathbb{R}}\}.$$

A rational section corresponding to a point $m \in M$ is global if and only if $m \in \Delta_D$. The global sections corresponding to the lattice points of Δ_D form a K-basis for the space of global sections of $\mathcal{O}(D)$. Identifying each character χ^m with the corresponding rational section ς_m of $\mathcal{O}(D)$, we have the decomposition

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{m \in \Delta_D \cap M} K \cdot \varsigma_m. \tag{2.5}$$

Now let $\Delta_1, \dots, \Delta_r$ be lattice polytopes in $M_{\mathbb{R}}$. For each Δ_i, we consider its *support function*, which is the piecewise linear concave function with lattice slopes $\Psi_{\Delta_i}\colon N_{\mathbb{R}} \to \mathbb{R}$ given by

$$\Psi_{\Delta_i}(u) = \min_{x \in \Delta_i} \langle x, u\rangle. \tag{2.6}$$

Let Σ be a regular complete fan in $N_{\mathbb{R}}$ compatible with the collection $\Delta_1, \dots, \Delta_r$, in the sense that the Ψ_{Δ_i}'s are virtual support functions on Σ. Such a fan can be constructed by taking any regular complete fan in $N_{\mathbb{R}}$ refining the complex of cones that are normal to the faces of Δ_i, for all i. Let X be the toric variety corresponding to this fan and D_i the toric Cartier divisor on X corresponding to these virtual support functions. By construction, Ψ_{Δ_i} is concave. Hence D_i is nef and $\mathcal{O}(D_i)$ is globally generated, and its associated polytope coincides with Δ_i.

Let vol_M be the Haar measure on $M_{\mathbb{R}}$ such that M has covolume 1, and take $r = n$. The *mixed volume* of $\Delta_1, \dots, \Delta_n$ is defined as the alternating sum

$$\mathrm{MV}_M(\Delta_1, \dots, \Delta_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \leq i_1 < \dots < i_j \leq n} \mathrm{vol}_M(\Delta_{i_1} + \dots + \Delta_{i_j}).$$

A fundamental result in toric geometry states that the degree of a toric variety with respect to a family of nef toric Cartier divisors is given by the mixed volume of its polytopes [Ful93, §5.4]. In our present setting, this amounts to the formula

$$\deg_{D_1, \dots, D_n}(X) = \mathrm{MV}_M(\Delta_1, \dots, \Delta_n). \tag{2.7}$$

We turn to 0-cycles of the torus defined by families of Laurent polynomials.

Definition 2.7. Let $f_1, \dots, f_n \in K[M]$ and denote by $V(f_1, \dots, f_n)_0$ the set of isolated closed points in the variety defined by this family of Laurent polynomials. For each $p \in V(f_1, \dots, f_n)_0$, let $\mathfrak{m}_p$ be the maximal ideal of $K[M]$ corresponding to p and set

$$\mu_p = \dim_K(K[M]_{\mathfrak{m}_p}/(f_1, \dots, f_n)).$$

The *0-cycle associated to* $f_1, \dots, f_n$ is defined as

$$Z(f_1, \dots, f_n) = \sum_{p \in V(f_1,\dots,f_n)_0} \mu_p \, p \in Z_0(\mathbb{T}).$$

Let $f = \sum_{m \in M} \alpha_m \chi^m \in K[M]$ be a Laurent polynomial. Its *support* is defined as the finite subset of M of the exponents of its nonzero terms, that is $\operatorname{supp}(f) = \{m \mid \alpha_m \neq 0\}$. The *Newton polytope* of f is the lattice polytope in $M_{\mathbb{R}}$ given by the convex hull of its support, that is $\mathrm{N}(f) = \operatorname{conv}(\operatorname{supp}(f))$.

Proposition 2.8. *Let $f_1, \dots, f_n \in K[M]$. Let Σ be a regular complete fan in $N_{\mathbb{R}}$ compatible with the Newton polytopes of the f_i's and, for $i = 1, \dots, n$, let D_i be the Cartier divisor on X_Σ associated to $\mathrm{N}(f_i)$ and s_i the global section of $\mathcal{O}(D_i)$ corresponding to f_i as in* (2.5). *Write $\prod_{i=1}^n \operatorname{div}(s_i) = \sum \nu_p \, p$, where the sum is over the closed points of X_Σ. Then*

(1) *for every $p \in V(f_1, \dots, f_n)_0$, we have $\nu_p = \dim_K(K[M]_{\mathfrak{m}_p}/(f_1, \dots, f_n))$;*
(2) *the inequality $Z(f_1, \dots, f_n) \le \prod_{i=1}^n \operatorname{div}(s_i)$ holds.*

Proof. We have that $\bigcap_{i=1}^n |\operatorname{div}(s_i)| = V(f_1, \dots, f_n)$. Since $\mathbb{T}$ is Cohen-Macaulay, Proposition 2.3 gives the first statement. Since the sections s_i are global, the 0-cycle $\prod_{i=1}^n \operatorname{div}(s_i)$ is effective. Hence, the second statement follows directly from the first one. □

Finally, we prove the version of the Bernštein-Kušnirenko theorem in (1.1).

Theorem 2.9. *Let $f_1, \dots, f_n \in K[M]$. Then*

$$\deg(Z(f_1, \dots, f_n)) \le \mathrm{MV}_M(\mathrm{N}(f_1), \dots, \mathrm{N}(f_n)).$$

Proof. This follows from Proposition 2.8(2), Corollary 2.6 and the formula (2.7). □

3. Adelic fields and finite extensions

In this section, we consider adelic fields following [BPS14]. We also give a new notion of adelic field extension that behaves better than the one in *loc. cit.*. With this definition, the product formula is preserved when passing to finite extensions.

Definition 3.1. Let $\mathbb{K}$ be an infinite field and $\mathfrak{M}$ a set of places. Each place $v \in \mathfrak{M}$ is a pair consisting of an absolute value $|\cdot|_v$ and a positive real weight n_v. We say that $(\mathbb{K}, \mathfrak{M})$ is an *adelic field* if

(1) for each $v \in \mathfrak{M}$, the absolute value $|\cdot|_v$ is either Archimedean or associated to a nontrivial discrete valuation;
(2) for each $\alpha \in \mathbb{K}^\times$, we have that $|\alpha|_v = 1$ for all but a finite number of $v \in \mathfrak{M}$.

An adelic field $(\mathbb{K}, \mathfrak{M})$ satisfies the *product formula* if, for every $\alpha \in \mathbb{K}^\times$,

$$\prod_{v \in \mathfrak{M}} |\alpha|_v^{n_v} = 1.$$

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field. For each place $v \in \mathfrak{M}$, we denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ with respect to the absolute value $|\cdot|_v$. By a theorem of Ostrowski, if v is Archimedean, then $\mathbb{K}_v$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ [Cas86, Chapter 3, Theorem 1.1]. In particular, an adelic field has only a finite number of Archimedean places.

Example 3.2. Let $\mathfrak{M}_{\mathbb{Q}}$ be the set of places of $\mathbb{Q}$ consisting of the Archimedean and p-adic absolute values of $\mathbb{Q}$, normalized in the standard way, and with all the weights equal to 1. The adelic field $(\mathbb{Q}, \mathfrak{M}_{\mathbb{Q}})$ satisfies the product formula.

Example 3.3. Let $\mathrm{K}(C)$ denote the function field of a regular projective curve C over a field κ. To each closed point $v \in C$ we associate the absolute value and weight given, for $f \in \mathrm{K}(C)^{\times}$, by

$$|f|_v = c_\kappa^{-\operatorname{ord}_v(f)} \quad \text{and} \quad n_v = [\mathrm{K}(v) : \kappa], \tag{3.1}$$

where $\operatorname{ord}_v(f)$ denotes the order of vanishing of f at v and

$$c_\kappa = \begin{cases} \mathrm{e} & \text{if } \#\kappa = \infty, \\ \#\kappa & \text{if } \#\kappa < \infty. \end{cases} \tag{3.2}$$

The set of places $\mathfrak{M}_{\mathrm{K}(C)}$ is indexed by the closed points of C, and consists of these absolute values and weights. The pair $(\mathrm{K}(C), \mathfrak{M}_{\mathrm{K}(C)})$ is an adelic field which satisfies the product formula.

Lemma 3.4. *Let $\mathbb{F}$ be a finite extension of $\mathbb{K}$ and $v \in \mathfrak{M}$. Then*

$$\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v \simeq \bigoplus_w E_w, \tag{3.3}$$

where the sum is over the absolute values $|\cdot|_w$ on $\mathbb{F}$ whose restriction to $\mathbb{K}_v$ coincides with $|\cdot|_v$, and where the E_w's are local Artinian $\mathbb{K}_v$-algebras with maximal ideal $\mathfrak{p}_w$. For each w, we have $E_w/\mathfrak{p}_w \simeq \mathbb{F}_w$.

Proof. Since $\mathbb{K} \hookrightarrow \mathbb{F}$ is a finite extension, the tensor product $\mathbb{F} \otimes \mathbb{K}_v$ is an Artinian $\mathbb{K}_v$-algebra. By the structure theorem for Artinian algebras,

$$\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v \simeq \bigoplus_{i \in I} E_i,$$

where I is a finite set and the E_i's are local Artinian $\mathbb{K}_v$-algebras. Let $\mathfrak{p}_i$ be the maximal ideal of E_i, for each i. These are the only prime ideals of $\mathbb{F} \otimes \mathbb{K}_v$, and so $\operatorname{rad}(\mathbb{F} \otimes \mathbb{K}_v) = \bigcap_{i \in I} \mathfrak{p}_i$.

Each w in the decomposition (3.3) corresponds to an absolute value $|\cdot|_w$ on $\mathbb{F}$ extending $|\cdot|_v$, and there is a natural inclusion $\mathbb{F} \hookrightarrow \mathbb{F}_w$. The diagonal morphism $\mathbb{F} \to \bigoplus_w \mathbb{F}_w$ extends to a map of $\mathbb{K}_v$-vector spaces

$$\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v \longrightarrow \bigoplus_w \mathbb{F}_w.$$

By [Bou64, Chapitre VI, §8.2 Proposition 11(b)], this morphism is surjective and its kernel is the radical ideal of $\mathbb{F} \otimes \mathbb{K}_v$. Therefore

$$\bigoplus_{i \in I} E_i/\mathfrak{p}_i = \Big(\bigoplus_{i \in I} E_i\Big) \Big/ \operatorname{rad}(\mathbb{F} \otimes \mathbb{K}_v) \simeq \bigoplus_w \mathbb{F}_w. \tag{3.4}$$

The summands in both extremes of (3.4) are fields over $\mathbb{K}_v$, and so local Artinian $\mathbb{K}_v$-algebras. By the uniqueness of the decomposition in the structure theorem for Artinian algebras, there is a bijection between the elements in I and the w's, identifying each $i \in I$ with the unique w such that $E_i/\mathfrak{p}_i \simeq \mathbb{F}_w$. $\square$

The following definition was introduced by Gubler in the context of M-fields, see [Gub97, Remark 2.5].

Definition 3.5. Let $(\mathbb{K},\mathfrak{M})$ be an adelic field and $\mathbb{F}$ a finite extension of $\mathbb{K}$. For every place $v\in\mathfrak{M}$, we denote by $\mathfrak{N}_v$ the set of absolute values $|\cdot|_w$ on $\mathbb{F}$ that extend $|\cdot|_v$ with weight given by

$$n_w=\frac{\dim_{\mathbb{K}_v}(E_w)}{[\mathbb{F}:\mathbb{K}]}\,n_v,$$

where the E_w's are the local Artinian $\mathbb{K}_v$-algebras in the decomposition of $\mathbb{F}\otimes_{\mathbb{K}}\mathbb{K}_v$ from Lemma 3.4. Set $\mathfrak{N}=\bigsqcup_{v\in\mathfrak{M}}\mathfrak{N}_v$. The pair $(\mathbb{F},\mathfrak{N})$ is an adelic field. The adelic fields of this form are called *adelic field extensions* of $(\mathbb{K},\mathfrak{M})$.

