# QUANTITATIVE EQUIDISTRIBUTION FOR THE SOLUTIONS OF SYSTEMS OF SPARSE POLYNOMIAL EQUATIONS

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Abstract. For a system of Laurent polynomials $f_1, \dots, f_n \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ whose coefficients are not too big with respect to its directional resultants, we show that the solutions in the algebraic torus $(\mathbb{C}^\times)^n$ of the system of equations $f_1 = \cdots = f_n = 0$, are approximately equidistributed near the unit polycircle. This generalizes to the multivariate case a classical result due to Erdös and Turán on the distribution of the arguments of the roots of a univariate polynomial.

We apply this result to bound the number of real roots of a system of Laurent polynomials, and to study the asymptotic distribution of the roots of systems of Laurent polynomials over $\mathbb{Z}$ and of random systems of Laurent polynomials over $\mathbb{C}$.

## 1. Introduction and statement of results

A celebrated result due to Erdös and Turán says that, for a univariate polynomial over $\mathbb{C}$ whose middle coefficients are not too big with respect to its extremal coefficients, the arguments of its roots are approximately equidistributed [ET50]. Combined with a recent result of Hughes and Nikeghbali [HN08], this shows that the roots of such a polynomial cluster near the unit circle.

We introduce some notation to make this result precise. Let $Z$ be an effective cycle of $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ of dimension 0, that is, a formal finite sum

$$Z = \sum_{\xi} m_\xi [\xi]$$

with $\xi \in \mathbb{C}^\times$ and $m_\xi \in \mathbb{N}$ with $m_\xi = 0$ for all but finitely many $\xi$, as in [Ful98, §1.3]. The degree of $Z$, denoted $\deg(Z)$, is defined as the sum of its multiplicities $m_\xi$. We assume that $Z \neq 0$ or, equivalently, that $\deg(Z) \geq 1$.

For each $-\pi \leq \alpha < \beta \leq \pi$, consider the cycle

$$Z_{\alpha,\beta} = \sum_{\alpha < \arg(\xi) \leq \beta} m_\xi [\xi],$$

where $\arg(\xi)$ denotes the argument of $\xi$. The *angle discrepancy* of $Z$ is defined as

$$\Delta_{\mathrm{ang}}(Z) = \sup_{-\pi \leq \alpha < \beta \leq \pi} \left| \frac{\deg(Z_{\alpha,\beta})}{\deg(Z)} - \frac{\beta - \alpha}{2\pi} \right|.$$

For example, when $Z$ is the zero set of $x^d - 1$ in $\mathbb{C}^\times$, we have that $\Delta_{\mathrm{ang}}(Z) = \frac{1}{d}$.

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For $0<\varepsilon<1$, consider also the cycle

$$Z_\varepsilon=\sum_{1-\varepsilon<|\xi|<(1-\varepsilon)^{-1}} m_\xi[\xi].$$

The *radius discrepancy* of $Z$ with respect to $\varepsilon$ is defined as

$$\Delta_{\mathrm{rad}}(Z,\varepsilon)=1-\frac{\deg(Z_\varepsilon)}{\deg(Z)}.$$

For example, when $Z$ is the zero set of $x^d-1$ in $\mathbb{C}^\times$, we have that $\Delta_{\mathrm{rad}}(Z,\varepsilon)=0$ for all $\varepsilon$.

For a polynomial $f\in\mathbb{C}[x]\setminus\{0\}$, we denote by $Z(f)$ the 0-dimensional effective cycle of $\mathbb{C}^\times$ defined by its roots and their corresponding multiplicities. We also set $\|f\|_{\sup}=\sup_{|z|=1}|f(z)|$.

**Theorem 1.1.** *Let $f=a_0+\dots+a_dx^d\in\mathbb{C}[x]$ with $d\geq 1$ and $a_0a_d\neq 0$, and $0<\varepsilon<1$. Then*

$$\Delta_{\mathrm{ang}}(Z(f))\leq c\sqrt{\frac{1}{d}\log\Big(\frac{\|f\|_{\sup}}{\sqrt{|a_0a_d|}}\Big)},\quad \Delta_{\mathrm{rad}}(Z(f),\varepsilon)\leq\frac{2}{\varepsilon d}\log\Big(\frac{\|f\|_{\sup}}{\sqrt{|a_0a_d|}}\Big),$$

*with $c=\sqrt{2\pi/G}=2.5619\dots$, where $G=\sum_{m=0}^\infty\frac{(-1)^m}{(2m+1)^2}=0.915965594\dots$ is Catalan's constant.*

The more interesting (and hardest) part is the bound for the angle discrepancy. The original Erdös-Turán result states [ET50]

$$\Delta_{\mathrm{ang}}(Z(f))\leq 16\sqrt{\frac{1}{d}\log\Big(\frac{\sum_j|a_j|}{\sqrt{|a_0a_d|}}\Big)}.$$

A few years after that paper, Ganelius [Gan54] replaced the $\ell^1$-norm $\sum_j|a_j|$ by the smaller quantity $\|f\|_{\sup}$ and improved the value of the constant to $c\leq\sqrt{2\pi/G}$. On the other hand, Amoroso and Mignotte [AM96] showed that the optimal value of $c$ cannot be smaller than $\sqrt{2}$. The bound for the radius discrepancy is due to Hughes and Nikeghbali [HN08].

Here, we study the distribution of the solutions of a system of multivariate polynomial equations in the algebraic torus $(\mathbb{C}^\times)^n$. For instance, consider the following system of bivariate polynomials:

$$f_1=x_1^{13}+x_1x_2^{12}+x_2^{13}+1,\quad f_2=x_1^{12}x_2-x_2^{13}-x_1x_2+1\quad\in\mathbb{C}[x_1,x_2]. \tag{1.2}$$

These are polynomials with moderate degree and small integer coefficients. By direct computation, we can verify that the solutions in $(\mathbb{C}^\times)^2$ of the system of equations $f_1=f_2=0$ are aproximately equidistributed near the unit polycircle $S^1\times S^1$ (Figure 1). This example and others of the same kind suggest that Theorem 1.1 has an extension to higher dimensions.

The study of the distribution of the solutions of a system of multivariate polynomial equations has been addressed from different perspectives. For instance, Khovanskii's theorem on complex fewnomials [Kho91, §3.13, Theorem 2] gives an estimate for the distribution of the arguments of these solutions in terms of the number of monomials and the Newton polytopes of the input system. There are also several interesting results by Shiffman, Zelditch and Bloom on the asymptotic distribution of the solutions

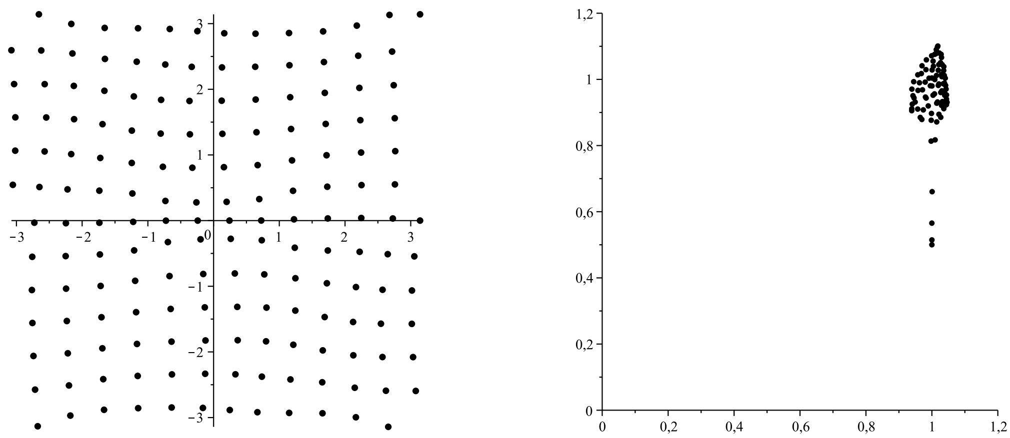


FIGURE 1. Angle and radius distribution of the zeros of the system (1.2)

of a random system of polynomial equations, see for instance [SZ04, BS07] and the references therein.

Our purpose in this text is to obtain an extension of Theorem 1.1 to systems of Laurent polynomials with a given support. For $i = 1, \dots, n$, let $\mathcal{A}_i$ be a non-empty finite subset of $\mathbb{Z}^n$ and $Q_i = \operatorname{conv}(\mathcal{A}_i) \subset \mathbb{R}^n$ its convex hull. Set $D = \mathrm{MV}_{\mathbb{R}^n}(Q_1, \dots, Q_n)$ for the mixed volume of these lattice polytopes, and assume that $D \geq 1$. For each $i$, let $f_i$ be a Laurent polynomial with support contained in $\mathcal{A}_i$, that is,

$$f_i = \sum_{\boldsymbol{a} \in \mathcal{A}_i} \alpha_{i,\boldsymbol{a}}\, \boldsymbol{x}^{\boldsymbol{a}} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

with $\alpha_{i,\boldsymbol{a}} \in \mathbb{C}$ and $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \dots x_n^{a_n}$ for each $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathcal{A}_i$. We write $\boldsymbol{f} = (f_1, \dots, f_n)$ for short. We assume that, for all vectors $\boldsymbol{v} \in \mathbb{Z}^n$, the directional resultant $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \dots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}})$ (Definition 3.4) is nonzero. This condition holds for a generic choice of $\boldsymbol{f}$ in the space of coefficients and, by Bernstein's theorem [Ber75, Theorem B], it implies that all the solutions of $f_1 = \dots = f_n = 0$ are isolated and that their number, counted with multiplicities, is equal to $D$.

For a vector $\boldsymbol{w} \in S^{n-1}$ in the unit sphere of $\mathbb{R}^n$, we denote by $\boldsymbol{w}^{\perp} \subset \mathbb{R}^n$ its orthogonal subspace and by $\pi_{\boldsymbol{w}} \colon \mathbb{R}^n \to \boldsymbol{w}^{\perp}$ the corresponding orthogonal projection. We denote by $\mathrm{MV}_{\boldsymbol{w}^{\perp}}$ the mixed volume of convex bodies of $\boldsymbol{w}^{\perp}$ induced by the Euclidean measure on $\boldsymbol{w}^{\perp}$ and, for $i = 1, \dots, n$, we set

$$D_{\boldsymbol{w},i} = \mathrm{MV}_{\boldsymbol{w}^{\perp}}\left(\pi_{\boldsymbol{w}}(Q_1), \dots, \pi_{\boldsymbol{w}}(Q_{i-1}), \pi_{\boldsymbol{w}}(Q_{i+1}), \dots, \pi_{\boldsymbol{w}}(Q_n)\right).$$

We then define the *Erdös-Turán size* of $\boldsymbol{f}$ as

$$\eta(\boldsymbol{f}) = \frac{1}{D} \sup_{\boldsymbol{w} \in S^{n-1}} \log\left( \frac{\prod_{i=1}^{n} \|f_i\|_{\sup}^{D_{\boldsymbol{w},i}}}{\prod_{\boldsymbol{v}} |\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \dots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}})|^{\frac{|\langle \boldsymbol{v}, \boldsymbol{w}\rangle|}{2}}} \right), \tag{1.3}$$

where the second product is over all primitive vectors $\boldsymbol{v} \in \mathbb{Z}^n$ that is, vectors whose coordinates do not have a non-trivial common factor, and $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^n$. This product is finite because $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\dots,f_n^{\boldsymbol{v}}) \neq 1$ only if $\boldsymbol{v}$ is an inner normal to a facet of the Minkowski sum $Q_1 + \cdots + Q_n$.

The Erdös-Turán size is a generalization to the multivariate case of the quantity $\frac{1}{d}\log\Big(\frac{\|f\|_{\sup}}{\sqrt{|a_0 a_d|}}\Big)$ that appears in Theorem 1.1 since, for $n = 1$, it is easily checked that $\eta(\boldsymbol{f})$ is exactly the preceding quantity (Proposition 3.14).

Let $Z(\boldsymbol{f})$ denote the 0-dimensional effective cycle of $(\mathbb{C}^\times)^n$ defined by the roots of $\boldsymbol{f}$ and their multiplicities. The angle and radius discrepancies of cycles of $(\mathbb{C}^\times)^n$ are the obvious generalization of those for the univariate case (see Definition 2.1).

Our main result is the following:

**Theorem 1.4.** *For $n \geq 2$, let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be non-empty finite subsets of $\mathbb{Z}^n$, set $Q_i = \mathrm{conv}(\mathcal{A}_i)$ and assume that $\mathrm{MV}_{\mathbb{R}^n}(Q_1,\dots,Q_n) \geq 1$. Let $f_1,\dots,f_n \in \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ with $\mathrm{supp}(f_i) \subset \mathcal{A}_i$ and such that $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\dots,f_n^{\boldsymbol{v}}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Then*

$$\Delta_{\mathrm{ang}}(Z(\boldsymbol{f})) \leq 66\, n\, 2^n (18 + \log^+(\eta(\boldsymbol{f})^{-1}))^{\frac{2}{3}(n-1)} \eta(\boldsymbol{f})^{\frac{1}{3}}$$

*with $\log^+(x) = \log(\max(1,x))$ for $x > 0$. Also, for $0 < \varepsilon < 1$,*

$$\Delta_{\mathrm{rad}}(Z(\boldsymbol{f}), \varepsilon) \leq \frac{2n}{\varepsilon}\eta(\boldsymbol{f}).$$

Theorem 1.1 shows that these bounds for the angle and the radius discrepancy also hold in the case $n = 1$. By analogy with the one-dimensional case, it is natural to ask if, in the setting of our result, a stronger inequality of the form

$$\Delta_{\mathrm{ang}}(Z(\boldsymbol{f})) \leq c(n)\, \eta(\boldsymbol{f})^{\frac{1}{2}}$$

holds, with $c(n) > 0$ not depending on $\boldsymbol{f}$. It would be interesting to settle this question.

Theorem 1.4 has several consequences. For instance, we can derive from it a bound for the number of positive real solutions of a system of polynomial equations, in terms of its Erdös-Turán size. For a cycle $Z = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}[\boldsymbol{\xi}]$ of $(\mathbb{C}^\times)^n$, set

$$Z_+ = \sum_{\boldsymbol{\xi} \in (\mathbb{R}_{>0})^n} m_{\boldsymbol{\xi}}[\boldsymbol{\xi}].$$

The following statement follows immediately from Theorem 1.4 and the definition of the angle discrepancy.

**Corollary 1.5.** *Let notation be as in Theorem 1.4. Then*

$$\deg(Z(\boldsymbol{f})_+) \leq 66\, n\, 2^n (18 + \log^+(\eta(\boldsymbol{f})^{-1}))^{\frac{2}{3}(n-1)} \eta(\boldsymbol{f})^{\frac{1}{3}} \deg(Z(\boldsymbol{f})).$$

We can also apply our result to study the asymptotic distribution of the roots of a sequence of systems of polynomials over $\mathbb{Z}$ with growing supports and whose coefficients are not too big. To be more precise, let $Q_i$, $i = 1,\dots,n$, be lattice polytopes in $\mathbb{R}^n$ such that $\mathrm{MV}_{\mathbb{R}^n}(Q_1,\dots,Q_n) \geq 1$. For each integer $\kappa \geq 1$ and $i = 1,\dots,n$, consider the finite subset of $\mathbb{Z}^n$ given by

$$\mathcal{A}_{\kappa,i} = \kappa Q_i \cap \mathbb{Z}^n. \tag{1.6}$$

For a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we denote by $\mathrm{supp}(f)$ its support, defined as the subset of $\mathbb{Z}^n$ of its exponent vectors. We also set

$$\|f\|_{\sup} = \sup_{|w_1|=1,\dots,|w_n|=1} |f(w_1, \dots, w_n)|.$$

For a nonzero cycle $Z = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} [\boldsymbol{\xi}]$ of $(\mathbb{C}^\times)^n$, we consider the discrete probability measure on $(\mathbb{C}^\times)^n$ defined by

$$\delta_Z = \frac{1}{\deg(Z)} \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \delta_{\boldsymbol{\xi}},$$

where $\delta_{\boldsymbol{\xi}}$ is the Dirac measure supported on the point $\boldsymbol{\xi}$. Let $\nu_{\mathrm{Haar}}$ be the measure on $(\mathbb{C}^\times)^n$ supported on $(S^1)^n$ and whose restriction to this polycircle coincides with its Haar measure of total mass 1.

Recall that a sequence of measures $(\nu_\kappa)_{\kappa \in \mathbb{N}}$ on $(\mathbb{C}^\times)^n$ converges weakly to $\nu_{\mathrm{Haar}}$ if, for every continuous function with compact support $h\colon (\mathbb{C}^\times)^n \to \mathbb{R}$, it holds

$$\lim_{\kappa \to \infty} \int_{(\mathbb{C}^\times)^n} h \, \mathrm{d}\nu_\kappa = \int_{(\mathbb{C}^\times)^n} h \, \mathrm{d}\nu_{\mathrm{Haar}}.$$

If this is the case, we write $\lim_{\kappa \to \infty} \nu_\kappa = \nu_{\mathrm{Haar}}$.

**Theorem 1.7.** *For $\kappa \geq 1$, let $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ be a family of Laurent polynomials in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that $\mathrm{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $\log \|f_{\kappa,i}\|_{\sup} = o(\kappa)$, and $\mathrm{Res}_{\mathcal{A}_{\kappa,1}^{\boldsymbol{v}}, \dots, \mathcal{A}_{\kappa,n}^{\boldsymbol{v}}}(f_{\kappa,1}^{\boldsymbol{v}}, \dots, f_{\kappa,n}^{\boldsymbol{v}}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Then*

$$\lim_{\kappa \to \infty} \delta_{Z(\boldsymbol{f}_\kappa)} = \nu_{\mathrm{Haar}}.$$

This result admits a quantitative version giving information on the rate of convergence, which we state in Proposition 4.2. Theorem 1.7 is related to Bilu's equidistribution theorem for the Galois orbit of algebraic points in $(\mathbb{C}^\times)^n$ of small height [Bil97] which, at least for $n = 1$, also admits quantitative versions [Pet05, FRL06].

We can also apply Theorem 1.4 to study the distribution of the roots of a random system of Laurent polynomials over $\mathbb{C}$. We will show that, under some mild conditions and without assuming any independence or equidistribution condition on the coefficients of the system, these roots tend to cluster uniformly near $(S^1)^n$.

To state this result, let us keep notation as above and set $\boldsymbol{\mathcal{A}}_\kappa = (\mathcal{A}_{\kappa,1}, \dots, \mathcal{A}_{\kappa,n})$ with $\mathcal{A}_{\kappa,i}$ as in (1.6). Each point of the projective space $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ can be identified with a system $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ of Laurent polynomials such that $\mathrm{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $i = 1, \dots, n$, modulo a multiplicative scalar. The associated cycle $Z(\boldsymbol{f}_\kappa)$ is well-defined, since it does not depend on this multiplicative scalar.

Let $\mu_\kappa$ be the normalized Fubini-Study measure on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ of total mass 1, and $\lambda_\kappa$ a probability density function on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ (see §4 for details). Let $\boldsymbol{f}_\kappa$ be a random system of Laurent polynomials with $\mathrm{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $i = 1, \dots, n$, distributed according to the probability law given by $\lambda_\kappa$ with respect to $\mu_\kappa$. The *expected zero density measure* of $\boldsymbol{f}_\kappa$ is the measure on $(\mathbb{C}^\times)^n$ defined, for a Borel subset $U$, as

$$\mathbb{E}(Z(\boldsymbol{f}_\kappa); \lambda_\kappa)(U) = \int_{\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})} \deg(Z(\boldsymbol{f}_\kappa)|_U) \, \lambda_\kappa(\boldsymbol{f}_\kappa) \, \mathrm{d}\mu_\kappa,$$

where $Z(\boldsymbol{f}_\kappa)|_U$ denotes the cycle $\sum_{\boldsymbol{\xi} \in V(\boldsymbol{f}_\kappa)_0 \cap U} m_{\boldsymbol{\xi}} [\boldsymbol{\xi}]$.

**Theorem 1.8.** *For $\kappa \geq 1$, let $\lambda_\kappa$ be a probability density function on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ with respect to the measure $\mu_\kappa$, and $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ a random system of Laurent polynomials with* $\operatorname{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $i = 1, \dots, n$, *distributed according to the probability law given by $\lambda_\kappa$. Assume that the sequence $(\lambda_\kappa)_{\kappa \geq 1}$ is uniformly bounded. Then*

$$\lim_{\kappa \to \infty} \frac{\mathbb{E}(Z(\boldsymbol{f}_\kappa); \lambda_\kappa)}{\kappa^n \operatorname{MV}_{\mathbb{R}^n}(Q_1, \dots, Q_n)} = \nu_{\mathrm{Haar}}.$$

As an application, consider a random system of Laurent polynomials $\boldsymbol{f}_\kappa$ with $\operatorname{supp}(f_{\kappa,i}) \subset \kappa Q_i$ whose coefficients are independent complex Gaussian random variables with mean 0 and variance 1. The random cycle $Z(\boldsymbol{f}_\kappa)$ might be described by the uniform distribution on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ (see Example 4.14 for details). Then, Theorem 1.8 implies that the roots of $\boldsymbol{f}_\kappa$ converge weakly to the equidistribution on $(S^1)^n$, and we recover in this way a result of Bloom and Shiffman [BS07, Example 3.5].

Our strategy for proving Theorem 1.4 consists of reducing to the univariate case. In §2, we consider the problem of studying the angle and radius discrepancies of an arbitrary 0-dimensional effective cycle $Z$ in $(\mathbb{C}^\times)^n$ in terms of the angle and radius discrepancies of its direct images under all monomial projections of $(\mathbb{C}^\times)^n$ onto $\mathbb{C}^\times$. By applying a tomography process based on Fourier analysis, we show that the distribution of $Z$ can be controlled in terms of the distribution of its projections (Theorem 2.3). In §3, we consider cycles defined by a system of Laurent polynomials with given support and we compute their direct image under monomial projections, in terms of sparse resultants. Theorem 1.4 then follows by applying Erdös-Turán's theorem combined with Theorem 2.3, and the basic properties of the sparse resultant.

In § 4, we study the asymptotic distribution of the roots of a sequence of systems of Laurent polynomials over $\mathbb{Z}$ and of random systems of Laurent polynomials over $\mathbb{C}$. In both situations, the key step consists of bounding from below the size of the relevant directional resultants. In the case of systems over $\mathbb{Z}$, this is trivial since these directional resultants are nonzero integer numbers. In the case of random systems over $\mathbb{C}$, the result follows from an estimate of the volume of a tube around an algebraic variety due to Beltrán and Pardo [BP07].

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## 2. Angle and radius distribution in the multivariate case

In this section, we show that the angle and radius discrepancies of an effective 0-dimensional cycle in the algebraic torus $(\mathbb{C}^\times)^n$ can be bounded in terms of the angle discrepancy of its image under monomial maps from $(\mathbb{C}^\times)^n$ to $\mathbb{C}^\times$.