Remark 3.6. With notation as in Lemma 3.4,

$$\dim_{\mathbb{K}_v}(E_w)=l_{E_w}(E_w)[\mathbb{F}_w:\mathbb{K}_v],$$

where $l_{E_w}(E_w)$ is the length of E_w as a module over itself. This follows from [Ful84, Lemma A.1.3] applied to the morphism $\mathbb{K}_v\to E_w$. Hence, the weights in Definition 3.5 can be alternatively written as

$$n_w=l_{E_w}(E_w)\frac{[\mathbb{F}_w:\mathbb{K}_v]}{[\mathbb{F}:\mathbb{K}]}\,n_v.$$

Proposition 3.7. *Let $(\mathbb{K},\mathfrak{M})$ be an adelic field and $(\mathbb{F},\mathfrak{N})$ an adelic field extension of $(\mathbb{K},\mathfrak{M})$. Then*

1. *the equality $\sum_{w\in\mathfrak{N}_v}n_w=n_v$ holds for every place $v\in\mathfrak{M}$;*
2. *if $(\mathbb{K},\mathfrak{M})$ satisfies the product formula, then $(\mathbb{F},\mathfrak{N})$ also does.*

Proof. From the definition of adelic field extension and Lemma 3.4,

$$\sum_{w\in\mathfrak{N}_v}n_w=\sum_{w\in\mathfrak{N}_v}\frac{\dim_{\mathbb{K}_v}(E_w)}{[\mathbb{F}:\mathbb{K}]}n_v=\frac{\dim_{\mathbb{K}_v}(\mathbb{F}\otimes\mathbb{K}_v)}{[\mathbb{F}:\mathbb{K}]}n_v=n_v,$$

which proves statement (1). To prove the second statement, let $\alpha\in\mathbb{F}^\times$ and consider the multiplication map $\eta_\alpha\colon\mathbb{F}\to\mathbb{F}$ given by $\eta_\alpha(x)=\alpha x$. The norm $N_{\mathbb{F}/\mathbb{K}}(\alpha)\in\mathbb{K}^\times$ is defined as the determinant of this $\mathbb{K}$-linear map. Moreover, η_α extends to the $\mathbb{K}_v$-linear map

$$\eta_\alpha\otimes 1_{\mathbb{K}_v}\colon\mathbb{F}\otimes\mathbb{K}_v\longrightarrow\mathbb{F}\otimes\mathbb{K}_v,$$

which has the same determinant. Using the decomposition in (3.3), write $\alpha\otimes 1_{\mathbb{K}_v}=(\alpha_w)_w$ with $\alpha_w\in E_w$. Hence $\eta_\alpha\otimes 1_{\mathbb{K}_v}=\bigoplus_w\eta_{\alpha_w}$ and

$$N_{\mathbb{F}/\mathbb{K}}(\alpha)=\det(\eta_\alpha\otimes 1_{\mathbb{K}_v})=\prod_{w\in\mathfrak{N}_v}N_{E_w/\mathbb{K}_v}(\alpha_w).$$

By [Bou70, Chapitre III, §9.2, Proposition 1], $N_{E_w/\mathbb{K}_v}(\alpha_w)=N_{\mathbb{F}_w/\mathbb{K}_v}(\alpha_w)^{l_{E_w}(E_w)}$. Moreover, by [Lan02, VI Proposition 5.6],

$$N_{\mathbb{F}_w/\mathbb{K}_v}(\alpha_w)=\prod_\sigma\sigma(\alpha_w)^{[\mathbb{F}_w:\mathbb{K}_v]_i},$$

where the product is over the different embeddings σ of $\mathbb{F}_w$ in an algebraic closure of $\mathbb{K}_v$, and $[\mathbb{F}_w:\mathbb{K}_v]_i$ denotes the inseparability degree of the extension $\mathbb{K}_v\hookrightarrow\mathbb{F}_w$. Furthermore, the number of such embeddings is equal to the separability degree $[\mathbb{F}_w:\mathbb{K}_v]_s$. For every embedding σ, we have $|\sigma(\alpha_w)|_v=|\alpha|_w$ because the base field $\mathbb{K}_v$ is complete. Since $[\mathbb{F}_w:\mathbb{K}_v]_i[\mathbb{F}_w:\mathbb{K}_v]_s=[\mathbb{F}_w:\mathbb{K}_v]$, we get

$$|N_{\mathbb{F}/\mathbb{K}}(\alpha)|_v^{n_v}=\prod_{w\in\mathfrak{N}_v}|\sigma(\alpha_w)|_v^{l_{E_w}(E_w)[\mathbb{F}_w:\mathbb{K}_v]n_v}=\prod_{w\in\mathfrak{N}_v}|\alpha|_w^{[\mathbb{F}:\mathbb{K}]n_w}.$$

Since $N_{\mathbb{F}/\mathbb{K}}(\alpha) \in \mathbb{K}^{\times}$, if $(\mathbb{K}, \mathfrak{M})$ satisfies the product formula, then

$$\prod_{w \in \mathfrak{N}} |\alpha|_w^{n_w} = \Big(\prod_{v \in \mathfrak{M}} |N_{\mathbb{F}/\mathbb{K}}(\alpha)|_v^{n_v} \Big)^{\frac{1}{[\mathbb{F}:\mathbb{K}]}} = 1,$$

concluding the proof. □

Example 3.8. Let $\mathbb{F}$ be a number field. This is a separable extension of $\mathbb{Q}$. By [Bou64, Chapitre VI, §8.5, Corollaire 3], we have that $\mathbb{F} \otimes \mathbb{Q}_v \simeq \bigoplus_{w \in \mathfrak{N}_v} \mathbb{F}_w$ for all $v \in \mathfrak{M}_{\mathbb{Q}}$. Therefore, the weight associated to each place $w \in \mathfrak{N}_v$ is

$$n_w = \frac{[\mathbb{F}_w : \mathbb{Q}_v]}{[\mathbb{F} : \mathbb{Q}]}.$$

Example 3.9. Let $(\mathrm{K}(C), \mathfrak{M}_{\mathrm{K}(C)})$ be the function field of a regular projective curve C over a field κ with the structure of adelic field as in Example 3.3. The places of $\mathrm{K}(C)$ correspond to the closed points of C with absolute values and weights given by (3.1). Let $\mathbb{F}$ be a finite extension of $\mathrm{K}(C)$ and $\mathfrak{N}$ the set of places of $\mathbb{F}$ as in Definition 3.5. There is a regular projective curve B over κ and a finite map $\pi\colon B \to C$ such that the extension $\mathrm{K}(C) \hookrightarrow \mathbb{F}$ identifies with the morphism $\pi^*\colon \mathrm{K}(C) \hookrightarrow \mathrm{K}(B)$. For each place $v \in \mathfrak{M}_{\mathrm{K}(C)}$, the absolute values of $\mathbb{F}$ that extend $|\cdot|_v$ are in bijection with the fiber $\pi^{-1}(v)$.

For each closed point $v \in C$, the integral closure in $\mathrm{K}(B)$ of $\mathcal{O}_{v,C}$ coincides with $\mathcal{O}_{\pi^{-1}(v),B}$, the local ring of B along the fiber $\pi^{-1}(v)$. The ring $\mathcal{O}_{\pi^{-1}(v),B}$ is of finite type over $\mathcal{O}_{v,C}$. With notation as in Lemma 3.4, by [Bou64, Chapter VI, §8.5, Corollaire 3], we have $E_w \simeq \mathbb{F}_w$ for all $w \in \mathfrak{N}_v$. Hence, the weight of w is given by

$$n_w = \frac{[\mathbb{F}_w : \mathrm{K}(C)_v]}{[\mathbb{F} : \mathrm{K}(C)]} [\mathrm{K}(v) : \kappa].$$

Let $e(w/v)$ denote the ramification index of w over v. By [Bou64, Chapter VI, §8.5, Corollaire 2], we have that $[\mathbb{F}_w : \mathrm{K}(C)_v] = e(w/v)\,[\mathrm{K}(w) : \mathrm{K}(v)]$. Therefore, for each place $w \in \mathfrak{N}_v$, the weight of w can also be expressed as

$$n_w = \frac{e(w/v)\,[\mathrm{K}(w) : \kappa]}{[\mathbb{F} : \mathrm{K}(C)]}.$$

Following [BPS14], a *global field* is a finite extension of the field of rational numbers or of the function field of a regular projective curve, with the structure of adelic field described in Examples 3.8 and 3.9. For these fields, Proposition 3.7 is already a known result, see for instance [BPRS15, Proposition 2.1].

By a result of Artin and Whaples, global fields can be characterized as the adelic fields having an absolute value that is either Archimedean or associated to a discrete valuation whose residue field has finite order over the field of constants [AW45, Theorems 2 and 3].

Function fields of varieties of higher dimension provide examples of adelic fields satisfying the product formula, and that are not global fields.

Example 3.10. Let $\mathrm{K}(S)$ be the function field of an irreducible normal variety S over a field κ of dimension $s \geq 1$, and $E_1, \dots, E_{s-1}$ nef Cartier divisors on S. Set $S^{(1)}$ for the set of irreducible hypersurfaces of S. For each $V \in S^{(1)}$, the local ring $\mathcal{O}_{V,S}$ is a discrete valuation ring. We associate to V the absolute value and weight given, for $f \in \mathrm{K}(S)$, by

$$|f|_V = c_\kappa^{-\mathrm{ord}_V(f)} \quad \text{and} \quad n_v = \deg_{E_1,\dots,E_{s-1}}(V),$$

with c_κ as in (3.2). The set of places $\mathfrak{M}_{\mathrm{K}(S)}$ is indexed by $S^{(1)}$, and consists of these absolute values and weights. For $f \in \mathrm{K}(S)^\times$,

$$\sum_{V \in S^{(1)}} n_V \log|f|_v = \log(c_k) \sum_{V \in S^{(1)}} \deg_{E_1,\dots,E_{s-1}}(V)\,\mathrm{ord}_V(f) = \deg_{E_1,\dots,E_{s-1}}(\mathrm{div}(f)) = 0,$$

because the Cartier divisor $\mathrm{div}(f)$ is principal. Hence $(\mathrm{K}(S), \mathfrak{M}_{\mathrm{K}(S)})$ satisfies the product formula.

4. Height of cycles

In this section, we introduce a notion of global height for cycles of a variety over an adelic field, with respect to a family of metrized divisors generated by small sections. We also recall the notion of local height of cycles from [BPS14, Chapter 1] and give a more explicit description of this construction in the 0-dimensional case.

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula, and X a normal projective variety over $\mathbb{K}$. For each place $v \in \mathfrak{M}$, we denote by X_v^{an} the v-adic analytification of X. In the Archimedean case, if $\mathbb{K}_v \simeq \mathbb{C}$, then X_v^{an} is an analytic space over $\mathbb{C}$ whereas, if $\mathbb{K}_v \simeq \mathbb{R}$, then X_v^{an} is an analytic space over $\mathbb{R}$, that is, an analytic space over $\mathbb{C}$ together with an antilinear involution, as explained in [BPS14, Remark 1.1.5]. In the non-Archimedean case, X_v^{an} is a Berkovich space over $\mathbb{K}_v$ as in [BPS14, §1.2].

Fix $v \in \mathfrak{M}$ and set

$$X_v = X \times \mathrm{Spec}(\mathbb{K}_v).$$

Given a 0-cycle Y of X_v, a usual construction in Arakelov geometry associates a signed measure on X_v^{an}, denoted by δ_Y, that is supported on $|Y|^{\mathrm{an}}$ and has total mass equal to $\deg(Y)$, see for instance [BPS14, Definition 1.3.15] for the non-Archimedean case. In what follows, we explicit this construction.

Let q be a closed point of X_v. The function field $\mathrm{K}(q)$ is a finite extension of $\mathbb{K}_v$ and $\deg(q) = [\mathrm{K}(q) : \mathbb{K}_v]$. If v is Archimedean, then $\deg(q)$ is either equal to 1 or 2. In the first case, the analytification of q is a point of X_v^{an} whereas, in the second case, it is a pair of conjugate points. If v is non-Archimedean, choose an affine open neighborhood $U = \mathrm{Spec}(A)$ of q and $A \to \mathrm{K}(q)$ the corresponding morphism of $\mathbb{K}_v$-algebras. The analytification of q is the point $q^{\mathrm{an}} \in U^{\mathrm{an}} \subset X_v^{\mathrm{an}}$ corresponding to the multiplicative seminorm given by the composition

$$A \longrightarrow \mathrm{K}(q) \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0},$$

where $|\cdot|$ is the unique extension to $\mathrm{K}(q)$ of the absolute value $|\cdot|_v$.

Since the measure δ_q is supported on the point q^{an} and has total mass $\deg(q)$, it follows that

$$\delta_q = [\mathrm{K}(q) : \mathbb{K}_v]\, \delta_{q^{\mathrm{an}}}, \tag{4.1}$$

where $\delta_{q^{\mathrm{an}}}$ denotes the Dirac delta measure on q^{an}. For an arbitrary 0-cycle Y of X_v, the signed measure δ_Y is obtained from (4.1) by linearity. It is a discrete signed measure of total mass equal to $\deg(Y)$.

Let D be a Cartier divisor on X. A *metric* on the analytic line bundle $\mathcal{O}(D)_v^{\mathrm{an}}$ is an assignment that, to each open subset $U \subset X_v^{\mathrm{an}}$ and local section s on U, associates a continuous function

$$\|s(\cdot)\|_v \colon U \longrightarrow \mathbb{R}_{\geq 0}$$

that is compatible with restrictions to open subsets, vanishes only when the local section does, and respects multiplication of local sections by analytic functions, see [BPS14, Definitions 1.1.1 and 1.3.1]. This notion allows to define local heights of 0-cycles.

Definition 4.1. Let D be a Cartier divisor on X, and $\|\cdot\|_v$ a metric on $\mathcal{O}(D)_v^{\mathrm{an}}$. For a 0-cycle Y of X_v and a rational section s of $\mathcal{O}(D)$ that is regular and non-vanishing on the support of Y, the *local height* of Y with respect to the pair $(\|\cdot\|_v, s)$ is defined as

$$\mathrm{h}_{\|\cdot\|_v}(Y;s) = -\int_{X_v^{\mathrm{an}}} \log \|s\|_v \, \delta_Y.$$

We now study the behavior of these objects with respect to adelic field extensions. Let $(\mathbb{F}, \mathfrak{N})$ be an extension of the adelic field $(\mathbb{K}, \mathfrak{M})$ (Definition 3.5) and fix a place $w \in \mathfrak{N}_v$, so that $\mathbb{F}_w$ is a finite extension of the local field $\mathbb{K}_v$. Let q be a closed point of X_v and consider the subscheme q_w of $X_w = X \times \mathrm{Spec}(\mathbb{F}_w)$ obtained by base change. Decompose

$$\mathrm{K}(q) \otimes_{\mathbb{K}_v} \mathbb{F}_w = \bigoplus_{j \in I} G_j$$

as a finite sum of local Artinian $\mathbb{F}_w$-algebras and, for each $j \in I$, denote by q_j the corresponding closed point of X_w. Thus the associated cycle is given by $[q_w] = \sum_{j\in I} l_{G_j}(G_j)\, q_j$. Hence, by (4.1) and Remark 3.6,

$$\delta_{[q_w]} = \sum_{j \in I} \dim_{\mathbb{F}_w}(G_j)\, \delta_{q_j^{\mathrm{an}}}.$$

The inclusion $\mathbb{K}_v \hookrightarrow \mathbb{F}_w$ induces a map of the corresponding analytic spaces

$$\pi \colon X_w^{\mathrm{an}} \longrightarrow X_v^{\mathrm{an}}. \tag{4.2}$$

In the non-Archimedean case, this map of Berkovich spaces is defined locally by restricting seminorms.