Let $Z$ be a nonzero effective 0-dimensional cycle of $(\mathbb{C}^\times)^n$, which we write as a finite sum

$$Z = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} [\boldsymbol{\xi}]$$

with $m_{\boldsymbol{\xi}} \in \mathbb{N}$ and $\boldsymbol{\xi} \in (\mathbb{C}^\times)^n$. The *support* of $Z$ is the finite subset of $(\mathbb{C}^\times)^n$ defined as $|Z| = \{\boldsymbol{\xi} \mid m_{\boldsymbol{\xi}} \geq 1\}$, and the *degree* of $Z$ is the positive number $\deg(Z) = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}$.

**Definition 2.1.** Let $Z = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}[\boldsymbol{\xi}]$ be a nonzero effective 0-dimensional cycle of $(\mathbb{C}^\times)^n$. For each $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ with $-\pi \leq \alpha_j < \beta_j \leq \pi$, $j = 1, \dots, n$, consider the cycle

$$Z_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{-\alpha_j < \arg(\xi_j) \leq \beta_j} m_{\boldsymbol{\xi}}[\boldsymbol{\xi}].$$

The *angle discrepancy* of $Z$ is defined as

$$\Delta_{\mathrm{ang}}(Z) = \sup_{\boldsymbol{\alpha},\boldsymbol{\beta}} \left| \frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\deg(Z)} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} \right|.$$

Let $0 < \varepsilon < 1$ and consider also the cycle

$$Z_\varepsilon = \sum_{1-\varepsilon < |\xi_j| < (1-\varepsilon)^{-1}} m_{\boldsymbol{\xi}}[\boldsymbol{\xi}],$$

where $\xi_j$ is the $j$-th coordinate of $\boldsymbol{\xi}$. The *radius discrepancy* of $Z$ with respect to $\varepsilon$ is defined as

$$\Delta_{\mathrm{rad}}(Z, \varepsilon) = 1 - \frac{\deg(Z_\varepsilon)}{\deg(Z)}.$$

We have $0 < \Delta_{\mathrm{ang}}(Z) \leq 1$. Observe also that $0 \leq \Delta_{\mathrm{rad}}(Z, \varepsilon) \leq 1$ and $\Delta_{\mathrm{rad}}(Z, \varepsilon) = 0$ for all $\varepsilon$ if and only if $|Z| \subset (S^1)^n$.

For a lattice point $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we denote by $\chi^{\boldsymbol{a}} \colon (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ the associated character, defined as $\chi^{\boldsymbol{a}}(\boldsymbol{\xi}) = \xi_1^{a_1} \dots \xi_n^{a_n}$ for $\boldsymbol{\xi} \in (\mathbb{C}^\times)^n$. The direct image of $Z$ under $\chi^{\boldsymbol{a}}$ is the cycle of $\mathbb{C}^\times$ given by

$$\chi^{\boldsymbol{a}}_*(Z) = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \chi^{\boldsymbol{a}}(\boldsymbol{\xi}).$$

We also set

$$\theta(Z) = \sup_{\boldsymbol{a} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}} \frac{\Delta_{\mathrm{ang}}(\chi^{\boldsymbol{a}}_*(Z))}{\|\boldsymbol{a}\|_2^{\frac{1}{2}}}, \quad \rho(Z, \varepsilon) = \sum_{j=1}^{n} \Delta_{\mathrm{rad}}(\chi^{\boldsymbol{e}_j}_*(Z), \varepsilon), \tag{2.2}$$

where $\boldsymbol{e}_j$ denotes the $j$-th vector in the standard basis of $\mathbb{Z}^n$, and $\|\boldsymbol{a}\|_2$ is the Euclidean norm of the vector $\boldsymbol{a} \in \mathbb{Z}^n$. We have $0 < \theta(Z) \leq 1$ and $0 \leq \rho(Z, \varepsilon) \leq 1$.

**Theorem 2.3.** *Let $Z$ be a nonzero effective 0-dimensional cycle of $(\mathbb{C}^\times)^n$. Then*

$$\Delta_{\mathrm{ang}}(Z) \leq 22n \left(\frac{8}{3}\right)^n (9 - \log(\theta(Z)))^{\frac{2}{3}(n-1)} \, \theta(Z)^{\frac{2}{3}}$$

*and, for $0 < \varepsilon < 1$,*

$$\Delta_{\mathrm{rad}}(Z, \varepsilon) \leq \rho(Z, \varepsilon).$$

The rest of this section is devoted to the proof of this result. Given two vectors $\boldsymbol{u} = (u_1, \dots, u_n), \boldsymbol{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ we write $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{j=1}^n u_j v_j$ for their standard inner product, and for $\boldsymbol{\xi} \in (\mathbb{C}^\times)^n$ we set $\arg(\boldsymbol{\xi}) = (\arg(\xi_1), \dots, \arg(\xi_n)) \in (-\pi, \pi]^n$.

**Lemma 2.4.** *Let $Z$ be a nonzero effective 0-dimensional cycle of $(\mathbb{C}^\times)^n$ and $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Then*

$$\left| \frac{1}{\deg(Z)} \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{a}, \arg(\boldsymbol{\xi}) \rangle} \right| \leq 2\pi \Delta_{\mathrm{ang}}(\chi^{\boldsymbol{a}}_*(Z)).$$

Note that for $\boldsymbol{a}=\mathbf{0}$ we get $\frac{1}{\deg(Z)}\sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}\,\mathrm{e}^{\mathrm{i}\langle \boldsymbol{a},\arg(\boldsymbol{\xi})\rangle}=1$.

*Proof.* Set $D_0=\#|Z|$ and $D=\deg(Z)$ for short. Let $-\pi\le \nu_k<\pi$, $k=1,\dots,D_0$, denote the inner products $\langle \boldsymbol{a},\arg(\boldsymbol{\xi})\rangle$ modulo $2\pi$ for the different points $\boldsymbol{\xi}$ in the support of $Z$, and let $m_k$ denote their corresponding multiplicity. We suppose that these numbers are arranged in increasing order, that is, $\nu_1\le\cdots\le\nu_{D_0}$.

For $-\pi<\nu\le\pi$ set

$$N(\nu)=\sum_{k|\nu_k\le\nu} m_k.$$

We have

$$\int_{-\pi}^{\pi} N(\nu)\,\mathrm{e}^{\mathrm{i}\nu}\,d\nu=\sum_{k=1}^{D_0}\int_{\nu_k}^{\nu_{k+1}}\Big(\sum_{l\le k} m_l\Big)\mathrm{e}^{\mathrm{i}\nu}\,d\nu \tag{2.5}$$

$$=-\mathrm{i}\sum_k\Big(\sum_{l\le k} m_l\Big)\,\mathrm{e}^{\mathrm{i}\nu}\Big|_{\nu_k}^{\nu_{k+1}}=\mathrm{i}\Big(D+\sum_k m_k\,\mathrm{e}^{\mathrm{i}\nu_k}\Big),$$

where we have set $\nu_{D_0+1}=\pi$. On the other hand, an easy calculation shows that

$$\int_{-\pi}^{\pi}\frac{\nu+\pi}{2\pi}\,\mathrm{e}^{\mathrm{i}\nu}\,d\nu=\mathrm{i}. \tag{2.6}$$

Combining (2.5) and (2.6), we deduce that

$$\frac{1}{D}\sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}\,\mathrm{e}^{\mathrm{i}\langle \boldsymbol{a},\arg(\boldsymbol{\xi})\rangle}=\frac{1}{D}\sum_{k=1}^{D_0} m_k\,\mathrm{e}^{\mathrm{i}\nu_k}=\mathrm{i}\int_{-\pi}^{\pi}\Big(\frac{\nu+\pi}{2\pi}-\frac{N(\nu)}{D}\Big)\mathrm{e}^{\mathrm{i}\nu}\,d\nu.$$

Hence

$$\Big|\frac{1}{D}\sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}\,\mathrm{e}^{\mathrm{i}\langle \boldsymbol{a},\arg(\boldsymbol{\xi})\rangle}\Big|\le\int_{-\pi}^{\pi}\Big|\frac{\nu+\pi}{2\pi}-\frac{N(\nu)}{D}\Big|\,d\nu$$

$$=\int_{-\pi}^{\pi}\Big|\frac{\nu+\pi}{2\pi}-\frac{\deg(\chi_*^{\boldsymbol{a}}(Z)_{-\pi,\nu})}{\deg(\chi_*^{\boldsymbol{a}}(Z))}\Big|\,d\nu\le 2\pi\Delta_{\mathrm{ang}}(\chi_*^{\boldsymbol{a}}(Z)),$$

which concludes the proof. □

Let $\alpha,\beta,\tau\in\mathbb{R}$ such that $\alpha\le\beta$ and $\tau>0$. We consider the function $h_{\alpha,\beta,\tau}\colon\mathbb{R}\to\mathbb{R}$ defined, for $x\in\mathbb{R}$, by

$$h_{\alpha,\beta,\tau}(x)=\begin{cases}0 & \text{if } x\le\alpha-\tau,\\ g\big(\frac{x-\alpha+\tau}{\tau}\big) & \text{if}\,\alpha-\tau<x\le\alpha,\\ 1 & \text{if } \alpha<x\le\beta,\\ g\big(\frac{\beta+\tau-x}{\tau}\big) & \text{if } \beta<x\le\beta+\tau,\\ 0 & \text{if } \beta+\tau<x,\end{cases}$$

with $g(x)=-2x^3+3x^2$. Lemma 2.7 below shows that $h_{\alpha,\beta,\tau}$ is an approximation of the characteristic function of the interval $[\alpha,\beta]$.

For $m\in\mathbb{N}$, we denote by $\mathcal{C}^m(\mathbb{R})$ the space of functions $f\colon\mathbb{R}\to\mathbb{R}$ having $m$ continuous derivatives.

**Lemma 2.7.** *Let $\alpha,\beta,\tau\in\mathbb{R}$ such that $\alpha\le\beta$ and $\tau>0$. Then*

(1) $h_{\alpha,\beta,\tau}\in\mathcal{C}^1(\mathbb{R})$;

(2) $h_{\alpha,\beta,\tau}(x) = 1$ *for* $x \in [\alpha, \beta]$, $h_{\alpha,\beta,\tau}(x) = 0$ *for* $x \in (-\infty, \alpha - \tau] \cup [\beta + \tau, \infty)$, *and* $0 \le h_{\alpha,\beta,\tau}(x) \le 1$ *for all* $x \in \mathbb{R}$*;*
(3) $\int_{-\infty}^{\infty} h_{\alpha,\beta,\tau} dx = \beta - \alpha + \tau$ *and, moreover,* $\int_{\alpha-\tau}^{\alpha} h_{\alpha,\beta,\tau} dx = \int_{\beta}^{\beta+\tau} h_{\alpha,\beta,\tau} dx = \frac{\tau}{2}$*;*
(4) $\int_{-\infty}^{\infty} |h'_{\alpha,\beta,\tau}| dx = 2$*;*
(5) $\int_{-\infty}^{\infty} |h''_{\alpha,\beta,\tau}| dx = \frac{6}{\tau}$.

*Proof.* By a direct calculation, we verify that the function $g$ satisfies the following properties:

- $g(x) \ge 0$ for all $x \in [0, 1]$;
- $g(0) = g'(0) = 0,\ g(1) = 1,\ g'(1) = 0$;
- $\int_0^1 g\, dx = \frac{1}{2}$;
- $\int_0^1 |g'|\, dx = 1$;
- $\int_0^1 |g''|\, dx = 3$.

The claim follows easily from these properties and the definition of $h_{\alpha,\beta,\tau}$. □

Suppose furthermore that $\beta - \alpha + 2\tau < 2\pi$. The support of $h_{\alpha,\beta,\tau}$ is then contained in an interval of length bounded by $2\pi$, and so this function can be regarded as a function on $\mathbb{R}/2\pi\mathbb{Z}$. For $a \in \mathbb{Z}$ set $c_a = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_{\alpha,\beta,\tau}(x)\, \mathrm{e}^{-\mathrm{i}ax}\, \mathrm{d}x$, so that its Fourier series is given by

$$\sum_{a \in \mathbb{Z}} c_a\, \mathrm{e}^{\mathrm{i}ax} .$$

**Lemma 2.8.** *Let* $\alpha \le \beta$ *and* $\tau > 0$ *such that* $\beta - \alpha + 2\tau < 2\pi$*. Then* $\sum_{a\in\mathbb{Z}} c_a\, \mathrm{e}^{\mathrm{i}ax}$ *converges absolutely and uniformly on* $\mathbb{R}/2\pi\mathbb{Z}$ *to* $h_{\alpha,\beta,\tau}$*. Moreover,* $c_0 = \frac{\beta-\alpha+\tau}{2\pi}$*, and, for* $a \neq 0$*,*

$$|c_a| \le \min\left\{\frac{1}{\pi\, a}, \frac{3}{\pi\, \tau\, a^2}\right\}. \tag{2.9}$$

*Proof.* Lemma 2.7(1) implies that the series $\sum_{a\in\mathbb{Z}} c_a\, \mathrm{e}^{\mathrm{i}ax}$ converges absolutely and uniformly on $\mathbb{R}/2\pi\mathbb{Z}$ to the function $h_{\alpha,\beta,\tau}$. The computation of $c_0$ follows from Lemma 2.7(3). Integrating by parts, we deduce for $a \in \mathbb{Z} \setminus \{0\}$ that

$$c_a = \frac{1}{-2\pi \mathrm{i}a} \int_{-\pi}^{\pi} h'_{\alpha,\beta,\tau}(x)\, \mathrm{e}^{-\mathrm{i}ax}\, dx = \frac{1}{2\pi(-\mathrm{i}a)^2} \int_{-\pi}^{\pi} h''_{\alpha,\beta,\tau}(x)\, \mathrm{e}^{-\mathrm{i}ax}\, dx.$$

Hence, $|c_a| \le \frac{1}{2\pi a} \int_{-\pi}^{\pi} |h'_{\alpha,\beta,\tau}| dx$ and also $|c_a| \le \frac{1}{2\pi a^2} \int_{-\pi}^{\pi} |h''_{\alpha,\beta,\tau}| dx$. Then (2.9) follows by bounding these integrals with Lemma 2.7(4-5). □

Next, we apply Fourier analysis to control the angle discrepancy of $Z$ in terms of the angle discrepancy of its direct image under monomial projections.

**Lemma 2.10.** *Let* $n \ge 2$, $q \in \mathbb{Z}_{\ge 1}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ *and* $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ *with* $\alpha_j, \beta_j \in \mathbb{R}$ *such that* $-\pi \le \alpha_j < \beta_j < \pi$ *and* $\beta_j - \alpha_j + \frac{2}{q} < 2\pi$*. Then*

$$\left| \frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\deg(Z)} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} \right| \le 2\, n\, \theta(Z) + \frac{3\, n}{2\, \pi\, q} + n \frac{2^{n+3}\sqrt{3}}{\pi^{n-1}} q^{\frac{1}{2}} (9 + \log(q))^{n-1}\, \theta(Z).$$

*Proof.* Set $\tau = \frac{1}{q}$ and $h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}(\boldsymbol{\nu}) = \prod_{j=1}^{n} h_{\alpha_j,\beta_j,\tau}(\nu_j)$ for $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$. Set also $D = \deg(Z)$ and

$$\Sigma_1 = \left| \frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{D} - \frac{1}{D} \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}(\arg(\boldsymbol{\xi})) \right|,$$

$$\Sigma_2 = \left| \frac{1}{D} \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}(\arg(\boldsymbol{\xi})) - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j + \tau}{2\pi} \right|,$$

$$\Sigma_3 = \left| \prod_{j=1}^{n} \frac{\beta_j - \alpha_j + \tau}{2\pi} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} \right|.$$

We will bound each of these quantities. For $\Sigma_1$, we consider the subset of $\mathbb{R}^n$ given by $I_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau} = \prod_{j=1}^{n} [\alpha_j - \tau, \beta_j + \tau] \setminus \prod_{j=1}^{n} [\alpha_j, \beta_j]$. Then

$$\Sigma_1 = \left| \frac{1}{D} \sum_{\arg(\boldsymbol{\xi}) \in I_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}} m_{\boldsymbol{\xi}} h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}(\arg(\boldsymbol{\xi})) \right|.$$

For each $\boldsymbol{\xi}$ such that $\arg(\boldsymbol{\xi}) \in I_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}$, there is $1 \le j \le n$ such that either $\alpha_j - \tau < \arg(\xi_j) \le \alpha_j$ or $\beta_j < \arg(\xi_j) < \beta_j + \tau$. Since $0 \le h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}(\boldsymbol{\nu}) \le 1$ for all $\boldsymbol{\nu}$, we have that $\Sigma_1$ is bounded from above by

$$\frac{1}{D} \sum_{j=1}^{n} \big( \deg(\chi_*^{\boldsymbol{e}_j}(Z)_{\alpha_j - \tau, \alpha_j}) + \deg(\chi_*^{\boldsymbol{e}_j}(Z)_{\beta_j, \beta_j + \tau}) \big).$$

By using the definition of $\Delta_{\mathrm{ang}}$ and Lemma 2.4, we get

$$\frac{1}{D} \deg(\chi_*^{\boldsymbol{e}_j}(Z)_{\alpha_j - \tau, \alpha_j}) \le \Delta_{\mathrm{ang}}(\chi_*^{\boldsymbol{e}_j}(Z)) + \frac{\tau}{2\pi} \le \theta(Z) + \frac{\tau}{2\pi} = \theta(Z) + \frac{1}{2\pi q},$$

and a similar bound holds for $\deg(\chi_*^{\boldsymbol{e}_j}(Z)_{\beta_j, \beta_j + \tau})$. Hence,

$$\Sigma_1 \le 2n\theta(Z) + \frac{n}{\pi q}. \tag{2.11}$$

Now we turn to $\Sigma_2$. Due to the conditions imposed on $\tau$, we can regard $h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}$ as a function on $\mathbb{R}^n / 2\pi\mathbb{Z}^n \simeq (-\pi, \pi]^n$. Let $\sum_{\boldsymbol{a} \in \mathbb{Z}^n} c_{\boldsymbol{a}} \, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{a}, \boldsymbol{\nu} \rangle}$ be its multivariate Fourier series. For $j = 1, \dots, n$, we denote with $\sum_{a_j \in \mathbb{Z}} c_{j,a_j} \, \mathrm{e}^{\mathrm{i} a_j \nu_j}$ the Fourier series of $h_{\alpha_j,\beta_j,\tau}$. Then, for each $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$,

$$c_{\boldsymbol{a}} = \prod_{j=1}^{n} c_{j,a_j}.$$

In particular, $c_{\boldsymbol{0}} = \prod_{j=1}^{n} \frac{\beta_j - \alpha_j + \tau}{2\pi}$. The Fourier series of each $h_{\alpha_j,\beta_j,\tau}$ converges absolutely to this function, and so the same holds for the Fourier series of $h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}$. Hence,

$$h_{\boldsymbol{\alpha},\boldsymbol{\beta},\tau}(\boldsymbol{\nu}) = \sum_{\boldsymbol{a} \in \mathbb{Z}^n} c_{\boldsymbol{a}} \, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{a}, \boldsymbol{\nu} \rangle}$$

for $\boldsymbol{\nu} \in (-\pi, \pi]^n$. Applying Lemma 2.4, we obtain

$$\Sigma_2 = \left| \frac{1}{D} \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \sum_{\boldsymbol{a} \neq \boldsymbol{0}} c_{\boldsymbol{a}} \, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{a}, \arg(\boldsymbol{\xi})\rangle} \right| \leq \sum_{\boldsymbol{a} \neq 0} |c_{\boldsymbol{a}}| \left| \frac{1}{D} \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \, \mathrm{e}^{\mathrm{i}\langle \boldsymbol{a}, \arg(\boldsymbol{\xi})\rangle} \right|$$
$$\leq 2\pi \, \theta(Z) \sum_{\boldsymbol{a} \neq 0} |c_{\boldsymbol{a}}| \|\boldsymbol{a}\|_2^{\frac{1}{2}} \leq 2^{n+1} \pi \, \theta(Z) \sum_{\boldsymbol{a} \geq \boldsymbol{0}} \Big( \sum_{s=1}^{n} \sqrt{a_s} \Big) |c_{\boldsymbol{a}}| \tag{2.12}$$

For $s = 1$,

$$\sum_{\boldsymbol{a} \geq \boldsymbol{0}} \sqrt{a_1} |c_{\boldsymbol{a}}| = \Big( \sum_{a_1 \geq 0} \sqrt{a_1} |c_{1,a_1}| \Big) \prod_{j=2}^{n} \Big( \sum_{a_j \geq 0} |c_{j,a_j}| \Big).$$

Using the bounds in (2.9) we get, for $j = 2, \ldots, n$,

$$\begin{aligned} \sum_{a_j \geq 0} |c_{j,a_j}| &\leq |c_{j,0}| + \sum_{a_j=1}^{3q} \frac{1}{a_j \pi} + \sum_{a_j=3q+1}^{\infty} \frac{3}{\pi \tau a_j^2} \\ &\leq \frac{\beta_j - \alpha_j + \tau}{2\pi} + \Big( 1 + \frac{1}{\pi} \int_1^{3q} \frac{dx}{x} \Big) + \frac{3}{\pi \tau} \int_{3q}^{\infty} \frac{dx}{x^2} \\ &\leq \frac{2\pi}{2\pi} + 1 + \frac{1}{\pi} \log \Big( \frac{3}{\tau} \Big) + \frac{3}{\pi\tau} \frac{\tau}{3} \\ &\leq \frac{1}{\pi} \big( 9 + \log q \big). \end{aligned} \tag{2.13}$$

Similarly, we now bound

$$\begin{aligned} \sum_{a_1 \geq 0} \sqrt{a_1} |c_{1,a_1}| &\leq \sum_{a_1=1}^{3q} \frac{1}{\pi \sqrt{a_1}} + \sum_{a_1=3q+1}^{\infty} \frac{3}{\pi \tau \, a_1 \sqrt{a_1}} \\ &\leq \frac{1}{\pi} \int_0^{3q} x^{-\frac{1}{2}} dx + \frac{3}{\pi\tau} \int_{3q}^{\infty} x^{-\frac{3}{2}} dx \\ &\leq \frac{2}{\pi} \Big( \frac{\tau}{3} \Big)^{-\frac{1}{2}} + \frac{6}{\pi\tau} \Big( \frac{\tau}{3} \Big)^{\frac{1}{2}} \\ &\leq \frac{4\sqrt{3}}{\pi} q^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

It follows from (2.12), (2.13) and (2.14) that

$$\Sigma_2 \leq n \frac{2^{n+3}\sqrt{3}}{\pi^{n-1}} q^{\frac{1}{2}} (9 + \log(q))^{n-1} \, \theta(Z). \tag{2.15}$$