The following proposition gives the behavior of the measure associated to a 0-cycle with respect to field extensions.

Proposition 4.2. *With notation as above, let Y be a 0-cycle of X_v and set Y_w for the 0-cycle of X_w obtained by base change. Then*

$$\pi_* \, \delta_{Y_w} = \delta_Y.$$

Proof. By the compatibility of the map π with restriction to subschemes, we have that $\pi(q_j^{\mathrm{an}}) = q^{\mathrm{an}}$ for all $j \in I$. It follows that

$$\pi_* \, \delta_{[q_w]} = \sum_{j\in I} \dim_{\mathbb{F}_w}(G_j)\, \pi_* \, \delta_{q_j^{\mathrm{an}}} = \Big(\sum_{j\in I} \dim_{\mathbb{F}_w}(G_j)\Big) \delta_{q^{\mathrm{an}}} = [\mathrm{K}(q) : \mathbb{K}_v]\, \delta_{q^{\mathrm{an}}} = \delta_q.$$

□

Let D be a Cartier divisor on X and $\|\cdot\|_v$ a metric on $\mathcal{O}(D)_v^{\mathrm{an}}$. The extension of this metric to a metric $\|\cdot\|_w$ on the analytic line bundle $\mathcal{O}(D)_w^{\mathrm{an}}$ on X_w^{an} is obtained by taking the inverse image with respect to the map π in (4.2), that is

$$\|\cdot\|_w = \pi^* \|\cdot\|_v. \tag{4.3}$$

Proposition 4.2 implies directly the invariance of the local height with respect to adelic field extensions.

Proposition 4.3. *With notation as above, let Y be a 0-cycle of X_v and s a rational section of $\mathcal{O}(D)_v^{\mathrm{an}}$ that is regular and non-vanishing on the support of Y. Set Y_w and $s_w = \pi^* s$ for the 0-cycle and rational section obtained by base extension. Then*

$$\mathrm{h}_{\|\cdot\|_w}(Y_w, s_w) = \mathrm{h}_{\|\cdot\|_v}(Y, s).$$

To define global heights of cycles over an adelic field, we consider adelic families of metrics on the Cartier divisor D satisfying a certain compatibility condition.

Definition 4.4. An *(adelic) metric* on D is a collection $\|\cdot\|_v$ of metrics on $\mathcal{O}(D)_v^{\mathrm{an}}$, $v \in \mathfrak{M}$, such that, for every point $p \in X(\overline{\mathbb{K}})$ and a choice of a rational section s of $\mathcal{O}(D)$ that is regular and non-vanishing at p and of an adelic field extension $(\mathbb{F}, \mathfrak{N})$ such that $p \in X(\mathbb{F})$,

$$\|s(p_w^{\mathrm{an}})\|_w = 1 \tag{4.4}$$

for all but a finite number of $w \in \mathfrak{N}$. We denote by $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ the corresponding *(adelically) metrized divisor* on X.

In addition, $\overline{D}$ is *semipositive* if each of its v-adic metrics is semipositive in the sense of [BPS14, Definition 1.4.1].

The condition (4.4) does not depend on the choice of the rational section s and of the adelic field extension $(\mathbb{F}, \mathfrak{N})$.

Remark 4.5. When $\mathbb{K}$ is a global field, the classical notion of compatibility for a collection of metrics $\|\cdot\|_v$ on $\mathcal{O}(D)_v^{\mathrm{an}}$, $v \in \mathfrak{M}$, is that of being quasi-algebraic, in the sense that there is an integral model that induces all but a finite number of these metrics [BPS14, Definition 1.5.13].

By Proposition 1.5.14 in *loc. cit.*, a quasi-algebraic metrized divisor $\overline{D}$ is adelic in the sense of Definition 4.4. The converse is not true, as it is easy to construct toric adelic metrized divisors that are not quasi-algebraic (Remark 5.4).

For a 0-cycle Y of X and a place $v \in \mathfrak{M}$, we denote by Y_v the 0-cycle of X_v defined by base change. When $Y = p$ is a closed point of X, by Lemma 3.4 applied to the finite extension $\mathrm{K}(p)$ of $\mathbb{K}$, the 0-dimensional subscheme $p_v = p \times \mathrm{Spec}(\mathbb{K}_v)$ of X_v decomposes as

$$p_v = \mathrm{Spec}(\mathrm{K}(p) \otimes_{\mathbb{K}} \mathbb{K}_v) \simeq \coprod_{w\in\mathfrak{N}_v} \mathrm{Spec}(E_w),$$

where the E_i's are the local Artinian $\mathbb{K}_v$-algebras in (3.3). Let q_w, $w \in \mathfrak{N}_v$, be the irreducible components of this subscheme. Then, the associated 0-cycle of X_v writes down as

$$[p_v] = \sum_{w\in\mathfrak{N}_v} l_{E_w}(E_w)\, q_w$$

and, for each $w \in \mathfrak{N}_v$, we have $\mathrm{K}(q_w) \simeq \mathrm{K}(p)_w$. For an arbitrary Y, the 0-cycle Y_v is obtained by linearity.

Let $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ be a metrized divisor on X, Y a 0-cycle of X and s a rational section of $\mathcal{O}(D)$ that is is regular and non-vanishing on the support of Y. For each place $v \in \mathfrak{M}$, we set

$$\mathrm{h}_{\overline{D},v}(Y; s) = \mathrm{h}_{\|\cdot\|_v}(Y_v; s),$$

where Y_v is the 0-cycle of X_v obtained by base change. The condition that $\overline{D}$ is adelic implies that $\mathrm{h}_{\overline{D},v}(Y; s) = 0$ for all but a finite number of places.

If s' is another rational section of $\mathcal{O}(D)$ that is regular and non-vanishing on $|Y|$, then $s' = fs$ with $f \in \mathrm{K}(X)^\times$ and, for $v \in \mathfrak{M}$,

$$\mathrm{h}_{\overline{D},v}(Y; s') = \mathrm{h}_{\overline{D},v}(Y; s) - \log|\gamma|_v \tag{4.5}$$

where $Y = \sum_p \mu_p\, p$ and $\gamma = \prod_p f(p)^{\mu_p} \in \mathbb{K}^\times$.

Definition 4.6. Let $\overline{D}$ be a metrized divisor on X and Y a 0-cycle of X. The *global height* of Y with respect to $\overline{D}$ is defined as

$$\mathrm{h}_{\overline{D}}(Y) = \sum_{v\in\mathfrak{M}} n_v \mathrm{h}_{\overline{D},v}(Y; s), \tag{4.6}$$

with s a rational section of $\mathcal{O}(D)$ that is is regular and non-vanishing on $|Y|$.

The local heights in (4.6) are zero for all but a finite number of places, and so this sum is finite. The equality (4.5) together with the product formula imply that this sum does not depend on the rational section s.

Given a metrized divisor $\overline{D}$ on X and an adelic field extension $(\mathbb{F}, \mathfrak{N})$, we denote by $\overline{D}_{\mathbb{F}}$ the metrized divisor on $X_{\mathbb{F}}$ obtained by extending the v-adic metrics of $\overline{D}$ as in (4.3).

Proposition 4.7. *Let $\overline{D}$ be a metrized divisor on X, Y a 0-cycle of X and $(\mathbb{F}, \mathfrak{N})$ an adelic field extension of $(\mathbb{K}, \mathfrak{M})$. Then*

$$\mathrm{h}_{\overline{D}_{\mathbb{F}}}(Y_{\mathbb{F}}) = \mathrm{h}_{\overline{D}}(Y).$$

Proof. Let s be a rational section of $\mathcal{O}(D)$ that is is regular and non-vanishing on $|Y|$ and $v \in \mathfrak{M}$. By Propositions 4.3 and 3.7(1),

$$\sum_{w\in\mathfrak{N}_v} n_w\, \mathrm{h}_{\overline{D}_{\mathbb{F}},w}(Y_{\mathbb{F}}, s) = \sum_{w\in\mathfrak{N}_v} n_w\, \mathrm{h}_{\overline{D},v}(Y, s) = n_v\, \mathrm{h}_{\overline{D},v}(Y, s).$$

The statement follows by summing over all the places of $\mathbb{K}$. □

Since the global height is invariant under field extension, it induces a notion of global height for algebraic points, that is, a well-defined function

$$\mathrm{h}_{\overline{D}} \colon X(\overline{\mathbb{K}}) \longrightarrow \mathbb{R}.$$

When $\mathbb{K}$ is a global field, this notion coincides with the one in [BPRS15, Definition 2.2].

Now we turn to cycles of arbitrary dimension. Let V be a k-dimensional irreducible subvariety of X and $\overline{D}_0, \dots, \overline{D}_{k-1}$ a family of k semipositive metrized divisors on X. For each place $v \in \mathfrak{M}$, we can associate to this data a measure on X_v^{an} denoted by

$$\mathrm{c}_1(\overline{D_0}) \wedge \cdots \wedge \mathrm{c}_1(\overline{D}_{k-1}) \wedge \delta_{V_v^{\mathrm{an}}}$$

and called the *v-adic Monge-Ampère measure* of V and $\overline{D}_0, \dots, \overline{D}_{k-1}$, see [Cha06, Définition 2.4] or [BPS14, Definition 1.4.6]. For a k-cycle Y of X, this notion extends by linearity to a signed measure on X_v^{an}, denoted by $\mathrm{c}_1(\overline{D_0}) \wedge \cdots \wedge \mathrm{c}_1(\overline{D}_{k-1}) \wedge \delta_{Y_v^{\mathrm{an}}}$. It is supported on $|Y_v|^{\mathrm{an}}$ and has total mass equal to the degree $\deg_{D_0,\dots,D_{k-1}}(Y)$.

We recall the notion of local height of cycles from [BPS14, Definition 1.4.11].

Definition 4.8. Let Y be a k-cycle of X and, for $i = 0, \dots, k$, let $(\overline{D}_i, s_i)$ be a semipositive metrized divisor on X and a rational section of $\mathcal{O}(D_i)$ such that $\mathrm{div}(s_0), \dots, \mathrm{div}(s_k)$ intersect Y properly (Definition 2.1). For $v \in \mathfrak{M}$, the *local height* of Y with respect to $(\overline{D}_0, s_0), \dots, (\overline{D}_k, s_k)$ is inductively defined by the rule

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k) = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1},v}(\mathrm{div}(s_k) \cdot Y; s_0, \dots, s_{k-1}) \\ - \int_{X_v^{\mathrm{an}}} \log \|s_k\|_{k,v} \, \mathrm{c}_1(\overline{D}_0) \wedge \dots \wedge \mathrm{c}_1(\overline{D}_{k-1}) \wedge \delta_{Y_v^{\mathrm{an}}}$$

and the convention that the local height of the cycle $0 \in Z_{-1}(X)$ is zero.

Remark 4.9.

(1) The local height is linear with respect to the group structure of $Z_k(X)$. In particular, the local heights of the cycle $0 \in Z_k(X)$ are zero.
(2) For a 0-cycle Y of X and $v \in \mathfrak{M}$, the v-adic Monge-Ampère measure coincides with the measure associated to the 0-cycle Y_v of X_v at the beginning of this section. Hence, Definition 4.8 applied to a 0-cycle coincides with Definition 4.1.

The following notion is the arithmetic analogue of global sections of a line bundle, and Proposition 4.11 below is an analogue for local heights of Proposition 2.5.

Definition 4.10. Let $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ be a metrized divisor on X. A global section s of $\mathcal{O}(D)$ is *$\overline{D}$-small* if, for all $v \in \mathfrak{M}$,

$$\sup_{q\in X_v^{\mathrm{an}}} \|s(q)\|_v \le 1.$$

Proposition 4.11. *Let Y be an effective k-cycle of X and, for $i = 0, \dots, k$, let $(\overline{D}_i, s_i)$ be a semipositive metrized divisor on X and a rational section of $\mathcal{O}(D_i)$ such that $\mathrm{div}(s_0), \dots, \mathrm{div}(s_k)$ intersect Y properly and such that s_k is $\overline{D}_k$-small. Then, for all $v \in \mathfrak{M}$,*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1},v}(\mathrm{div}(s_k) \cdot Y; s_0, \dots, s_{k-1}) \le \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k).$$

Proof. Since the cycle Y is effective and the metrized divisors $\overline{D}_i$ are semipositive, their v-adic Monge-Ampere measure is a measure, that is, it takes only nonnegative values. Since the global section s_k is $\overline{D}_k$-small, $\log \|s_k(q)\|_{k,v} \le 0$ for all $q \in X_v^{\mathrm{an}}$. The inequality follows then from the inductive definition of the local height. □

Our next step is to define global heights for cycles over an adelic field. We first state an auxiliary result specifying the behavior of local heights with respect to change of sections, extending (4.5) to the higher dimensional case. The following lemma and its proof are similar to [Gub97, Corollary 3.8].

Lemma 4.12. *Let Y be a k-cycle of X and $\overline{D}_0, \dots, \overline{D}_k$ semipositive metrized divisors on X. Let s_i, s_i' be rational sections of $\mathcal{O}(D_i)$, $i = 0, \dots, k$, such that both $\mathrm{div}(s_0), \dots, \mathrm{div}(s_k)$ and $\mathrm{div}(s_0'), \dots, \mathrm{div}(s_k')$ intersect Y properly. Then there exists $\gamma \in \mathbb{K}^\times$ such that, for all $v \in \mathfrak{M}$,*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0', \dots, s_k') = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k) - \log |\gamma|_v. \tag{4.7}$$

Proof. Let s_i'' be rational sections of $\mathcal{O}(D_i)$, $i = 0, \dots, k$, such that $(s_0'', \dots, s_k'')$ is generic enough so that, for every subset $J \subset \{0, \dots, k\}$, the family of divisors

$$\{\mathrm{div}(s_j) \mid j \in J\} \cup \{\mathrm{div}(s_j'') \mid j \notin J\} \tag{4.8}$$

intersects Y properly.