Next we consider $\Sigma_3$. Set $\phi(t) = \prod_{j=1}^{n} \frac{\beta_j - \alpha_j + \tau t}{2\pi}$ for $t \in \mathbb{R}$. There exists $0 < t_0 < 1$ such that $\Sigma_3 = |\phi(1) - \phi(0)| = |\phi'(t_0)|$ and so

$$\Sigma_3 \leq \sup_{0 < t_0 < 1} |\phi'(t_0)| \leq \sum_{j=1}^{n} \frac{\tau}{2\pi} \prod_{\ell \neq j} \frac{\beta_\ell - \alpha_\ell + \tau}{2\pi} \leq n\tau \frac{(2\pi)^{n-1}}{(2\pi)^n} = \frac{n}{2\pi q}.$$

Finally, we collect (2.11), (2.15) and the above inequality to get

$$\begin{aligned}\left|\frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\deg(Z)}-\prod_{j=1}^{n}\frac{\beta_j-\alpha_j}{2\pi}\right| &\leq \Sigma_1+\Sigma_2+\Sigma_3\\ &\leq 2n\theta(Z)+\frac{n}{\pi q}+n\frac{2^{n+3}\sqrt{3}}{\pi^{n-1}}q^{\frac12}(9+\log(q))^{n-1}\,\theta(Z)+\frac{n}{2\pi q}\\ &=2n\theta(Z)+\frac{3}{2}\frac{n}{\pi q}+n\frac{2^{n+3}\sqrt{3}}{\pi^{n-1}}q^{\frac12}(9+\log(q))^{n-1}\,\theta(Z).\end{aligned}$$

□

*Proof of Theorem 2.3.* For the radius discrepancy, we have

$$|Z_\varepsilon|=\bigcap_{j=1}^{n}\Big\{\boldsymbol{\xi}\in|Z|\mid 1-\varepsilon<|\xi_j|<\frac{1}{1-\varepsilon}\Big\}.$$

By taking complements in this equality and considering the corresponding multiplicities, we deduce that

$$\deg(Z)-\deg(Z_\varepsilon)\leq\sum_{j=1}^{n}\deg(\chi_*^{\boldsymbol{e}_j}(Z))-\deg(\chi_*^{\boldsymbol{e}_j}(Z)_\varepsilon).$$

Hence, $\Delta_{\mathrm{rad}}(Z,\varepsilon)\leq\sum_{j=1}^{n}\Delta_{\mathrm{rad}}\big(\chi_*^{\boldsymbol{e}_j}(Z,\varepsilon)\big)=\rho(Z,\varepsilon)$, as stated.

We now consider the bound for the angle discrepancy. For $n=1$, $\Delta_{\mathrm{ang}}(Z)\leq\theta(Z)$ and so the bound in the claim is trivial. Hence, we suppose that $n\geq 2$. Put then $\zeta(Z)=(9-\log(\theta(Z)))^{\frac{2}{3}(n-1)}\,\theta(Z)^{2/3}\in\mathbb{R}_{>0}$ for short and set

$$q=\left\lfloor\frac{9^{\frac23(n-1)}}{\zeta(Z)}\right\rfloor.$$

Suppose also that $q\geq 1$. Then

$$\frac{9^{\frac23(n-1)}}{2\zeta(Z)}<q\leq\frac{9^{\frac23(n-1)}}{\zeta(Z)}\leq\frac{1}{\theta(Z)}. \tag{2.16}$$

Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$ and $\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)$ with $-\pi\leq\alpha_j<\beta_j\leq\pi$. Consider first the case where $\beta_j-\alpha_j\leq\pi$. In particular, $\beta_j-\alpha_j+\frac{2}{q}<2\pi$. Applying Lemma 2.10 and the inequalities (2.16), we deduce that the quantity $\left|\frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\deg(Z)}-\prod_{j=1}^{n}\frac{\beta_j-\alpha_j}{2\pi}\right|$ is bounded from above by

$$2n\frac{\zeta(Z)}{9^{\frac23(n-1)}}+\frac{3\,n}{2\,\pi}\frac{2\,\zeta(Z)}{9^{\frac23(n-1)}}+n\frac{2^{n+3}\sqrt{3}}{\pi^{n-1}}\left(\frac{9^{\frac23(n-1)}}{\zeta(Z)}\right)^{\frac12}(9-\log(\theta(Z)))^{n-1}\,\theta(Z).$$

Since $n\geq 2$, this quantity can be bounded by

$$\begin{aligned}&n\left(\frac{2}{9^{\frac23(n-1)}}+\frac{3}{9^{\frac23(n-1)}\pi}+\frac{2^{n+3}9^{\frac13(n-1)}\sqrt{3}}{\pi^{n-1}}\right)\zeta(Z)\\ &\qquad\leq n\left(1+\frac{2^3\sqrt{3}\pi}{9^{\frac13}}\left(\frac{2\cdot 9^{\frac13}}{\pi}\right)^{n}\right)\zeta(Z)\leq 22\,n\left(\frac{4}{3}\right)^{n}\zeta(Z),\end{aligned}$$

as it can be easily verified that $\frac{2\cdot 9^{\frac13}}{\pi}<\frac43$ and $\frac{2^3\sqrt{3}\pi}{9^{\frac13}}<21$.

If $q = 0$, then $\frac{9^{\frac{2}{3}(n-1)}}{\zeta(Z)} < 1$, which implies that $\Delta_{\mathrm{ang}}(Z) \leq 1 \leq \zeta(Z)$. Hence, in the case where $-\pi \leq \alpha_j < \beta_j \leq \pi$ for all $j$, we have

$$\Big| \frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\deg(Z)} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} \Big| \leq 22\, n \left(\frac{4}{3}\right)^n \zeta(Z). \tag{2.17}$$

Now, if $\beta_j - \alpha_j > \pi$ for some $j$, we subdivide each of those intervals $(\alpha_j, \beta_j]$ into two subintervals of length $\leq \pi$. We can then decompose $Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as the sum of at most $2^n$ cycles of the form $Z_{\boldsymbol{\nu}_{0,\sigma},\boldsymbol{\nu}_{1,\sigma}}$ where the $j$th coordinate of $\boldsymbol{\nu}_{0,\sigma}$ (respectively $\boldsymbol{\nu}_{1,\sigma}$) is either $\alpha_j$ (respectively $\frac{\alpha_j+\beta_j}{2}$) or $\frac{\alpha_j+\beta_j}{2}$ (respectively $\beta_j$). Also, we can expand the product

$$\prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi}$$

as the sum of the volumes of the sets $\prod_{j=1}^{m} (\frac{\nu_{0,\sigma,j}}{2\pi}, \frac{\nu_{1,\sigma,j}}{2\pi}]$. From here, we easily get that

$$\left| \frac{\deg(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\deg(Z)} - \prod_{j=1}^{n} \frac{\beta_j - \alpha_j}{2\pi} \right| \leq \sum_{\sigma} \left| \frac{\deg(Z_{\boldsymbol{\nu}_{0,\sigma},\boldsymbol{\nu}_{1,\sigma}})}{\deg(Z)} - \prod_{j=1}^{n} \frac{\nu_{1,\sigma,j} - \nu_{0,\sigma,j}}{2\pi} \right|.$$

The claim follows applying the bound (2.17), which has to be multiplied by $2^n$. Altogether, we get

$$\Delta_{\mathrm{ang}}(Z) \leq 22\, n \left(\frac{8}{3}\right)^n \zeta(Z),$$

which concludes the proof. □

## 3. Bounds for the discrepancy in terms of sparse resultants

In this section, we consider cycles defined by a system of Laurent polynomials with given support. We compute their direct image under monomial projections, in terms of sparse resultants, and we derive Theorem 1.4 from the Erdös-Turán's theorem and the results in the previous section. We also establish some basic properties of the Erdös-Turán size.

We first recall the definition of the sparse resultant following [DS13]. Let $\mathcal{A}_0, \dots, \mathcal{A}_n$ be a family of $n+1$ non-empty finite subsets of $\mathbb{Z}^n$ and put $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$. Let $\boldsymbol{u}_i = \{u_{i,\boldsymbol{a}}\}_{a \in \mathcal{A}_i}$ be a group of $\#\mathcal{A}_i$ variables, $i = 0, \dots, n$, and set $\boldsymbol{u} = \{\boldsymbol{u}_0, \dots, \boldsymbol{u}_n\}$. For each $i$, let $F_i$ be the general polynomial with support $\mathcal{A}_i$, that is

$$F_i = \sum_{\boldsymbol{a} \in \mathcal{A}_i} u_{i,\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}} \in \mathbb{C}[\boldsymbol{u}][x_1^{\pm 1}, \dots, x_n^{\pm 1}], \tag{3.1}$$

and consider the incidence variety

$$W_{\boldsymbol{\mathcal{A}}} = \Big\{ (\boldsymbol{x}, \boldsymbol{u}) \in (\mathbb{C}^{\times})^n \times \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \Bigm| F_i(\boldsymbol{u}_i, \boldsymbol{x}) = 0 \Big\}.$$

The direct image of $W_{\boldsymbol{\mathcal{A}}}$ under the projection $\pi\colon (\mathbb{C}^{\times})^n \times \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \to \prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$ is the Weil divisor of $\prod_{i=0}^{n} \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$ given by

$$\pi_*(W_{\mathcal{A}}) = \begin{cases} \deg(\pi|_{W_{\boldsymbol{\mathcal{A}}}}) \overline{\pi(W_{\boldsymbol{\mathcal{A}}})} & \text{if } \operatorname{codim}(\overline{\pi(W_{\boldsymbol{\mathcal{A}}})}) = 1, \\ 0 & \text{if } \operatorname{codim}(\overline{\pi(W_{\boldsymbol{\mathcal{A}}})}) \geq 2. \end{cases}$$

The *sparse resultant* associated to $\boldsymbol{\mathcal{A}}$, denoted $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$, is defined as any primitive equation in $\mathbb{Z}[\boldsymbol{u}]$ of this Weil divisor. It is well-defined up to a sign.

According to this definition, sparse resultants are not necessarily irreducible. If we denote by $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$ what is classically called the sparse resultant [GKZ94, CLO05, PS93], then $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$ if and only if $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ and, if this is the case,

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{\deg(\pi|_{W_{\boldsymbol{\mathcal{A}}}})}.$$

For instance, for $\mathcal{A}_0 = \{0\}, \mathcal{A}_1 = \{0,1,2\} \subset \mathbb{Z}$, we have that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm u_{0,0}^2, \quad \mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm u_{0,0},$$

see [DS13, Example 3.14].

To recall the basic properties of the sparse resultant that we will need in the sequel, we need to introduce some definitions. Let $H$ be a linear subspace of $\mathbb{R}^n$ of dimension $m$ and $P_i$, $i = 1, \dots, m$, convex bodies of $H$. The *mixed volume* of these convex bodies is defined as

$$\mathrm{MV}_H(P_1, \dots, P_m) = \sum_{j=1}^{m} (-1)^{m-j} \sum_{1 \le i_1 < \cdots < i_j \le m} \mathrm{vol}_H(P_{i_1} + \cdots + P_{i_j}),$$

where $\mathrm{vol}_H$ denotes the Euclidean volume of $H$. We refer to [CLO05] for further background on the mixed volume of convex bodies.

Write $\mathbb{C}[\boldsymbol{x}^{\pm1}] = \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$ for short. The *height* of a Laurent polynomial $f = \sum_{\boldsymbol{a} \in \mathbb{Z}^n} \alpha_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}} \in \mathbb{C}[\boldsymbol{x}^{\pm1}]$ is defined as

$$\mathrm{h}(f) = \log\big(\max_{\boldsymbol{a}} |\alpha_{\boldsymbol{a}}|\big).$$

Given a finite subset $\mathcal{B}$ of $\mathbb{Z}^n$, we denote by $\mathrm{conv}(\mathcal{B})$ its convex hull, which is a lattice polytope of $\mathbb{R}^n$.