We proceed to prove the formula (4.7) with the s_i'''s in the place of the s_i''s. Hence, we want to prove that there is $\widetilde{\gamma} \in \mathbb{K}^\times$ such that, for every $v \in \mathfrak{M}$,

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0'', \dots, s_k'') = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k) - \log|\widetilde{\gamma}|_v. \tag{4.9}$$

To this end, consider first the particular case when $s_i = s_i''$, $i = 0, \dots, k-1$. Set $s_k'' = f s_k$ with $f \in \mathrm{K}(X)^\times$, and $\big(\prod_{i=0}^{k-1} \mathrm{div}(s_i)\big) \cdot Y = \sum_p \mu_p p$. By [BPS14, Theorem 1.4.17(3)], the equality (4.7) holds with $\widetilde{\gamma}_k \in \mathbb{K}^\times$ given by

$$\widetilde{\gamma}_k = \prod_p f(p)^{\mu_p}.$$

By [BPS14, Theorem 1.4.17(1)], the local height is symmetric in the pairs $(\overline{D_i}, s_i)$. By the hypothesis (4.8), we can reorder the metrized line bundles and rational sections, and iterate the above construction for every $i = 0, \dots, k$. This proves (4.9) with $\widetilde{\gamma} = \prod_{i=0}^k \widetilde{\gamma}_i$.

Assuming that the s_i'''s are generic enough so that the condition in (4.8) also holds with the s_i''s instead of the s_i's, similarly there exists $\widetilde{\gamma}' \in \mathbb{K}^\times$ such that, for every $v \in \mathfrak{M}$,

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0'', \dots, s_k'') = \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0', \dots, s_k') - \log|\widetilde{\gamma}'|_v. \tag{4.10}$$

The statement follows by combining (4.9) and (4.10). □

We consider the following notions of positivity of metrized divisors.

Definition 4.13. Let $\overline{D}$ be a metrized divisor on X.

(1) $\overline{D}$ is *nef* if D is nef, $\overline{D}$ is semipositive, and $\mathrm{h}_{\overline{D}}(p) \geq 0$ for every closed point p of X.
(2) $\overline{D}$ is *generated by small sections* if, for every closed point $p \in X$, there is a $\overline{D}$-small section s such that $p \notin |\mathrm{div}(s)|$.

Lemma 4.14. *Let Y be an effective k-cycle of X and $(\overline{D}_i, s_i)$ semipositive metrized divisors on X together with rational sections of $\mathcal{O}(D_i)$, $i = 0, \dots, k$, such that* $\mathrm{div}(s_0), \dots, \mathrm{div}(s_k)$ *intersect Y properly. Suppose that $\overline{D}_i$, $i = 1, \dots, k$, are generated by small sections. Then there exists $\zeta \in \mathbb{K}^\times$ such that, for all $v \in \mathfrak{M}$,*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k) \geq \log|\zeta|_v + \mathrm{h}_{\overline{D}_0,v}\bigg(\bigg(\prod_{i=1}^k \mathrm{div}(s_i)\bigg) \cdot Y, s_0\bigg).$$

Proof. For $k = 0$, the statement is obvious, so we only consider the case when $k \geq 1$. By Lemma 4.12, it is enough to prove the statement for any particular choice of rational sections s_i, provided that their associated Cartier divisors intersect Y properly.

We can also reduce without loss of generality to the case when $Y = V$ is an irreducible variety of dimension k. We can then choose rational sections s_i, $i = 0, \dots, k$, such that s_i is $\overline{D}_i$-small. By Proposition 4.11,

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(V; s_0, \dots, s_k) \geq \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1},v}(\mathrm{div}(s_k) \cdot V; s_0, \dots, s_{k-1}).$$

Since $\mathrm{div}(s_k) \cdot V$ is an effective $(k-1)$-cycle, the statement follows by induction on k. □

Proposition-Definition 4.15. *Let Y be an effective k-cycle of X, and $\overline{D}_0, \dots, \overline{D}_k$ semipositive metrized divisors on X such that $\overline{D}_1, \dots, \overline{D}_k$ are generated by small sections. Let s_i be rational sections of $\mathcal{O}(D_i)$, $i = 0, \dots, k$, such that* $\operatorname{div}(s_0), \dots, \operatorname{div}(s_k)$ *intersects Y properly. The* global height *of Y with respect to $\overline{D}_0, \dots, \overline{D}_k$ is defined as the sum*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k}(Y) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k). \tag{4.11}$$

This sum converges to an element in $\mathbb{R} \cup \{+\infty\}$, and its value does not depend on the choice of the s_i's.

Proof. The existence of rational sections s_i such that $\operatorname{div}(s_0), \dots, \operatorname{div}(s_k)$ intersects Y properly follows from the moving lemma, with the hypothesis that X is projective.

By Lemma 4.14 and the fact that the local heights of 0-cycles are zero for all but a finite number of places, the local heights in (4.11) are nonnegative, except for a finite number of v's. Hence, the sum converges to an element in $\mathbb{R} \cup \{+\infty\}$. Lemma 4.12 and the product formula imply that the value of this sum does not depend on the choice of the s_i's. □

This definition generalizes the notion of global height of cycles of varieties over global fields in [BPS14, §1.5], to cycles of varieties over an arbitrary adelic field, in the case when the considered metrized divisors are generated by small sections.

In principle, the sum in (4.11) might contain an infinite number of nonzero terms. Nevertheless, we are not aware of any example where this phenomenon actually happens. Moreover, for varieties over global fields, the local heights of a given cycle are zero for all but a finite number of places [BPS14, Proposition 1.5.14], and so their global height is a real number given as a weighted sum of a finite number local heights.

In this context, we propose the following question.

Question 4.16. *Let Y be a k-cycle of X and, for each $i = 0, \dots, k$, let $(\overline{D}_i, s_i)$ be a semipositive metrized divisor on X and a rational section of $\mathcal{O}(D_i)$ such that* $\operatorname{div}(s_0), \dots, \operatorname{div}(s_k)$ *intersect Y properly. Is it true that*

$$\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k,v}(Y; s_0, \dots, s_k) = 0$$

for all but a finite number of $v \in \mathfrak{M}_{\mathbb{K}}$?

A positive answer would imply that, for a variety over an adelic field and a family of semipositive metrized divisors, the global height of a cycle is a well-defined real number, given as a weighted sum of a finite number local heights.

The following results are arithmetic analogues of Proposition 2.5 and Corollary 2.6.

Proposition 4.17. *Let Y be an effective k-cycle of X, and $\overline{D}_0, \dots, \overline{D}_k$ semipositive metrized divisors on X such that $\overline{D}_0$ is nef and $\overline{D}_1, \dots, \overline{D}_k$ are generated by small sections. Let s_k be a $\overline{D}_k$-small section. Then*

$$0 \le \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_{k-1}}(\operatorname{div}(s_k) \cdot Y) \le \mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k}(Y).$$

Proof. We reduce without loss of generality to the case when $Y = V$ is an irreducible subvariety of dimension k. If $V \subset |\operatorname{div}(s_k)|$, the first inequality is clear. For the second inequality, we choose rational sections s_i, $i = 0, \dots, k-1$, and s'_k such that $\operatorname{div}(s_0), \dots, \operatorname{div}(s_{k-1}), \operatorname{div}(s'_k)$ intersect Y properly. Using Lemmas 4.12 and 4.14, the product formula and the fact that $\overline{D}_0$ is nef, we deduce that $\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_k}(Y) \ge 0$.

Otherwise, $V \not\subset |\operatorname{div}(s_k)|$ and we choose rational sections s_i, $i = 0, \dots, k-1$, such that $\operatorname{div}(s_0), \dots, \operatorname{div}(s_k)$ intersect Y properly. The first inequality follows by applying the argument above to $\operatorname{div}(s_k) \cdot Y$, whereas the second one is given by Proposition 4.11. □

Corollary 4.18. *Let $\overline{D}_0, \dots, \overline{D}_n$ be semipositive metrized divisors on X such that $\overline{D}_0$ is nef and $\overline{D}_1, \dots, \overline{D}_n$ are generated by small sections. Let s_i be $\overline{D}_i$-small sections, $i = 1, \dots, n$. Then*

$$0 \le \mathrm{h}_{\overline{D}_0}\left(\prod_{i=1}^{n} \operatorname{div}(s_i)\right) \le \mathrm{h}_{\overline{D}_0, \dots, \overline{D}_n}(X).$$

5. Metrics and heights on toric varieties

In this section, we recall the necessary background on the arithmetic geometry of toric varieties following [BPS14, BMPS16]. In the second part of §2, we presented elements of the algebraic geometry of toric varieties over a field. In the sequel, we will freely use the notation introduced therein.

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula. Let $M \simeq \mathbb{Z}^n$ be a lattice and $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m},\mathbb{K}}^n$ its associated torus over $\mathbb{K}$ as in (2.4). For $v \in \mathfrak{M}$, we denote by $\mathbb{T}_v^{\mathrm{an}}$ the v-adic analytification of $\mathbb{T}$, and by $\mathbb{S}_v$ its compact torus. In the Archimedean case, $\mathbb{S}_v$ is isomorphic to the polycircle $(S^1)^n$, whereas in the non-Archimedean case, it is a compact analytic group, see [BPS14, §4.2] for a description. Moreover, there is a map $\operatorname{val}_v \colon \mathbb{T}_v^{\mathrm{an}} \to N_{\mathbb{R}}$ defined, in a given splitting, as

$$\operatorname{val}_v(x_1, \dots, x_n) = (-\log|x_1|_v, \dots, -\log|x_n|_v).$$

This map does not depend on the choice of the splitting, and the compact torus $\mathbb{S}_v$ coincides with its fiber over the point $0 \in N_{\mathbb{R}}$.

Let X be a projective toric variety with torus $\mathbb{T}$ given by a regular complete fan Σ on $N_{\mathbb{R}}$, and D a toric Cartier divisor on X given by a virtual support function Ψ_D on Σ. Recall that X contains $\mathbb{T}$ as a dense open subset. Let $\|\cdot\|_v$ be a toric v-adic metric on D, that is, a metric on the analytic line bundle $\mathcal{O}(D)_v^{\mathrm{an}}$ that is invariant under the action of $\mathbb{S}_v$. The associated *v-adic metric function* is the continuous function $\psi_{\|\cdot\|_v} \colon N_{\mathbb{R}} \to \mathbb{R}$ given by

$$\psi_{\|\cdot\|_v}(u) = \log\|s_D(p)\|_v, \tag{5.1}$$

for any $p \in \mathbb{T}_v^{\mathrm{an}}$ with $\operatorname{val}_v(p) = u$ and where s_D is the distinguished rational section of $\mathcal{O}(D)$. This function satisfies that $|\psi_{\|\cdot\|_v} - \Psi_D|$ is bounded on $N_{\mathbb{R}}$ and moreover, this difference extends to a continuous function on N_Σ, the compactification of $N_{\mathbb{R}}$ induced by the fan Σ. Indeed, the assignment

$$\|\cdot\|_v \longmapsto \psi_{\|\cdot\|_v} \tag{5.2}$$

is a one-to-one correspondence between the set of toric v-adic metrics on D and the set of such continuous functions on $N_{\mathbb{R}}$ [BPS14, Proposition 4.3.10]. In particular, the toric v-adic metric on D associated to the virtual support function Ψ_D is called the *canonical* v-adic toric metric of D and is denoted by $\|\cdot\|_{v,\mathrm{can}}$.

Furthermore, when $\|\cdot\|_v$ is semipositive, $\psi_{\|\cdot\|_v}$ is a concave function and it verifies that $|\psi_{\|\cdot\|_v} - \Psi_D|$ is bounded on $N_{\mathbb{R}}$, and the assignment in (5.2) gives a one-to-one correspondence between the set of semipositive toric v-adic metrics on D and the set of such concave functions on $N_{\mathbb{R}}$.

When $\|\cdot\|_v$ is semipositive, we also consider a continuous concave function on the polytope $\vartheta_{\|\cdot\|_v}\colon \Delta_D \to \mathbb{R}$ defined as the Legendre-Fenchel dual of $\psi_{\|\cdot\|_v}$, that is

$$\vartheta_{\|\cdot\|_v}(x) = \inf_{u\in N_{\mathbb{R}}} \langle x, u\rangle - \psi_{\|\cdot\|_v}(u).$$

The assignment $\|\cdot\|_v \mapsto \vartheta_{\|\cdot\|_v}$ is a one-to-one correspondence between the set of semipositive toric v-adic metrics on D and that of continuous concave functions on Δ_D. Under this assignment, the canonical v-adic toric metric of D corresponds to the zero function on Δ_D.

Definition 5.1. An *(adelic) toric metric* on D is a collection of toric v-adic metrics $(\|\cdot\|_v)_{v\in\mathfrak{M}}$, such that $\|\cdot\|_v = \|\cdot\|_{v,\mathrm{can}}$ for all but a finite number of $v \in \mathfrak{M}$. We denote by $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ the corresponding *(adelic) toric metrized divisor* on X.

Example 5.2. The collection $(\|\cdot\|_{v,\mathrm{can}})_{v\in\mathfrak{M}}$ of v-adic toric metrics on D is adelic in the sense of Definition 5.1. We denote by $\overline{D}^{\mathrm{can}}$ the corresponding *canonical* toric metrized divisor on X.

Let $\overline{D}$ be a toric metrized divisor on X. For each $v \in \mathfrak{M}$, we set

$$\psi_{\overline{D},v} = \psi_{\|\cdot\|_v} \quad \text{and} \quad \vartheta_{\overline{D},v} = \vartheta_{\|\cdot\|_v}$$

for the associated *v-adic metric function* and *v-adic roof function*, respectively.

Proposition 5.3. *Let $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}})$ be toric divisor together with a collection of toric v-adic metrics. If $\overline{D}$ is adelic in the sense of Definition 5.1, then it is also adelic in the sense of Definition 4.4. Moreover, both definitions coincide in the semipositive case.*

Proof. Let $p \in X(\overline{\mathbb{K}})$ and choose an adelic field extension $(\mathbb{F}, \mathfrak{N})$ such that $p \in X(\mathbb{F})$. Then $p_{\mathbb{F}}$ is a rational point of $X_{\mathbb{F}}$ and the inclusion

$$\iota\colon p_{\mathbb{F}} \hookrightarrow X_{\mathbb{F}}$$

is an equivariant map. Hence the inverse image $\iota^*\overline{D}$ is an adelic toric metric on $p_{\mathbb{F}}$ and so, for $w \in \mathfrak{N}$,

$$\log \|p_{\mathbb{F}}\|_w = \psi_{\iota^*\overline{D},w}(0),$$

and this quantity vanishes for all but the finite number of $w \in \mathfrak{N}$ such that $\|\cdot\|_w$ is not the canonical metric. Since this holds for all $p \in X(\overline{\mathbb{K}})$, we conclude that $\overline{D}$ is adelic in the sense of Definition 4.4.

For the second statement, assume that $\overline{D}$ is semipositive and adelic in the sense of Definition 4.4. Let $x_i \in M$, $i = 1, \dots, s$, be the vertices of the lattice polytope Δ_D. By [BPS14, Example 2.5.13], for each i there is an n-dimensional cone $\sigma_i \in \Sigma$ corresponding to x_i under the Legendre-Fenchel correspondence, $i = 1, \dots, s$. Each of these n-dimensional cones corresponds to a 0-dimensional orbit p_i of X. Denote by $\iota_i\colon p_i \hookrightarrow X$ the inclusion of this orbit.

Fix $1 \le i \le s$. Modulo a translation, we can assume without loss of generality that $x_i = 0$. By [BPS14, Proposition 4.8.9], for $v \in \mathfrak{M}$,

$$\vartheta_{\overline{D},v}(x_i) = \vartheta_{\iota_i^*\overline{D},v}(0) = -\log \|s_D(p_i)\|_v.$$

Hence $\vartheta_{\overline{D},v}(x_i) = 0$ for all but a finite number of v's.

On the other hand, let x_0 be the distinguished point of X, which coincides with the neutral element of $\mathbb{T}$, and denote by $\iota_0\colon x_0 \hookrightarrow X$ its inclusion. By [BPS14, Proposition 4.8.10],

$$\max_{x\in\Delta_D} \vartheta_{\overline{D},v}(x) = \vartheta_{\iota_0^*\overline{D},v}(0) = -\log\|s_D(x_0)\|_v.$$

Hence $\max_{x\in\Delta_D}\vartheta_{\overline{D},v}(x) = 0$ for all but a finite number of v's.

For every $v\in\mathfrak{M}$ such that $\vartheta_{\overline{D},v}(x_i) = 0$ for all i and $\max_{x\in\Delta_D}\vartheta_{\overline{D},v}(x) = 0$, we have that $\vartheta_{\overline{D},v}\equiv 0$ because this local roof function is a concave function on Δ_D. Hence, $\|\cdot\|_v$ coincides with the v-adic canonical metric of D for all these places. □

Remark 5.4. In the general non-semipositive case, Definitions 5.1 and 4.4 do not coincide. For instance, when $X = \mathbb{P}^1_{\mathbb{K}}$, a collection of metrics $\|\cdot\|_v$, $v\in\mathfrak{M}$, satisfies Definition 4.4 if and only if its associated metric functions satisfy that

$$\psi_{\overline{D},v}(0) = 0 \quad\text{and}\quad \lim_{u\to\pm\infty}\psi_{\overline{D},v}(u) - \Psi_D(u) = 0$$

for all but a finite number of places. In the absence of convexity, these conditions do not imply that $\psi_{\overline{D},v} = \Psi_D$ for all but a finite number of places.

A classical example of toric metrized divisors are those given by the inverse image of an equivariant map to a projective space equipped with the canonical metric on its universal line bundle. Below we describe this example and we refer to [BPS14, Example 5.1.16] for the technical details.

Let $\boldsymbol{m} = (m_0,\dots,m_r)\in M^{r+1}$ and $\boldsymbol{\alpha} = (\alpha_0,\dots,\alpha_r)\in(\mathbb{K}^\times)^{r+1}$, with $r\geq 0$. The monomial map associated to this data is defined as

$$\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}\colon \mathbb{T}\longrightarrow \mathbb{P}^r_{\mathbb{K}},\quad p\longmapsto (\alpha_0\chi^{m_0}(p):\dots:\alpha_r\chi^{m_r}(p)). \tag{5.3}$$

For a toric variety X with torus $\mathbb{T}$ corresponding to fan that is compatible with the polytope $\Delta = \mathrm{conv}(m_0,\dots,m_r)\subset M_{\mathbb{R}}$, this extends to an equivariant map $X\to\mathbb{P}^r_{\mathbb{K}}$, also denoted by $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$.

Example 5.5. With notation as above, let $\overline{E}^{\mathrm{can}}$ be the divisor of the hyperplane at infinity of $\mathbb{P}^r_{\mathbb{K}}$, equipped with the canonical metric at all places. Then $D = \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}}E$ is the nef toric Cartier divisor on X corresponding to the translated polytope $\Delta - m_0$. We consider the semipositive toric metrized divisor $\overline{D} = \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}}\overline{E}$ on X.

For each $v\in\mathfrak{M}$, the v-adic metric function of $\overline{D}$ is given by

$$\psi_{\overline{D},v}\colon N_{\mathbb{R}}\longrightarrow\mathbb{R},\quad u\longmapsto \min_{0\leq j\leq r}\Big(\langle m_j - m_0, u\rangle - \log\Big|\frac{\alpha_j}{\alpha_0}\Big|_v\Big).$$

The polytope corresponding to D is $\Delta - m_0$ and, for each $v\in\mathfrak{M}$, the v-adic roof function of $\overline{D}$ is given by

$$\vartheta_{\overline{D},v}(x) = \max_{\boldsymbol{\lambda}}\sum_{j=0}^r \lambda_j\log|\alpha_j|_v - \log|\alpha_0|_v,$$

the maximum being over the vectors $\boldsymbol{\lambda} = (\lambda_0,\dots,\lambda_r)\in\mathbb{R}^{r+1}_{\geq 0}$ with $\sum_{j=0}^r\lambda_j = 1$ such that $\sum_{j=0}^r\lambda_j(m_j - m_0) = x$. In other words, this is the piecewise affine concave function on $\Delta - m_0$ parametrizing the upper envelope of the extended polytope

$$\mathrm{conv}\big((m_j - m_0, \log|\alpha_j/\alpha_0|_v)_{0\leq j\leq r}\big)\subset M_{\mathbb{R}}\times\mathbb{R}.$$

Definition 5.6. For $i = 0, \dots, n$, let $g_i : \Delta_i \to \mathbb{R}$ be a concave function on a convex body $\Delta_i \subset M_{\mathbb{R}}$. The *mixed integral* of $g_0, \dots, g_n$ is defined as

$$\mathrm{MI}_M(g_0, \dots, g_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0 \le i_0 < \dots < i_j \le n} \int_{\Delta_{i_0} + \dots + \Delta_{i_j}} g_{i_0} \boxplus \dots \boxplus g_{i_j} \, \mathrm{d}\,\mathrm{vol}_M,$$

where $\Delta_{i_0} + \dots + \Delta_{i_j}$ denotes the Minkowski sum of polytopes, and $g_{i_0} \boxplus \dots \boxplus g_{i_j}$ the sup-convolution of concave function, which is the function on $\Delta_{i_0} + \dots + \Delta_{i_j}$ defined as

$$g_{i_0} \boxplus \dots \boxplus g_{i_j}(x) = \sup \big(g_{i_0}(x_{i_0}) + \dots + g_{i_j}(x_{i_j}) \big),$$

where the supremum is taken over $x_{i_l} \in \Delta_{i_l}$, $l = 0, \dots, j$, such that $x_{i_0} + \dots + x_{i_j} = x$.

The mixed integral is symmetric and additive in each variable with respect to the sup-convolution. Moreover, for a concave function $g \colon \Delta \to \mathbb{R}$ on a convex body Δ, we have $\mathrm{MI}_M(g, \dots, g) = (n+1)! \int_\Delta g \, \mathrm{d}\,\mathrm{vol}_M$, see [PS08, §8] for details.

The following is a restricted version of a result by Burgos Gil, Philippon and the second author, giving the global height of a toric variety with respect to a family of semipositive toric metrized divisors in terms of the mixed integrals of the associated local roof functions [BPS14, Theorem 5.2.5].

Theorem 5.7. *Let $\overline{D}_i$, $i = 0, \dots, n$, be semipositive toric metrized divisors on X such that $\overline{D}_1, \dots, \overline{D}_n$ are generated by small sections. Then*

$$\mathrm{h}_{\overline{D}_0, \dots, \overline{D}_n}(X) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_{\overline{D}_0, v}, \dots, \vartheta_{\overline{D}_n, v}). \tag{5.4}$$

Remark 5.8. The result in [BPS14, Theorem 5.2.5] is more general. Given semipositive toric metrized divisors $\overline{D}_i$, $i = 0, \dots, n$, and rational sections s_i such that $\mathrm{div}(s_0), \dots, \mathrm{div}(s_n)$ intersect X properly, the corresponding local heights are zero except for a finite number of places, and the formula (5.4) holds without any extra positivity assumption.

6. Proof of Theorem 1.1

Let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula. Let $f \in \mathbb{K}[M]$ be a Laurent polynomial and $\Delta \subset M_{\mathbb{R}}$ its Newton polytope. Let X be a projective toric variety over $\mathbb{K}$ given by a fan on $N_{\mathbb{R}}$ that is compatible with Δ, and D the Cartier divisor on X given by this polytope. To prove Theorem 1.1, we first construct a toric metric on D such that the associated toric metrized divisor $\overline{D}$ is semipositive and generated by small sections, and the global section of $\mathcal{O}(D)$ associated to f is $\overline{D}$-small. We obtain this metrized divisor as the inverse image of a metrized divisor on a projective space.

For $r \ge 0$, let $\mathbb{P}^r_{\mathbb{K}}$ be the r-dimensional projective space over $\mathbb{K}$ and E the divisor of the hyperplane at infinity. We denote by $\overline{E}$ this Cartier divisor equipped with the ℓ^1-norm at the Archimedean places, and the canonical one at the non-Archimedean ones. This metric is defined, for $p = (p_0 : \dots : p_s) \in \mathbb{P}^s_{\mathbb{K}}(\overline{\mathbb{K}}_v)$ and a global section s of $\mathcal{O}(E)$ corresponding to a linear form $\rho_s \in \mathbb{K}[x_0, \dots, x_s]$, by

$$\|s(p)\|_v = \begin{cases} \dfrac{|\rho_s(p_0, \dots, p_s)|_v}{\sum_j |p_j|_v} & \text{if } v \text{ is Archimedean,} \\[2ex] \dfrac{|\rho_s(p_0, \dots, p_s)|_v}{\max_j |p_j|_v} & \text{if } v \text{ is non-Archimedean,} \end{cases} \tag{6.1}$$

The projective space $\mathbb{P}^r_{\mathbb{K}}$ has a standard structure of toric variety with torus $\mathbb{G}^r_{\mathrm{m},\mathbb{K}}$, included via the map $(z_1, \dots, z_r) \mapsto (1 : z_1 : \cdots : z_r)$. Thus $\overline{E}$ is a toric metrized divisor. It is a particular case of the weighted ℓ^p-metrized divisors on toric varieties studied in [BPS15, §5.2].

The following result summarizes the basic properties of this toric metrized divisor and its combinatorial data.

Proposition 6.1. *The toric metrized divisor $\overline{E}$ on $\mathbb{P}^r_{\mathbb{K}}$ is semipositive and generated by small sections. For $v \in \mathfrak{M}$, its v-adic metric function is given, for $\boldsymbol{u} = (u_1, \dots, u_r) \in \mathbb{R}^r$, by*

$$\psi_{\overline{E},v}(\boldsymbol{u}) = \begin{cases} -\log\Big(1 + \sum_{j=1}^r \mathrm{e}^{-u_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \min(0, u_1, \dots, u_r) & \text{if } v \text{ is non-Archimedean.} \end{cases} \tag{6.2}$$

The polytope corresponding to E is the standard simplex Δ^r of $\mathbb{R}^r$. For $v \in \mathfrak{M}$, the v-adic roof function of $\overline{E}$ is given, for $\boldsymbol{x} = (x_1, \dots, x_r) \in \Delta^r$, by

$$\vartheta_{\overline{E},v}(\boldsymbol{x}) = \begin{cases} -\sum_{j=0}^r x_j \log(x_j) & \text{if } v \text{ is Archimedean,} \\ 0 & \text{if } v \text{ is non-Archimedean,} \end{cases}$$

with $x_0 = 1 - \sum_{j=1}^r x_j$.

Proof. The distinguished rational section of the line bundle $\mathcal{O}(E)$ corresponds to the linear form $x_0 \in \mathbb{K}[x_0, \dots, x_r]$. Hence, for an Archimedean place v and a point $\boldsymbol{z} = (z_1, \dots, z_r) \in \mathbb{G}^r_{\mathrm{m},\mathbb{K}}(\overline{\mathbb{K}}_v)$,

$$\psi_{\overline{E},v}(\mathrm{val}_v(\boldsymbol{z})) = \log \|s_E(\boldsymbol{z})\|_v = -\log\Big(1 + \sum_{j=1}^r |z_j|\Big),$$

which gives the expression in (6.2) for this case. The non-Archimedean case is done similarly. We can easily check that these metric functions are concave. In the Archimedean case, this can be done by computing its Hessian and verifying that it is nonpositive and, in the non-Archimedean case, it is immediate from its expression. Hence, $\overline{E}$ is semipositive.