**Proposition 3.2.** *Let $\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$ be a family of $n+1$ non-empty finite subsets and set $Q_i = \mathrm{conv}(\mathcal{A}_i)$. Then*

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}_{\mathbb{R}^n}(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n), \quad i = 0, \dots, n,$$

*and*

$$\mathrm{h}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) \le \sum_{i=0}^{n} \mathrm{MV}_{\mathbb{R}^n}(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n) \log(\#\mathcal{A}_i).$$

*Proof.* The formula for the partial degrees is classical, see for instance [GKZ94]. The bound for the height is given by [Som04, Theorem 1] for the case where the resultant depends on all the groups of variables $\boldsymbol{u}_0, \dots, \boldsymbol{u}_n$. The general case can be found in [DS13, Proposition 3.15]. □

For a family of Laurent polynomials $f_i \in \mathbb{C}[\boldsymbol{x}^{\pm1}]$ with $\mathrm{supp}(f_i) \subset \mathcal{A}_i$, $i = 0, \dots, n$, we write

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(f_0, \dots, f_n)$$

for the evaluation of the resultant at their coefficients. The following is the multiplicativity formula for sparse resultants.

**Proposition 3.3.** *Let $0 \le i \le n$ and consider a family of non-empty finite subsets $\mathcal{A}_0, \dots, \mathcal{A}_n, \mathcal{A}_i' \subset \mathbb{Z}^n$. Let $f_j \in \mathbb{C}[\boldsymbol{x}^{\pm1}]$, be a Laurent polynomial with support contained* *in* $\mathcal{A}_j$*,* $j = 0, \dots, n$*, and* $f_i' \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ *a further Laurent polynomial with support contained in* $\mathcal{A}_i'$*. Then*

$$\begin{aligned}&\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_i+\mathcal{A}_i',\dots,\mathcal{A}_n}(f_0,\dots,f_if_i',\dots,f_n)\\ &\qquad = \pm\,\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_i,\dots,\mathcal{A}_n}(f_0,\dots,f_i,\dots,f_n)\,\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_i',\dots,\mathcal{A}_n}(f_0,\dots,f_i',\dots,f_n)\end{aligned}$$

*Proof.* The validity of this formula, with some restrictions, has been stablished first in [PS93, Proposition 7.1]. The general case can be found in [DS13, Corollary 4.6]. □

The *support function* of a compact subset $P \subset \mathbb{R}^n$ is the function $h_P\colon \mathbb{R}^n \to \mathbb{R}$ defined, for $\boldsymbol{v} \in \mathbb{R}^n$, as

$$h_P(\boldsymbol{v}) = \inf_{\boldsymbol{a}\in P} \langle \boldsymbol{a}, \boldsymbol{v}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^n$. Let $\mathcal{B} \subset \mathbb{Z}^n$ be a finite subset and $f = \sum_{\boldsymbol{b}\in\mathcal{B}} \beta_{\boldsymbol{b}} \boldsymbol{x}^{\boldsymbol{b}}$ a Laurent polynomial with support contained in $\mathcal{B}$. For $\boldsymbol{v} \in \mathbb{R}^n$, we set

$$\mathcal{B}^{\boldsymbol{v}} = \{\boldsymbol{b} \in \mathcal{B} \mid \langle \boldsymbol{b}, \boldsymbol{v}\rangle = h_{\mathrm{conv}(\mathcal{B})}(\boldsymbol{v})\}, \quad f^{\boldsymbol{v}} = \sum_{\boldsymbol{b}\in\mathcal{B}^{\boldsymbol{v}}} \beta_{\boldsymbol{b}} x^{\boldsymbol{b}}.$$

**Definition 3.4.** Let $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$ be a family of $n$ non-empty finite subsets, $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{0\}$, and $\boldsymbol{v}^\perp \subset \mathbb{R}^n$ the orthogonal subspace. Then $\mathbb{Z}^n \cap \boldsymbol{v}^\perp$ is a lattice of rank $n-1$ and, for $i = 1, \dots, n$, there exists $\boldsymbol{b}_{i,\boldsymbol{v}} \in \mathbb{Z}^n$ such that $\mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}} \subset \mathbb{Z}^n \cap \boldsymbol{v}^\perp$. The *resultant of* $\mathcal{A}_1, \dots, \mathcal{A}_n$ *in the direction of* $\boldsymbol{v}$, denoted $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}$, is defined as the resultant of the family of finite subsets $\mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}} \subset \mathbb{Z}^n \cap \boldsymbol{v}^\perp$.

Let $f_i \in \mathbb{C}[x_1^{\pm1}, \dots, x_n^{\pm1}]$, $i = 1, \dots, n$, with $\mathrm{supp}(f_i) \subset \mathcal{A}_i$. For each $i$, write $f_i^{\boldsymbol{v}} = \boldsymbol{x}^{\boldsymbol{b}_{i,\boldsymbol{v}}} g_{i,\boldsymbol{v}}$ for a Laurent polynomial $g_{i,\boldsymbol{v}} \in \mathbb{C}[\mathbb{Z}^n \cap \boldsymbol{v}^\perp] \simeq \mathbb{C}[y_1^{\pm1}, \dots, y_{n-1}^{\pm1}]$ with $\mathrm{supp}(g_{i,\boldsymbol{v}}) \subset \mathcal{A}_i^{\boldsymbol{v}} - \boldsymbol{b}_{i,\boldsymbol{v}}$. The expression

$$\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}})$$

is defined as the evaluation of this directional resultant at the coefficients of the $g_{i,\boldsymbol{v}}$. These constructions are independent of the choice of the vectors $\boldsymbol{b}_{i,\boldsymbol{v}}$.

We have that $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}} \neq 1$ only if $\boldsymbol{v}$ is an inner normal to a facet of the Minkowski sum $\sum_{i=1}^n \mathrm{conv}(\mathcal{A}_i)$. In particular, the number of non-trivial directional resultants of the family $\mathcal{A}_1, \dots, \mathcal{A}_n$ is finite.

With notation as in Definition 3.4, write $\boldsymbol{f} = (f_1, \dots, f_n)$ for short. We denote by $V(\boldsymbol{f})_0 \subset (\mathbb{C}^\times)^n$ the set of isolated solutions in the algebraic torus of the system of equations $f_1 = \cdots = f_n = 0$ and we set

$$Z(f_1, \dots, f_n) = \sum_{\boldsymbol{\xi}\in V(\boldsymbol{f})_0} \mathrm{mult}(\boldsymbol{\xi}|\boldsymbol{f})[\boldsymbol{\xi}]$$

for the associated 0-dimensional cycle, where $\mathrm{mult}(\boldsymbol{\xi}|\boldsymbol{f})$ denotes the intersection multiplicity of $\boldsymbol{f}$ at a point $\boldsymbol{\xi}$. For $f_0 \in \mathbb{C}[\boldsymbol{x}^{\pm1}]$, we set

$$f_0(Z(f_1, \dots, f_n)) = \prod_{\boldsymbol{\xi}} f_0(\boldsymbol{\xi})^{\mathrm{mult}(\boldsymbol{\xi}|\boldsymbol{f})}.$$

The following result is known as the Poisson formula for sparse resultants.

**Proposition 3.5.** *Let* $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n)$ *be a family of non-empty finite subsets of* $\mathbb{Z}^n$ *and* $f_i \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ *a Laurent polynomial with* $\mathrm{supp}(f_i) \subset \mathcal{A}_i$, $i = 0, \dots, n$. *Suppose that* $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \dots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}}) \neq 0$ *for all* $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. *Then*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(f_0, f_1, \dots, f_n) = \pm \bigg( \prod_{\boldsymbol{v}} \mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \dots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}})^{-h_{\mathcal{A}_0}(\boldsymbol{v})} \bigg) f_0(Z(f_1, \dots, f_n)),$$

*the product being over all primitive elements* $\boldsymbol{v} \in \mathbb{Z}^n$.

*Proof.* This formula has been obtained, under some restrictions on the supports, by Pedersen and Sturmfels in [PS93, Theorem 1.1]. The general case can be found in [DS13, Theorem 1.1]. □

From now on, we fix a family of non-empty finite subsets $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathbb{Z}^n$ such that $\mathrm{MV}_{\mathbb{R}^n}(Q_1, \dots, Q_n) \geq 1$, where $Q_i = \mathrm{conv}(\mathcal{A}_i)$. We consider also a family of Laurent polynomials $f_1, \dots, f_n \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ with $\mathrm{supp}(f_i) \subset \mathcal{A}_i$ and $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \dots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \dots, f_n^{\boldsymbol{v}}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{0\}$. By Bernstein's theorem [Ber75, Theorem B],

$$\deg(Z(\boldsymbol{f})) = \mathrm{MV}_{\mathbb{R}^n}(Q_1, \dots, Q_n) \geq 1.$$

Consider the projection $\pi_{\boldsymbol{e}_1} \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\pi_{\boldsymbol{e}_1}(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$. If we regard each $f_i$ as a Laurent polynomial in the group of variables $\boldsymbol{x}' := (x_2, \dots, x_n)$ with coefficients in the ring $\mathbb{C}[x_1{}^{\pm 1}]$, its support with respect to $\boldsymbol{x}'$ is contained in the finite subset $\pi_{\boldsymbol{e}_1}(\mathcal{A}_i)$ of $\mathbb{R}^{n-1}$. Set then

$$R(\boldsymbol{f}) = \mathrm{Res}_{\pi_{e_1}(\mathcal{A}_1), \dots, \pi_{e_1}(\mathcal{A}_n)} \big( f_1(x_1, \boldsymbol{x}'), \dots, f_n(x_1, \boldsymbol{x}') \big) \in \mathbb{C}[x_1^{\pm 1}]$$

for the evaluation of the resultant at these coefficients in $\mathbb{C}[x_1^{\pm 1}]$.

Recall that the sup-norm of a Laurent polynomial $f \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ is defined as $\|f\|_{\sup} = \sup_{\boldsymbol{w} \in (S^1)^n} |f(\boldsymbol{w})|$. In general, it holds that

$$\mathrm{h}(f) \leq \log \|f\|_{\sup} \tag{3.6}$$

This is a consequence of Cauchy's formula for the coefficients of the Laurent expansion of a holomorphic function on $(\mathbb{C}^{\times})^N$ (see for instance [Som04, page 1255]).

The following result gives a bound for the sup-norm of $R(\boldsymbol{f})$. Its proof is a variant of that for [Som04, Lemma 1.3].

**Lemma 3.7.** *Let notation be as above. Then*

$$\log \|R(\boldsymbol{f})\|_{\sup} \leq \sum_{j=1}^{n} \mathrm{MV}_{\boldsymbol{e}_1^{\perp}}(\{\pi_{\boldsymbol{e}_1}(Q_\ell)\}_{\ell \neq j}) \log \|f_j\|_{\sup}.$$

*Proof.* Let $k \geq 1$. Proposition 3.3 implies that

$$R(\boldsymbol{f})^{k^n} = \mathrm{Res}_{k\pi_{\boldsymbol{e}_1}(\mathcal{A}_1), \dots, k\pi_{\boldsymbol{e}_1}(\mathcal{A}_n)}(f_1(x_1, \boldsymbol{x}')^k, \dots, f_n(x_1, \boldsymbol{x}')^k), \tag{3.8}$$

where $k\pi_{\boldsymbol{e}_1}(\mathcal{A}_j)$ denotes the pointwise sum of $k$ copies of $\pi_{\boldsymbol{e}_1}(\mathcal{A}_j)$. For short, set $R_k = \mathrm{Res}_{k\pi_{\boldsymbol{e}_1}(\mathcal{A}_1), \dots, k\pi_{\boldsymbol{e}_1}(\mathcal{A}_n)}$ and $\boldsymbol{f}^k = (f_1^k, \dots, f_n^k)$, so that the identity above can be rewritten as $R(\boldsymbol{f})^{k^n} = R_k(\boldsymbol{f}^k)$. By Proposition 3.2, the partial degrees of this resultant are given by

$$\deg_{\boldsymbol{u}_j}(R_k) = \mathrm{MV}_{\boldsymbol{e}_1^{\perp}}(\{k\pi_{\boldsymbol{e}_1}(Q_\ell)\}_{\ell \neq j}) = k^{n-1} \mathrm{MV}_{\boldsymbol{e}_1^{\perp}}(\{\pi_{\boldsymbol{e}_1}(Q_\ell)\}_{\ell \neq j}),$$

where $\boldsymbol{u}_j$ is a group of $\#k\mathcal{A}_j$ variables, for $j = 1, \dots, n$. In particular, the logarithm of its number of monomials is bounded from above as

$$\log(\#\operatorname{supp}(R_k)) \le \log\Big(\prod_{j=1}^{n} \binom{\#k\mathcal{A}_j + \deg_{\boldsymbol{u}_j}(R_k)}{\#k\mathcal{A}_j}\Big)$$
$$\le \sum_{j=1}^{n} \deg_{\boldsymbol{u}_j}(R_k) \log(\#k\mathcal{A}_j + 1) = O(k^{n-1}\log(k+1)),$$

since $\#k\mathcal{A}_j \le (k+1)^{c_1 n}$ for a constant $c_1$ independent of $k$. By Proposition 3.2, the height of this resultant is bounded from above by

$$\mathrm{h}(R_k) \le \sum_{j=1}^{n} \deg_{\boldsymbol{u}_j}(R_k)\log(\#k\mathcal{A}_j) = O(k^{n-1}\log(k+1)).$$

Let $w_1 \in S^1$. Using (3.8), (3.6), the previous bounds, and the fact that $\|f_j^k\|_{\sup} = \|f_j\|_{\sup}^k$, we deduce that

$$k^n \log\|R(\boldsymbol{f})\|_{\sup} = \log\|R_k(\boldsymbol{f}^k)\|_{\sup}$$
$$\le \mathrm{h}(R_k) + \sum_{j=1}^{n} \deg_{\boldsymbol{u}_j}(R_j)\log\|f_j^k\|_{\sup} + \log(\#\operatorname{supp}(R_k))$$
$$= k^n\Big(\sum_{j=1}^{n} \mathrm{MV}_{\boldsymbol{e}_1^{\perp}}(\{\pi_{\boldsymbol{e}_1}(Q_\ell)\}_{\ell\ne j})\log\|f_j\|_{\sup}\Big) + O(k^{n-1}\log(k+1)).$$

The result then follows by dividing both sides of this inequality by $k^n$ and letting $k \to \infty$. □

For $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}$ and $z$ an additional variable, set

$$E_{\boldsymbol{a}}(\boldsymbol{f}) = \mathrm{Res}_{\{\boldsymbol{0},\boldsymbol{a}\},\mathcal{A}_1,\dots,\mathcal{A}_n}(z - \boldsymbol{x}^{\boldsymbol{a}}, f_1, \dots, f_n) \in \mathbb{C}[z].$$

Due to the Poisson formula for sparse resultants given in Proposition 3.5, we have that $Z(E_{\boldsymbol{a}}(\boldsymbol{f})) = \chi_*^{\boldsymbol{a}}(Z(\boldsymbol{f}))$, and so $E_{\boldsymbol{a}}(\boldsymbol{f})$ can be regarded as an elimination polynomial for the cycle $Z(\boldsymbol{f})$ with respect to the monomial projection $\chi^{\boldsymbol{a}}$.

By [DS13, Theorem 1.4], there exists $m \in \mathbb{Z}$ such that

$$E_{\boldsymbol{e}_1}(\boldsymbol{f})(x_1) = x_1^m R(\boldsymbol{f}). \tag{3.9}$$

Hence, Lemma 3.7 can be regarded as a bound for the sup-norm of the elimination polynomial $E_{\boldsymbol{e}_1}(\boldsymbol{f})$. Our next step is to extend this result to an arbitrary $\boldsymbol{a}$. Recall that $\pi_{\boldsymbol{a}}\colon \mathbb{R}^n \to \boldsymbol{a}^{\perp}$ denotes the orthogonal projection onto the hyperplane $\boldsymbol{a}^{\perp} \subset \mathbb{R}^n$.

**Lemma 3.10.** *Following the notation above,*

$$\log\|E_{\boldsymbol{a}}(\boldsymbol{f})\|_{\sup} \le \|\boldsymbol{a}\|_2 \sum_{j=1}^{n} \mathrm{MV}_{\boldsymbol{a}^{\perp}}(\{\pi_{\boldsymbol{a}}(Q_\ell)\}_{\ell\ne j})\log\|f_j\|_{\sup}.$$

*Proof.* Consider first the case where $\boldsymbol{a} \in \mathbb{Z}^n$ is primitive. The quotient $\mathbb{Z}^n/\boldsymbol{a}\mathbb{Z}$ is torsion-free and so $\boldsymbol{a}$ can be completed to a basis of $\mathbb{Z}^n$. Equivalently, there is an invertible matrix $A \in \mathrm{SL}_n(\mathbb{Z})$ with first row $\boldsymbol{a}$. Set $\boldsymbol{a}, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n$ and $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_n$ for the rows of $A$ and of $A^{-1}$, respectively. There is a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^\times)^n & \xrightarrow{\chi^{\boldsymbol{a}}} & \mathbb{C}^\times \\ {\scriptstyle \varphi_{A^{-1}}}\uparrow\ \downarrow{\scriptstyle \varphi_A} & \nearrow{\scriptstyle \chi^{\boldsymbol{e}_1}} & \\ (\mathbb{C}^\times)^n & & \end{array}$$

where $\varphi_A$ and $\varphi_{A^{-1}}$ are the monomial isomorphisms given by $\boldsymbol{x} \mapsto \boldsymbol{x}^A = (\boldsymbol{x}^{\boldsymbol{a}_1}, \dots, \boldsymbol{x}^{\boldsymbol{a}_n})$ and $\boldsymbol{x} \mapsto \boldsymbol{x}^{A^{-1}} = (\boldsymbol{x}^{\boldsymbol{b}_1}, \dots, \boldsymbol{x}^{\boldsymbol{b}_n})$, respectively. Let $\boldsymbol{y} = (y_1, \dots, y_n)$ denote the coordinates of the algebraic torus below. For $\ell = 1, \dots, n$, set

$$f_\ell^A(\boldsymbol{y}) = \varphi_{A^{-1}}^* f_\ell = f_\ell(\boldsymbol{y}^{\boldsymbol{b}_1}, \dots, \boldsymbol{y}^{\boldsymbol{b}_n}) \in \mathbb{C}[\boldsymbol{y}^{\pm 1}],$$

so that $(\varphi_A)_* Z(\boldsymbol{f}) = Z(\boldsymbol{f}^A)$. Hence, $E_{\boldsymbol{a}}(\boldsymbol{f}) = E_{\boldsymbol{f}^A, \boldsymbol{e}_1}$ and so Lemma 3.7 combined with (3.9) implies that

$$\log \|E_{\boldsymbol{a}}(\boldsymbol{f})\|_{\sup} \le \sum_{j=1}^n \mathrm{MV}_{\boldsymbol{e}_1^\perp}(\{\pi_{\boldsymbol{e}_1}(\mathrm{N}(f_\ell^A))\}_{\ell \ne j}) \log \|f_j^A\|_{\sup}, \tag{3.11}$$

where $\mathrm{N}(f_\ell^A)$ is the Newton polytope of $f_{\ell^A}$. We have $\pi_{\boldsymbol{e}_1}(\mathrm{N}(f_\ell^A)) = \widetilde{B}(\mathrm{N}(f_\ell)) \subset \widetilde{B}(Q_\ell)$ for the linear map $\widetilde{B}\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\widetilde{B}(\boldsymbol{x}) = (\langle \boldsymbol{x}, \boldsymbol{b}_2\rangle, \dots, \langle \boldsymbol{x}, \boldsymbol{b}_n\rangle)$. Let $\{\boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be an orthonormal basis of $\boldsymbol{a}^\perp$ and consider a second commutative diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\widetilde{B}} & \mathbb{R}^{n-1} \\ {\scriptstyle \pi_{\boldsymbol{a}}}\downarrow & & \downarrow{\scriptstyle C} \\ \boldsymbol{a}^\perp & \xrightarrow[U]{} & \boldsymbol{a}^\perp \end{array}$$

where $C$ is the linear map defined by $C(y_2, \dots, y_n) = y_2 \boldsymbol{v}_2 + \cdots + y_n \boldsymbol{v}_n$. It is easy to verify that $U \in \mathrm{GL}(\boldsymbol{a}^\perp)$ is uniquely determined by $\pi_{\boldsymbol{a}}$, $\widetilde{B}$ and $C$. Since $C$ maps the canonical basis of $\mathbb{R}^{n-1}$ into an orthonormal basis of $\boldsymbol{a}^\perp$, it is an isometry between these two spaces. On the other hand, a straightforward computation shows that

$$U(\boldsymbol{v}_j) = \sum_{k=2}^n \langle \boldsymbol{v}_j, \boldsymbol{b}_k\rangle \boldsymbol{v}_k = \boldsymbol{b}_j, \qquad j = 2, \dots, n.$$

We note that $\boldsymbol{b}_2, \dots, \boldsymbol{b}_n$ is a basis of the $\mathbb{Z}$-module $\boldsymbol{a}^\perp \cap \mathbb{Z}^n$. The Brill-Gordan formula [Bou70, Chapitre 3, § 11, Proposition 15] implies that $\mathrm{vol}(\boldsymbol{a}^\perp / (\boldsymbol{a}^\perp \cap \mathbb{Z}^n)) = \|\boldsymbol{a}\|_2$. Hence,

$$|\det(U)| = \mathrm{vol}(\boldsymbol{a}^\perp / (\boldsymbol{a}^\perp \cap \mathbb{Z}^n)) = \|\boldsymbol{a}\|_2. \tag{3.12}$$

Since $C$ is an isometry, [CLO05, Theorem 4.12(a)] implies that, for $j = 1, \dots, n$,

$$\mathrm{MV}_{\boldsymbol{e}_1^\perp}(\{\widetilde{B}(Q_\ell)\}_{\ell \ne j}) = \mathrm{MV}_{\boldsymbol{a}^\perp}(\{C \circ \widetilde{B}(Q_\ell)\}_{\ell \ne j}) = \mathrm{MV}_{\boldsymbol{a}^\perp}(\{U \circ \pi_{\boldsymbol{a}}(Q_\ell)\}_{\ell \ne j}).$$

By (3.12), $\|\boldsymbol{a}\|^{-1/(n-1)} U$ is a volume preserving map. Applying [CLO05, Theorem 4.12(a,b)], we deduce that

$$\mathrm{MV}_{\boldsymbol{a}^\perp}(\{U \circ \pi_{\boldsymbol{a}}(Q_\ell)\}_{\ell \ne j}) = \|\boldsymbol{a}\|_2 \, \mathrm{MV}_{\boldsymbol{a}^\perp}(\{\pi_{\boldsymbol{a}}(Q_\ell)\}_{\ell \ne j}).$$