Set s_j for the global sections corresponding to the linear forms $x_j \in \mathbb{K}[x_0, \dots, x_r]$, $j = 0, \dots, r$. We have that $\bigcap_{j=0}^r |\operatorname{div}(s_j)| = \emptyset$, and so this is a set of generating global sections. It follows from the definition of the metric in (6.1) that these global sections are $\overline{E}$-small. Hence, $\overline{E}$ is generated by small sections.

The fact that the polytope corresponding to E is the standard simplex is classical, see for instance [Ful93, page 27]. When v is Archimedean, the v-adic roof function can be computed similarly as the one for the Fubini-Study metric in [BPS14, Example 2.4.3]. When v is non-Archimedean, v-adic roof function is zero, because the metric $\|\cdot\|_v$ is canonical. □

Set $r \geq 0$. Take $\boldsymbol{m} = (m_0, \dots, m_r) \in M^{r+1}$ and $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$, and consider the polytope $\Delta = \mathrm{conv}(m_0, \dots, m_r) \subset M_{\mathbb{R}}$. Let X be a projective toric variety over $\mathbb{K}$ given by a fan on $N_{\mathbb{R}}$ that is compatible with Δ. Let $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}} \colon \mathbb{T} \to \mathbb{P}^r_{\mathbb{K}}$ be the monomial map in (5.3) and set

$$D_{\boldsymbol{m}} = \operatorname{div}(\chi^{-m_0}) + \varphi^*_{\boldsymbol{m},\boldsymbol{\alpha}} E,$$

which coincides with the Cartier divisor on X corresponding to Δ. For each $v \in \mathfrak{M}$, we consider the metric on $\mathcal{O}(D_{\boldsymbol{m}})_v^{\mathrm{an}} \simeq \mathcal{O}(\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}^* E)_v^{\mathrm{an}}$ defined by

$$\|\cdot\|_{\boldsymbol{m},\boldsymbol{\alpha},v} = |\alpha_0|_v^{-1} \varphi_{\boldsymbol{m},\boldsymbol{\alpha}}^* \|\cdot\|_{\overline{E},v}, \tag{6.3}$$

the homothecy by $|\alpha_0|_v$ of the inverse image by $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$ of the v-adic metric of $\overline{E}$. We then set

$$\overline{D}_{\boldsymbol{m},\boldsymbol{\alpha}} = (D_{\boldsymbol{m}}, (\|\cdot\|_{\boldsymbol{m},\boldsymbol{\alpha},v})_{v\in\mathfrak{M}}). \tag{6.4}$$

Since $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$ is an equivariant map and $\overline{E}$ is toric, this is a toric metrized divisor on X.

Proposition 6.2. *The toric metrized divisor $\overline{D} = \overline{D}_{\boldsymbol{m},\boldsymbol{\alpha}}$ on X is semipositive and generated by small sections. For $v \in \mathfrak{M}$, its v-adic metric is given, for $p \in \mathbb{T}(\overline{\mathbb{K}}_v)$, by*

$$\|s_D(p)\|_v = \begin{cases} \left(\displaystyle\sum_{j=0}^r |\alpha_j \chi^{m_j}(p)|_v\right)^{-1} & \text{if } v \text{ is Archimedean,} \\ \left(\displaystyle\max_{0\le j\le r} |\alpha_j \chi^{m_j}(p)|_v\right)^{-1} & \text{if } v \text{ is non-Archimedean.} \end{cases} \tag{6.5}$$

The v-adic metric function of $\overline{D}$ is given, for $u \in N_{\mathbb{R}}$, by

$$\psi_{\overline{D},v}(u) = \begin{cases} -\log\left(\displaystyle\sum_{j=0}^r |\alpha_j|_v \,\mathrm{e}^{-\langle m_j,u\rangle}\right) & \text{if } v \text{ is Archimedean,} \\ \displaystyle\min_{0\le j\le r} \langle m_j, u\rangle - \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean,} \end{cases} \tag{6.6}$$

and the v-adic roof function of $\overline{D}$ is given, for $x \in \Delta$, by

$$\vartheta_{\overline{D},v}(x) = \begin{cases} \displaystyle\max_{\boldsymbol{\lambda}} \sum_{j=0}^r \lambda_j \log\left(\frac{|\alpha_j|_v}{\lambda_j}\right) & \text{if } v \text{ is Archimedean,} \\ \displaystyle\max_{\boldsymbol{\lambda}} \sum_{j=0}^r \lambda_j \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean,} \end{cases} \tag{6.7}$$

the maximum being over the vectors $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_r) \in \mathbb{R}_{\ge 0}^{r+1}$ with $\sum_{j=0}^r \lambda_j = 1$ such that $\sum_{j=0}^r \lambda_j m_j = x$.

Proof. Set $\overline{D'} = \varphi_{\boldsymbol{m},\boldsymbol{\alpha}}^* \overline{E}$ for short. This is a toric metrized divisor on X that is semipositive and generated by small sections, due to Proposition 6.1 and the preservation of these properties under inverse image. Since the v-adic metrics of $\overline{D}$ are homothecies of those of $\overline{D'}$, it follows that $\overline{D}$ is semipositive too. Moreover, a global section ς of $\mathcal{O}(D') \simeq \mathcal{O}(D)$ is $\overline{D'}$-small if and only if the global section $\alpha_0\,\varsigma$ is $\overline{D}$-small. It follows that $\overline{D}$ is also generated by small sections.

Using (6.1) and the definition of the monomial map $\varphi_{\boldsymbol{m},\boldsymbol{\alpha}}$, for $v \in \mathfrak{M}$, the v-adic metric of $\overline{D'}$ is given, for $p \in \mathbb{T}(\overline{\mathbb{K}}_v)$, by

$$\|s_{D'}(p)\|_v = \begin{cases} \left(\displaystyle\sum_{j=0}^r \left|\frac{\alpha_j}{\alpha_0}\chi^{m_j-m_0}(p)\right|_v\right)^{-1} & \text{if } v \text{ is Archimedean,} \\ \left(\displaystyle\max_{0\le j\le r} \left|\frac{\alpha_j}{\alpha_0}\chi^{m_j-m_0}(p)\right|_v\right)^{-1} & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Since $D = \mathrm{div}(\chi^{-m_0}) + D'$, their distinguished rational sections are related by $s_D = \chi^{-m_0} s_{D'}$. It follows from (6.3) that

$$\|s_D(p)\|_v = |\alpha_0|_v^{-1} |\chi^{-m_0}(p)|_v \, \|s_{D'}(p)\|_v,$$

which implies the formulae in (6.5). As a consequence, we obtain also the expressions for the v-adic metric functions of $\overline{D}$.

For its roof function, consider first the linear map $H\colon N_{\mathbb{R}} \to \mathbb{R}^{r+1}$ given, for $u \in N_{\mathbb{R}}$, by $H(u) = (\langle m_0, u\rangle, \dots, \langle m_r, u\rangle)$. For each place v, consider the concave function $g_v\colon \mathbb{R}^{r+1} \to \mathbb{R}$ given, for $\boldsymbol{\nu} \in \mathbb{R}^{r+1}$, by

$$g_v(\boldsymbol{\nu}) = \begin{cases} -\log\Big(\displaystyle\sum_{j=0}^{r} |\alpha_j|_v \, \mathrm{e}^{-\nu_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \displaystyle\min_{0\le j\le r} \nu_j - \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Notice that $\psi_{\overline{D},v} = H^* g_v$. The domain of the Legendre-Fenchel dual of g_v is the simplex S given as the convex hull of the vectors in the standard basis of $\mathbb{R}^{r+1}$. This Legendre-Fenchel dual is given, for $\boldsymbol{\lambda} \in S$, by

$$g_v^{\vee}(\boldsymbol{\lambda}) = \begin{cases} \displaystyle\sum_{j=0}^{r} \lambda_j \log\Big(\frac{|\alpha_j|_v}{\lambda_j}\Big) & \text{if } v \text{ is Archimedean,} \\ \displaystyle\max_{\boldsymbol{\lambda}} \sum_{j=0}^{r} \lambda_j \log|\alpha_j|_v & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

For the Archimedean case, this formula follows from [BPS15, Proposition 5.8], whereas in the non-Archimedean case, it is given by Example 5.5.

By [BPS14, Proposition 2.3.8(3)], the v-adic roof function $\vartheta_{\overline{D},v}$ is the direct image under the dual map $H^{\vee}$ of the Legendre-Fenchel dual $g_v^{\vee}$, which gives the stated formulae in (6.7). □

Definition 6.3. Let $f \in \mathbb{K}[M]$ be a Laurent polynomial and X be a projective toric variety over $\mathbb{K}$ given by a fan on $N_{\mathbb{R}}$ that is compatible with the Newton polytope $\mathrm{N}(f)$. Write $f = \sum_{j=0}^{r} \alpha_j \chi^{m_j}$ with $m_j \in M$ and $\alpha_j \in \mathbb{K}^{\times}$. The *toric metrized divisor on X associated to f* is defined as

$$\overline{D}_f = \overline{D}_{\boldsymbol{m},\boldsymbol{\alpha}},$$

the toric metrized divisor in (6.4) for the data $\boldsymbol{m} = (m_0, \dots, m_r) \in M^{r+1}$ and $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_r) \in (\mathbb{K}^{\times})^{r+1}$. It does not depend on the ordering of the terms of f. For $v \in \mathfrak{M}$, we denote by $\psi_{f,v}$ and $\vartheta_{f,v}$ the v-adic metric and roof functions of $\overline{D}_f$, respectively.

Lemma 6.4. *With notation as in Definition 6.3, the global section of $\mathcal{O}(D_f)$ associated to f is $\overline{D}_f$-small.*

Proof. Set $\overline{D} = \overline{D}_f$ for short, and let $s = f s_D$ be the global section of $\mathcal{O}(D)$ associated to f. For $v \in \mathfrak{M}$ and $p \in \mathbb{T}(\overline{\mathbb{K}}_v)$,

$$\|s(p)\|_v = |f(p)|_v \, \|s_D(p)\|_v = \Big|\sum_{j=0}^{r} \alpha_j \chi^{m_j}(p)\Big|_v \|s_D(p)\|_v.$$

It follows from (6.5) that $\|s\|_v \le 1$ on $\mathbb{T}(\overline{\mathbb{K}}_v)$, and so s is $\overline{D}$-small. □

The following result corresponds to Theorem 1.1 in the introduction.

Theorem 6.5. *Let $f_1,\dots,f_n\in\mathbb{K}[M]$, and let X be a proper toric variety with torus $\mathbb{T}_M$ and $\overline{D}_0$ a nef toric metrized divisor on X. Let $\Delta_0\subset M_{\mathbb{R}}$ be the polytope of D_0 and, for $v\in\mathfrak{M}$, let $\vartheta_{0,v}\colon\Delta_i\to\mathbb{R}$ be the v-adic roof function of $\overline{D}_0$. For $i=1,\dots,n$, let $\Delta_i\subset M_{\mathbb{R}}$ be the Newton polytope of f_i and, for $v\in\mathfrak{M}$, let $\vartheta_{i,v}\colon\Delta_i\to\mathbb{R}$ be the v-adic roof function on the metric associated to f_i. Then*

$$\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n))\le\sum_{v\in\mathfrak{M}}n_v\,\mathrm{MI}(\vartheta_{0,v},\dots,\vartheta_{n,v}).$$

Proof. Let Σ be the complete fan corresponding to the proper toric variety X. By taking a refinement, we can assume without loss of generality that Σ is regular and compatible with the Newton polytopes Δ_i, $i=1,\dots,n$. Hence X is a projective toric variety and $\overline{D}_0$ a nef toric metrized divisor, and there are nef toric Cartier divisors D_i, $i=1,\dots,n$, corresponding to these Newton polytopes.

For $i=1,\dots,n$, we denote by $\overline{D}_i$ the toric metrized divisor associated to f_i (Definition 6.3). By Proposition 6.2, each $\overline{D}_i$ is semipositive and generated by small sections and, by Lemma 6.4, the global sections s_i of $\mathcal{O}(D_i)$ corresponding to f_i are $\overline{D}_i$-small. Applying Corollary 4.18 and Theorem 5.7,

$$\mathrm{h}_{\overline{D}_0}\Bigg(\prod_{i=1}^{n}\mathrm{div}(s_i)\Bigg)\le\mathrm{h}_{\overline{D}_0,\dots,\overline{D}_n}(X)=\sum_{v\in\mathfrak{M}}n_v\,\mathrm{MI}_M(\vartheta_{\overline{D}_0,v},\dots,\vartheta_{\overline{D}_n,v}).$$

Due to Proposition 2.8(2), the inequality $Z(f_1,\dots,f_n)\le\prod_{i=1}^{n}\mathrm{div}(s_i)$ holds. By the linearity of the global height and the nefness of $\overline{D}_0$,

$$\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n))\le\mathrm{h}_{\overline{D}_0}\Bigg(\prod_{i=1}^{n}\mathrm{div}(s_i)\Bigg),$$

which concludes the proof. □

Definition 6.6. Let $\boldsymbol{\alpha}=(\alpha_0,\dots,\alpha_r)\in(\mathbb{K}^\times)^r$ with $r\ge1$. For $v\in\mathfrak{M}$, the *v-adic (logarithmic) length* of $\boldsymbol{\alpha}$ is defined as

$$\ell_v(\boldsymbol{\alpha})=\begin{cases}\log(\sum_{j=0}^{r}|\alpha_j|_v) & \text{if } v \text{ is Archimedean,}\\ \log(\max_{0\le j\le r}|\alpha_j|_v) & \text{if } v \text{ is non-Archimedean.}\end{cases}$$

The *(logarithmic) length* of $\boldsymbol{\alpha}$ is defined as $\ell(\boldsymbol{\alpha})=\sum_{v\in\mathfrak{M}}n_v\ell_v(\boldsymbol{\alpha})$.

For a Laurent polynomial $f\in\mathbb{K}[M]$, we define its *v-adic (logarithmic) length*, denoted by $\ell_v(f)$, as the v-adic length of its vector of coefficients, $v\in\mathfrak{M}$. We also define its *(logarithmic) length*, denoted by $\ell(f)$, as the length of its vector of coefficients.

Lemma 6.7. *Let $\vartheta_i\colon\Delta_i\to\mathbb{R}$ be concave functions on convex bodies, $i=0,\dots,n$. Then*

$$\mathrm{MI}_M(\vartheta_0,\dots,\vartheta_n)\le\sum_{i=0}^{n}\Big(\max_{x\in\Delta_i}\vartheta_i(x)\Big)\,\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n)$$

Proof. Set $c_i=\max_{x\in\Delta_i}\vartheta_i(x)$ for short. By the monotonicity of the mixed integral [PS08, Proposition 8.1]

$$\mathrm{MI}_M(\vartheta_0,\dots,\vartheta_n)\le\mathrm{MI}_M(c_0|_{\Delta_0},\dots,c_n|_{\Delta_n}),$$

where $c_i|_{\Delta_i}$ denotes the constant function c_i on the convex body Δ_i. By [PS08, formula (8.3)]

$$\mathrm{MI}_M(c_0|_{\Delta_0},\dots,c_n|_{\Delta_n})=\sum_{i=0}^n c_i\,\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n),$$

giving the stated inequality. □

Corollary 6.8. *With notation as in Theorem 6.5,*

$$\mathrm{h}_{\overline{D}_0}(Z(f_1,\dots,f_n))\le\Big(\sum_{v\in\mathfrak{M}}n_v\max_{x\in\Delta_0}\vartheta_{0,v}(x)\Big)\mathrm{MV}_M(\Delta_1,\dots,\Delta_n)\\+\sum_{i=1}^n\ell(f_i)\,\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n).$$

In particular, for the canonical metric on D_0 (Example 5.2),

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(Z(f_1,\dots,f_n))\le\sum_{i=1}^n\ell(f_i)\,\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n).\tag{6.8}$$

Proof. For $1\le i\le n$ and $v\in\mathfrak{M}$, let $\vartheta_{i,v}$ be the v-adic roof function of the toric semipositive metric associated to f_i. Using (6.7), we compute the value of $\psi_{i,v}(0)=-\vartheta_{i,v}^\vee(0)$ to obtain that

$$\max_{x\in\Delta_i}\vartheta_{i,v}(x)=\ell_v(f_i).\tag{6.9}$$

The first statement follows then from Theorem 6.5 and Lemma 6.7. The second statement is a particular case of the first one, using the fact that the v-adic roof functions of $\overline{D}_0^{\mathrm{can}}$ are the zero functions on Δ_0. □

We readily derive from the previous corollary the following version of the arithmetic Bézout theorem.

Corollary 6.9. *Let $f_1,\dots,f_n\in\mathbb{K}[x_1,\dots,x_n]$ and let $\overline{D}^{\mathrm{can}}$ be the divisor at infinity of $\mathbb{P}^n_{\mathbb{K}}$ equipped with the canonical metric. Then*

$$\mathrm{h}_{\overline{D}^{\mathrm{can}}}(Z(f_1,\dots,f_n))\le\sum_{i=1}^n\Big(\prod_{j\ne i}\deg(f_j)\Big)\ell(f_i),$$

where $\deg(f_i)$ denotes the total degree of the polynomial f_i.

7. Comparisons, examples and applications

In this section, we first compare our main results (Theorem 6.5 and Corollary 6.8) with the previous ones. Next, we compute the bounds given by these results in two families of examples, and compare them with the actual height of the 0-cycles. The first family of examples illustrates a case in which these bounds do approach the height of the 0-cycle, while the second one shows a situation where the bound of Theorem 6.5 is sharp and that of Corollary 6.8 is not. Finally, we present an application bounding the height of the resultant of a 0-cycle defined by a system of Laurent polynomials.

The first arithmetic analogue of the BKK theorem was proposed by Maillot [Mai00, Corollaire 8.2.3]. With notations as in Theorem 6.5, suppose that $f_1,\dots,f_n\in\mathbb{Z}[M]$

and that D_0 is the nef toric divisor corresponding to the polytope $\Delta_0 = \sum_{i=1}^n \Delta_i$. Then Maillot's result amounts to the upper bound

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(Z(f_1,\dots,f_n)) \le \sum_{i=1}^n (\mathrm{m}(f_i)+L(\Delta_i))\,\mathrm{MV}_M(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n), \tag{7.1}$$

where $\mathrm{m}(f_i)$ denotes the logarithmic Mahler measure of f_i, and $L(\Delta_i)$ a constant associated to the polytope Δ_i.

This result is similar to Corollary 6.8 specialized to a system of Laurent polynomials with integer coefficients, and the toric divisor D_0 associated to the polytope given by the Minkowski sum $\sum_{i=1}^n \Delta_i$, equipped with the canonical metric. The factors $\mathrm{m}(f_i)+L(\Delta_i)$ in (7.1) and $\ell(f_i)$ in (6.8) are comparable, albeit the fact that the constant $L(\Delta_i)$ is not effective, see [Som05, Remark 4.2] for a discussion on this point.

Another previous result in this direction was obtained by the second author [Som05, Théorème 0.3]. Using again the notation in Theorem 6.5, suppose that $f_1,\dots,f_n \in \mathbb{Z}[M]$ and that the polytope Δ_0 associated to the nef toric divisor D_0 contains Δ_i, $i=1,\dots,n$. Then

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(Z(f_1,\dots,f_n)) \le n!\,\mathrm{vol}_M(\Delta_0)\sum_{i=1}^n \ell(f_i).$$

This result is equivalent to the specialization of the upper bound in (6.8) to a system of Laurent polynomials with integer coefficients and Newton polytopes contained in the polytope Δ_0.

We next turn to the computation of the bounds given by Theorem 6.5 and Corollary 6.8 in two families of examples.

We keep the notation of §6. We need the the following auxiliary computation of mixed volumes. For its proof, we recall that the mixed volume of a family of polytopes $\Delta_i \subset \mathbb{R}^n$, $i=1,\dots,n$, can be decomposed in terms of mixed volumes of their lower dimensional faces as

$$\mathrm{MV}_n(\Delta_1,\dots,\Delta_n) = -\sum_{u\in S^{n-1}} \Psi_{\Delta_1}(u)\,\mathrm{MV}_{n-1}(\Delta_2^u,\dots,\Delta_n^u), \tag{7.2}$$

where S^{n-1} is the unit sphere of $\mathbb{R}^n$, Ψ_{Δ_1} is the support function of Δ_1 as in (2.6), Δ_i^u is the unique face of Δ_i that minimizes the functional u on this polytope, and MV_n and MV_{n-1} denote the mixed volume functions associated to the Lebesgue measure of $\mathbb{R}^n$ and $u^\perp \simeq \mathbb{R}^{n-1}$, respectively. In fact, the sum ranges through the normal vectors of the facets of each polytope. We refer to [Sch93, formula (5.1.22)] for more details.

Lemma 7.1. *Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polytope, and $m_i \in M$, $i=2,\dots,n$, linearly independent lattice points. Denote by $\overline{0\,m_i}$ the segment between 0 and m_i, and $u\in N$ the smallest lattice point orthogonal to all the m_i's, which is unique up to a sign. Let $P=\sum_{i=2}^n \mathbb{Z}m_i \subset M$ be the sublattice generated by the m_i's, and P^{sat} its saturation. Then*

$$\mathrm{MV}_M(\Delta,\overline{0\,m_2},\dots,\overline{0\,m_n}) = [P^{\mathrm{sat}}:P]\,\mathrm{vol}_{\mathbb{Z}}\langle\Delta,u\rangle,$$

where $\langle\Delta,u\rangle$ is the image of Δ under the functional $u\colon M_{\mathbb{R}}\to\mathbb{R}$.

Proof. Choosing a basis, we identify $M=\mathbb{Z}^n$. With this identification, $\mathrm{MV}_M=\mathrm{MV}_n$, the mixed volume associated to the Lebesgue measure of $\mathbb{R}^n$. The formula in (7.2) applied to the polytopes $\Delta, \overline{0\,m_2}, \dots, \overline{0\,m_n}$ implies that

$$\mathrm{MV}_n(\Delta, \overline{0\,m_2}, \dots, \overline{0\,m_n}) = -\Big(\Psi_\Delta\Big(\frac{u}{\|u\|}\Big) + \Psi_\Delta\Big(-\frac{u}{\|u\|}\Big)\Big)\,\mathrm{MV}_{n-1}(\overline{0\,m_2}, \dots, \overline{0\,m_n})$$

$$= -\frac{1}{\|u\|}(\Psi_\Delta(u) + \Psi_\Delta(-u))\,\mathrm{MV}_{n-1}(\overline{0\,m_2}, \dots, \overline{0\,m_n}), \tag{7.3}$$

where $\|u\|$ is the Euclidean norm. We have that

$$\Psi_\Delta(u) + \Psi_\Delta(-u) = \min_{x\in\Delta}\langle x, u\rangle + \min_{x\in\Delta}\langle x, -u\rangle = -\operatorname{vol}_{\mathbb{Z}}\langle \Delta, u\rangle \tag{7.4}$$

By the Brill-Gordan duality theorem [HB84, Lemma 1], we have the equality $\|u\| = \operatorname{vol}_{n-1}(P_{\mathbb{R}}/P^{\mathrm{sat}})$, where vol_{n-1} denotes the Lebesgue measure of $u^\perp$. Hence

$$\frac{1}{\|u\|}\,\mathrm{MV}_{n-1}(\overline{0\,m_2}, \dots, \overline{0\,m_n}) = \mathrm{MV}_{P^{\mathrm{sat}}}(\overline{0\,m_2}, \dots, \overline{0\,m_n}) = [P^{\mathrm{sat}} : P]. \tag{7.5}$$

The result follows then from (7.3), (7.4) and (7.5). □

Example 7.2. Let $d, \alpha \geq 1$ be integers and consider the system of Laurent polynomials given by

$$f_1 = x_1 - \alpha, \quad f_2 = x_2 - \alpha x_1^d, \quad \dots, \quad f_n = x_n - \alpha x_{n-1}^d \quad \in \mathbb{Q}[x_1^{\pm1}, \dots, x_n^{\pm1}].$$

Its zero set in $\mathbb{T}_{\mathbb{Z}^n} = \mathbb{G}_{\mathrm{m},\mathbb{Q}}^n$ consists of the rational point

$$p = (\alpha, \alpha^{d+1}, \dots, \alpha^{d^{n-1}+d^{n-2}+\dots+1}) \in \mathbb{T}_{\mathbb{Z}^n}(\mathbb{Q}) = (\mathbb{Q}^\times)^n.$$

Let X be a proper toric variety over $\mathbb{Q}$, and $\overline{D}_0^{\mathrm{can}}$ a nef toric Cartier divisor on X equipped with the canonical metric. Let $\Delta_0 \subset \mathbb{R}^n$ be the polytope corresponding to D_0 and, for $i = 1, \dots, n$, set

$$u_i = e_i + d e_{i+1} + \dots + d^{n-i} e_n \in \mathbb{Z}^n,$$

where the e_j's are the vectors in the standard basis of $\mathbb{Z}^n$. The height of p with respect to $\overline{D}_0^{\mathrm{can}}$ is

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(p) = \Big(\operatorname{vol}_{\mathbb{Z}}\Big\langle \Delta_0, \sum_{i=1}^n u_i\Big\rangle\Big)\log(\alpha). \tag{7.6}$$

To prove this, let $v \in \mathfrak{M}_{\mathbb{Q}}$. By (5.1), the local height of p with respect to the pair $(\overline{D}_0^{\mathrm{can}}, s_{D_0})$ is given by

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}},v}(p, s_{D_0}) = -\log\|s_{D_0}(p)\|_{v,\mathrm{can}} = -\Psi_{\Delta_0}\big(\operatorname{val}_v(p)\big).$$

Set $u = \sum_{i=1}^n u_i$ for short. Since $\operatorname{val}_v(p) = -\log|\alpha|_v\, u$,

$$-\Psi_{\Delta_0}\big(\operatorname{val}_v(p)\big) = \begin{cases} \log|\alpha|_v \max\limits_{m\in\Delta_0\cap\mathbb{Z}^n}\langle m, u\rangle & \text{if } v = \infty, \\ \log|\alpha|_v \min\limits_{m\in\Delta_0\cap\mathbb{Z}^n}\langle m, u\rangle & \text{if } v \neq \infty. \end{cases}$$

By adding these contributions,

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(p) = \log(\alpha)\Big(\max_{m\in\Delta_0\cap\mathbb{Z}^n}\langle m, u\rangle - \min_{m\in\Delta_0\cap\mathbb{Z}^n}\langle m, u\rangle\Big),$$

which gives the formula in (7.6).

Next we compare the value of the height of p with the bounds given by Corollary 6.8. We have $\ell(f_i) = \log(\alpha + 1)$ for all i. Consider the dual basis of the u_i's, given by

$$m_1 = e_1, m_2 = e_2 - d e_1, \dots, m_n = e_n - d e_{n-1} \in \mathbb{Z}^n.$$

For $i = 1, \dots, n$, the Newton polytope Δ_i of f_i is a translate of the segment $\overline{0\,m_i}$, and u_i is the smallest lattice point in the line $(\sum_{j\neq i} \mathbb{R} m_j)^{\perp}$. Moreover the sublattice $\sum_{j\neq i} \mathbb{Z} m_i$ is saturated. By Lemma 7.1

$$\mathrm{MV}_{\mathbb{Z}^n}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) = \mathrm{vol}_{\mathbb{Z}}\langle \Delta_0, u_i\rangle.$$

Therefore, the bound given by Corollary 6.8 is

$$\mathrm{h}_{\overline{D}_0^{\mathrm{can}}}(p) \leq \Big(\sum_{i=1}^{n} \mathrm{vol}_{\mathbb{Z}}\langle \Delta_0, u_i\rangle\Big) \log(\alpha+1).$$

Example 1.2 in the introduction consists of the particular cases corresponding to the polytopes $\Delta_0 = \Delta^n$, the standard simplex of $\mathbb{R}^n$, and $\Delta_0 = \mathrm{conv}(0, m_1, \dots, m_n)$.

In the following example, we exhibit a situation where the difference between the bounds given by the results in §6 is noticeable. Recall that passing from Theorem 6.5 to Corollary 6.8 amounts to replacing the local roof functions by constant functions on the polytope bounding them from above. Hence, to maximize the discrepancy between these two concave functions, we look for local roof functions that are tent-shaped, which is the situation where the difference between the mean value and the maximum value of these functions is the greatest possible.

Example 7.3. Let $\alpha \geq 1$ be an integer and consider the system of Laurent polynomials

$$f_i = x_i - \alpha \in \mathbb{Q}[x_1^{\pm1}, \dots, x_n^{\pm1}], \quad i = 1, \dots, n,$$

Its zero set in $\mathbb{G}_{\mathrm{m},\mathbb{Q}}^n$ is the rational point $p = (\alpha, \dots, \alpha) \in (\mathbb{Q}^{\times})^n$. Let $X = \mathbb{P}^n_{\mathbb{Q}}$ and let $\overline{E}^{\mathrm{can}}$ be the divisor of the hyperplane at infinity equipped with the canonical metric. Then the height of p with respect to $\overline{E}^{\mathrm{can}}$ is

$$\mathrm{h}_{\overline{E}^{\mathrm{can}}}(p) = \log(\alpha).$$

Next we compare the value of this height with the bound given by Theorem 6.5. Since the explicit computation of the mixed integrals appearing in this bound is somewhat involved, instead of giving its exact value we are going to approximate them with an upper bound that is easier to compute.

The polytope associated to the toric Cartier divisor E is $\Delta_0 = \Delta^n$, the standard simplex of $\mathbb{R}^n$. For each $v \in \mathfrak{M}_{\mathbb{Q}}$, the v-adic roof function $\vartheta_{0,v}$ of $\overline{E}^{\mathrm{can}}$ is the zero function on this simplex.

For each $i = 1, \dots, n$, let $\Delta_i = \mathrm{N}(f_i) \subset \mathbb{R}^n$ be the Newton polytope of f_i, which coincides with the segment $\overline{0\,e_i}$. For $v \in \mathfrak{M}_{\mathbb{Q}}$, let $\vartheta_{i,v}$ be the v-adic roof function associated to f_i (Definition 6.3). This function is given, for $t\,e_i \in \Delta_i = \overline{0\,e_i}$, by

$$\vartheta_{i,\infty}(t\,e_i) = \begin{cases} (1-t)\log(\alpha) - t\log t - (1-t)\log(1-t) & \text{if } v = \infty, \\ (1-t)\log|\alpha|_v & \text{if } v \neq \infty. \end{cases}$$

For the Archimedean place, the v-adic roof functions are nonnegative, and so their mixed integral can be expressed as a mixed volume

$$\mathrm{MI}_{\mathbb{Z}^n}(\vartheta_{0,\infty}, \dots, \vartheta_{n,\infty}) = \mathrm{MV}_{\mathbb{Z}^{n+1}}(\widetilde{\Delta}_0, \dots, \widetilde{\Delta}_n), \tag{7.7}$$

with $\widetilde{\Delta}_i = \mathrm{conv}\big(\mathrm{graph}(\vartheta_{i,\infty}), \Delta_i \times \{0\}\big) \subset \mathbb{R}^n \times \mathbb{R}$. Consider the concave function $\vartheta : \Delta^n \to \mathbb{R}$ defined by

$$\boldsymbol{x} = (x_1, \dots, x_n) \longmapsto \log(2) + \log(\alpha)\Big(1 - \sum_{i=1}^{n} x_i\Big),$$

and set $\widetilde{\Delta} = \operatorname{conv}\big(\operatorname{graph}(\vartheta), \Delta^n \times \{0\}\big) \subset \mathbb{R}^n \times \mathbb{R}$. Notice that $\vartheta_{i,\infty} \le \vartheta$ on Δ_i, and so $\widetilde{\Delta}_i \subset \widetilde{\Delta}$, $i = 0, \dots, n$. By the monotony of the mixed volume,

$$\begin{aligned} \text{(7.8)} \quad \mathrm{MV}_{\mathbb{Z}^{n+1}}(\widetilde{\Delta}_0, \dots, \widetilde{\Delta}_n) &\le \mathrm{MV}_{\mathbb{Z}^{n+1}}(\widetilde{\Delta}, \dots, \widetilde{\Delta}) = (n+1)! \int_{\Delta^n} \vartheta \, \mathrm{d}\boldsymbol{x} \\ &= (n+1)! \Big(\log(2) \operatorname{vol}(\Delta^n) + \log(\alpha) \int_{\Delta^n} \sum_{i=1}^n x_i \, \mathrm{d}\boldsymbol{x} \Big) = (n+1)\log(2) + \log(\alpha). \end{aligned}$$

When v is non-Archimedean, we have that $|\alpha|_v \le 1$ because α is an integer. Hence $\vartheta_{i,v} \le 0$, and so the mixed integral of these concave functions is nonpositive. Theorem 6.5 together with (7.7) and (7.8) gives the upper bound

$$\mathrm{h}_{\overline{E}^{\mathrm{can}}}(p) \le (n+1)\log(2) + \log(\alpha).$$

To conclude the example, we compute the bound given by Corollary 6.8. For $i = 1, \dots, n$, we have that $\ell(f_i) = \log(\alpha + 1)$ and $\mathrm{MV}_{\mathbb{Z}^n}(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) = 1$. Hence, this bound reduces to

$$\mathrm{h}_{\overline{E}^{\mathrm{can}}}(p) \le n \log(\alpha + 1),$$

concluding the study of this example.

As an application of our results, we bound the size of the coefficients of the $\boldsymbol{u}$-resultant of the direct image under an equivariant map of the 0-cycle defined by a family of Laurent polynomials. As in the previous sections, let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $M \simeq \mathbb{Z}^n$ a lattice.

Definition 7.4. Let $W \in Z_0(\mathbb{P}^r_{\mathbb{K}})$ be a 0-cycle of a projective space over $\mathbb{K}$ and $\boldsymbol{u} = (u_0, \dots, u_r)$ a set of $r+1$ variables. Write $W_{\overline{\mathbb{K}}} = \sum_{\boldsymbol{q}} \mu_{\boldsymbol{q}} \, \boldsymbol{q} \in Z_0\big(\mathbb{P}^r_{\overline{\mathbb{K}}}\big)$ for the 0-cycle obtained from W by the base change $\mathbb{K} \hookrightarrow \overline{\mathbb{K}}$. The *$\boldsymbol{u}$-resultant* (or *Chow form*) of W is defined as

$$\operatorname{Res}(W) = \prod_{\boldsymbol{q}} (q_0 u_0 + \dots + q_r u_r)^{\mu_{\boldsymbol{q}}} \in \mathbb{K}(\boldsymbol{u})^\times,$$

the product being over the points $\boldsymbol{q} = (q_0 : \dots : q_r) \in \mathbb{P}^r_{\mathbb{K}}(\overline{\mathbb{K}})$ in the support of $W_{\overline{\mathbb{K}}}$. It is well-defined up to a factor in $\mathbb{K}^\times$.

The length of a Laurent polynomial (Definition 6.6) is invariant under adelic field extensions and multiplication by scalars. It is also submultiplicative, in the sense that it satisfies the inequality

$$\ell(fg) \le \ell(f) + \ell(g),$$

for $f, g \in \mathbb{K}[M]$. The following result corresponds to Theorem 1.3 in the introduction.

Theorem 7.5. *Let $f_1, \dots, f_n \in \mathbb{K}[M]$, $\boldsymbol{m}_0 \in M^{r+1}$ and $\boldsymbol{\alpha}_0 \in (\mathbb{K}^\times)^{r+1}$ with $r \ge 0$. Set $\Delta_0 = \operatorname{conv}(m_{0,0}, \dots, m_{0,r}) \subset M_{\mathbb{R}}$ and let $\varphi \colon \mathbb{T}_M \to \mathbb{P}^r_{\mathbb{K}}$ be the monomial map associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (5.3). For $i = 1, \dots, n$, let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of f_i, and $\boldsymbol{\alpha}_i$ the vector of nonzero coefficients of f_i. Then*

$$\ell(\operatorname{Res}(\varphi_* Z(f_1, \dots, f_n))) \le \sum_{i=0}^n \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) \, \ell(\boldsymbol{\alpha}_i).$$

Proof. Write $Z(f_1,\dots,f_n)_{\overline{\mathbb{K}}} = \sum_p \mu_p\, p$, the sum being over the points $p \in \mathbb{T}_M(\overline{\mathbb{K}})$. Since the length is invariant under adelic field extensions and submultiplicative, we deduce that

$$\ell(\mathrm{Res}(\varphi_* Z(f_1,\dots,f_n))) \le \sum_p \mu_p\, \ell\big(\alpha_{0,0}\chi^{m_{0,0}}(p)\, u_0 + \dots + \alpha_{0,r}\chi^{m_{0,r}}(p)\, u_r\big). \tag{7.9}$$

Let X be a proper toric variety over $\mathbb{K}$ defined by a fan that is compatible with Δ_i, $i = 0,\dots,n$, and let $\overline{D}_0$ be the toric metrized divisor on X associated to $\boldsymbol{m}_0$ and $\boldsymbol{\alpha}_0$ as in (6.4). Given a point $p \in \mathbb{T}_M(\overline{\mathbb{K}})$, we deduce from (6.5) that

$$\ell\big(\alpha_{0,0}\chi^{m_{0,0}}(p)\, u_0 + \dots + \alpha_{0,r}\chi^{m_{0,r}}(p)\, u_r\big) = \mathrm{h}_{\overline{D}_0}(p). \tag{7.10}$$

By Proposition 6.2, the toric metrized divisor is semipositive and generated by small sections. In particular, it is nef. Similarly as in (6.9), we also get from Proposition 6.2 that the v-adic roof functions of $\overline{D}_0$ satisfy $\sum_{v\in\mathfrak{M}} n_v \max \vartheta_{0,v} = \ell(\boldsymbol{\alpha}_0)$. Hence, Corollary 6.8 implies that

$$\sum_p \mu_p\, \mathrm{h}_{\overline{D}_0}(p) \le \sum_{i=0}^{n} \ell(\boldsymbol{\alpha}_i)\, \mathrm{MV}(\Delta_0,\dots,\Delta_{i-1},\Delta_{i+1},\dots,\Delta_n). \tag{7.11}$$

The statement follows then from (7.9), (7.10) and (7.11). □

References

[AW45] E. Artin and G. Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bull. Amer. Math. Soc. **51** (1945), 469–492.

[Ber75] D. N. Bernstein, *The number of roots of a system of equations*, Funkcional. Anal. i Priložen. **9** (1975), 1–4, English translation: Functional Anal. Appl. **9** (1975), 183–185.

[BG06] E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Math. Monogr., vol. 4, Cambridge Univ. Press, 2006.

[BMPS16] J. I. Burgos Gil, A. Moriwaki, P. Philippon, and M. Sombra, *Arithmetic positivity on toric varieties*, J. Algebraic Geom. **25** (2016), 201–272.

[Bou64] N. Bourbaki, *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations*, Hermann, 1964.

[Bou70] ———, *Éléments de mathématique. Algèbre. Chapitres 1 à 3*, Hermann, 1970.

[BPRS15] J. I. Burgos Gil, P. Philippon, J. Rivera-Letelier, and M. Sombra, *The distribution of Galois orbits of points of small height in toric varieties*, e-print arXiv:1509.01011v1, 2015.

[BPS14] J. I. Burgos Gil, P. Philippon, and M. Sombra, *Arithmetic geometry of toric varieties. Metrics, measures and heights*, Astérisque, vol. 360, Soc. Math. France, 2014.

[BPS15] ———, *Successive minima of toric height functions*, Ann. Inst. Fourier (Grenoble) **65** (2015), 2145–2197.

[Cas86] J. W. S. Cassels, *Local fields*, London Math. Soc. Stud. Texts, vol. 3, Cambridge Univ. Press, 1986.

[Cha06] A. Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235.

[Ful84] W. Fulton, *Intersection theory*, Ergeb. Math. Grenzgeb. (3), vol. 2, Springer-Verlag, 1984.

[Ful93] ———, *Introduction to toric varieties*, Ann. of Math. Stud., vol. 131, Princeton Univ. Press, 1993.

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Math. Theory Appl., Birkhäuser, 1994.

[Gub97] W. Gubler, *Heights of subvarieties over M-fields*, Arithmetic geometry (Cortona, 1994), Sympos. Math., XXXVII, Cambridge Univ. Press, 1997, pp. 190–227.

[HB84] D. R. Heath-Brown, *Diophantine approximation with square-free numbers*, Math. Z. **187** (1984), 335–344.

[Kuš76] A. G. Kušnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.

[Lan02] S. Lang, *Algebra. Revised third edition*, Graduate Texts in Math., vol. 121, Springer-Verlag, 2002.

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergeb. Math. Grenzgeb. (3), vol. 48, Springer-Verlag, 2004.

[Mai00] V. Maillot, *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables*, Mém. Soc. Math. France, vol. 80, Soc. Math. France, 2000.

[PS08] P. Philippon and M. Sombra, *A refinement of the Bernštein-Kušnirenko estimate*, Adv. Math. **218** (2008), 1370–1418.

[Sch93] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia Math. Appl., vol. 44, Cambridge Univ. Press, 1993.

[Som05] M. Sombra, *Minimums successifs des variétés toriques projectives*, J. Reine Angew. Math. **586** (2005), 207–233.

[Stu02] B. Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conf. Ser. in Math., vol. 97, Amer. Math. Soc., 2002.

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