In addition, $\varphi_A$ gives an automorphism of $(S^1)^n$ and so $\|f_\ell^A\|_{\sup} = \|f_\ell\|_{\sup}$. We conclude that, when $\boldsymbol{a}$ is primitive,

$$\log \|E_{\boldsymbol{a}}(\boldsymbol{f})\|_{\sup} \leq \|\boldsymbol{a}\|_2 \sum_{j=1}^{n} \mathrm{MV}_{\boldsymbol{a}^\perp}(\{\pi_{\boldsymbol{a}}(Q_\ell)\}_{\ell\neq j}) \log \|f_j\|_{\sup}. \tag{3.13}$$

Now let $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{0\}$ be any vector. Choose a primitive $\boldsymbol{a}' \in \mathbb{Z}^n$ and $m \in \mathbb{Z}_{\geq 1}$ such that $\boldsymbol{a} = m\boldsymbol{a}'$. Using Proposition 3.3, we deduce that

$$E_{\boldsymbol{a}}(\boldsymbol{f})(z) = \pm \prod_{\omega\in\mu_m} E_{\boldsymbol{a}'}(\boldsymbol{f})(\omega z)$$

where $\mu_m$ denotes the set of $m$-th roots of 1. Hence $\|\boldsymbol{a}\|_2 = m\|\boldsymbol{a}'\|_2$, $\pi_{\boldsymbol{a}} = \pi_{\boldsymbol{a}'}$ and $\log \|E_{\boldsymbol{a}}(\boldsymbol{f})\|_{\sup} \leq m \log \|E_{\boldsymbol{a}'}(\boldsymbol{f})\|_{\sup}$. The result follows from the bound (3.13) applied to $\boldsymbol{a}'$. □

*Proof of Theorem 1.4.* Let $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Applying Proposition 3.5 with $f_0 = z - \boldsymbol{x}^{\boldsymbol{a}}$ and $f_0 = z - \boldsymbol{x}^{-\boldsymbol{a}}$, we get that the product of the leading and the constant coefficients of $E_{\boldsymbol{a}}(\boldsymbol{f})$ is equal to

$$\pm \prod_{\boldsymbol{v}} \mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\dots,f_n^{\boldsymbol{v}})^{|\langle \boldsymbol{v},\boldsymbol{a}\rangle|}.$$

Recall also that $Z(E_{\boldsymbol{a}}(\boldsymbol{f})(z)) = \boldsymbol{\chi}_*^{\boldsymbol{a}}\big(Z(\boldsymbol{f})\big)$. The Erdös-Turán's theorem (Theorem 1.1) then implies that

$$\Delta_{\mathrm{ang}}\big(\boldsymbol{\chi}_*^{\boldsymbol{a}}(Z(\boldsymbol{f}))\big) \leq c \sqrt{\frac{1}{D}\log\left(\frac{\|E_{\boldsymbol{a}}(\boldsymbol{f})(z)\|_{\sup}}{\prod_{\boldsymbol{v}} |\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\dots,f_n^{\boldsymbol{v}})|^{\frac{|\langle \boldsymbol{v},\boldsymbol{a}\rangle|}{2}}}\right)},$$

with $c = 2.5619\dots$ Lemma 3.10 implies that $\log\left(\frac{\|E_{\boldsymbol{a}}(\boldsymbol{f})(z)\|_{\sup}}{\prod_{\boldsymbol{v}} |\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\dots,f_n^{\boldsymbol{v}})|^{\frac{|\langle \boldsymbol{v},\boldsymbol{a}\rangle|}{2}}}\right)$ is bounded from above by the quantity

$$\|\boldsymbol{a}\|_2 \log\left(\frac{\prod_{i=1}^n \|f_i\|_{\sup}^{D_{\boldsymbol{w},i}}}{\prod_{\boldsymbol{v}} |\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}},\dots,\mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}},\dots,f_n^{\boldsymbol{v}})|^{\frac{|\langle \boldsymbol{v},\boldsymbol{w}\rangle|}{2}}}\right),$$

for $\boldsymbol{w} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|_2} \in S^{n-1}$. From the definitions of $\theta(Z(\boldsymbol{f}))$ and of the Erdös-Turán size $\eta(\boldsymbol{f})$ given in (2.2) and (1.3), respectively, we get

$$\theta(Z(\boldsymbol{f})) \leq \min\{1, c\sqrt{\eta(\boldsymbol{f})}\}.$$

Applying Theorem 2.3 and the fact that the function $t^{\frac{2}{3}}(9-\log(t))$ is monotonically increasing in the interval $(0,1]$, we deduce that

$$\begin{aligned}
\Delta_{\mathrm{ang}}(Z(\boldsymbol{f})) &\leq 22n\left(\frac{8}{3}\right)^n \big(9 - \log(\min\{1, c\sqrt{\eta(\boldsymbol{f})}\})\big)^{\frac{2}{3}(n-1)} \min\{1, c\sqrt{\eta(\boldsymbol{f})}\}^{\frac{2}{3}} \\
&\leq 22n\left(\frac{8}{3}\right)^n 2^{-\frac{2}{3}(n-1)}(18 + \log^+(\eta(\boldsymbol{f})^{-1}))^{\frac{2}{3}(n-1)} c^{\frac{2}{3}} \eta(\boldsymbol{f})^{\frac{1}{3}} \\
&\leq 66\, n\, 2^n (18 + \log^+(\eta(\boldsymbol{f})^{-1}))^{\frac{2}{3}(n-1)} \eta(\boldsymbol{f})^{\frac{1}{3}},
\end{aligned}$$

which gives the bound for the angle discrepancy. For the radius discrepancy, we use the bounds given in Theorem 1.1, (2.2), and Theorem 2.3 to get, for $0 < \varepsilon < 1$,

$$\Delta_{\mathrm{rad}}(\boldsymbol{f}, \varepsilon) \leq \sum_{j=1}^{n} \Delta_{\mathrm{rad}}(\boldsymbol{\chi}_*^{\boldsymbol{e}_j}(Z(\boldsymbol{f})), \varepsilon) \leq \frac{2n}{\varepsilon}\, \eta(\boldsymbol{f}).$$

This concludes with the proof of the theorem. □

We next study a number of basic properties of the Erdös-Turán size. The following proposition shows that this notion generalizes the measure of polynomials that appears in the statement of Theorem 1.1.

**Proposition 3.14.** *Let $d \geq 1$ and $f = a_0 + \cdots + a_d x^d \in \mathbb{C}[x]$ with $a_0 a_d \neq 0$. Then*

$$\eta(f) = \frac{1}{d} \log\left(\frac{\|f\|_{\sup}}{\sqrt{|a_0 a_d|}}\right).$$

*Proof.* The directional resultants of $f$ are

$$\mathrm{Res}_{\boldsymbol{v}}(f^{\boldsymbol{v}}) \begin{cases} \pm a_0 & \text{for } \boldsymbol{v} = 1, \\ \pm a_d & \text{for } \boldsymbol{v} = -1. \end{cases}$$

Moreover, $D = \mathrm{MV}_{\mathbb{R}}([0, d]) = d$, $D_{\boldsymbol{w},1} = 1$ for $\boldsymbol{w} \in S^0 = \{\pm 1\}$, and $|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| = 1$ for all $\boldsymbol{v}, \boldsymbol{w} \in \{\pm 1\}$. The formula for $\eta(f)$ then boils down to $\frac{1}{d} \log\left(\frac{\|f\|_{\sup}}{\sqrt{|a_0 a_d|}}\right)$. □

We denote by $\Delta^n = \mathrm{conv}(\boldsymbol{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \subset \mathbb{R}^n$ the standard $n$-simplex.

**Proposition 3.15.** *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be a family of non-empty finite subsets of $\mathbb{Z}^n$ such that $\mathrm{MV}_{\mathbb{R}^n}(Q_1, \ldots, Q_n) \geq 1$ with $Q_i = \mathrm{conv}(\mathcal{A}_i)$. Let $f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with $\mathrm{supp}(f_i) \subset \mathcal{A}_i$ and such that $\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \ldots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \ldots, f_n^{\boldsymbol{v}}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{0\}$.*

1. *$\eta(\boldsymbol{f}) < \infty$.*
2. *Let $\gamma_1, \ldots, \gamma_n \in \mathbb{C}^\times$. Then $\eta(\gamma_1 f_1, \ldots, \gamma_n f_n) = \eta(f_1, \ldots, f_n)$.*
3. *Let $d_j \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{b}_j \in \mathbb{Z}^n$ such that $Q_j \subset d_j \Delta^n + \boldsymbol{b}_j$, $j = 1, \ldots, n$. Then*

$$\eta(\boldsymbol{f}) \leq \frac{1}{\mathrm{MV}_{\mathbb{R}^n}(Q_1, \ldots, Q_n)} \Bigg( (n + \sqrt{n}) \Big( \prod_{j=1}^{n} d_j \Big) \sum_{j=1}^{n} \frac{\log \|f_j\|_{\sup}}{d_j} + \sum_{\boldsymbol{v}} \frac{\|\boldsymbol{v}\|_2}{2} \log^+ |\mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \ldots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \ldots, f_n^{\boldsymbol{v}})^{-1}| \Bigg),$$

*the second sum being taken over all primitive vectors $\boldsymbol{v} \in \mathbb{Z}^n$. Moreover, if $f_1, \ldots, f_n \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then*

$$\eta(\boldsymbol{f}) \leq \frac{(n + \sqrt{n}) \big( \prod_{j=1}^{n} d_j \big)}{\mathrm{MV}_{\mathbb{R}^n}(Q_1, \ldots, Q_n)} \sum_{j=1}^{n} \frac{\log \|f_j\|_{\sup}}{d_j}.$$

*Proof.* The statement of (1) is clear, since $\eta(\boldsymbol{f})$ is defined as the supremum of a continuous function over the compact set $S^{n-1}$.

For (2), let $\boldsymbol{a} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. As explained in the proof of Theorem 1.4, the product of the leading and the constant coefficients of $E_{\boldsymbol{a}}(\boldsymbol{f})$ is equal to

$$\pm \prod_{\boldsymbol{v}} \mathrm{Res}_{\mathcal{A}_1^{\boldsymbol{v}}, \ldots, \mathcal{A}_n^{\boldsymbol{v}}}(f_1^{\boldsymbol{v}}, \ldots, f_n^{\boldsymbol{v}})^{|\langle \boldsymbol{v}, \boldsymbol{a} \rangle|}.$$

Hence, the denominator in the definition of $\eta(\boldsymbol{f})$ is multihomogeneous in the coefficients of each $f_j$, of partial degrees equal to $\|\boldsymbol{a}\|_2^{-1}$ times those of $E_{\boldsymbol{a}}(\boldsymbol{f})$. Hence,

$$\frac{1}{\|\boldsymbol{a}\|_2}\deg_{f_j}(E_{\boldsymbol{a}}(\boldsymbol{f}))=\mathrm{MV}_{\boldsymbol{a}^\perp}\left(\pi_{\boldsymbol{a}}(Q_1),\dots,\pi_{\boldsymbol{w}}(Q_{j-1}),\pi_{\boldsymbol{a}}(Q_{j+1}),\dots,\pi_{\boldsymbol{a}}(Q_n)\right)=D_{\boldsymbol{w},j}$$

for $\boldsymbol{w}=\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|_2}$, which implies the statement.

For (3), let $\boldsymbol{w}\in S^{n-1}$. Then $\pi_{\boldsymbol{w}}(Q_j)\subset\pi_{\boldsymbol{w}}(d_j\Delta^n+\boldsymbol{b}_j)$. Due to the monotonicity of the mixed volume with respect to the inclusion, plus its properties of homogeneity and invariance under translation, we deduce that, for $j=1,\dots,n$,

$$\begin{aligned}\mathrm{MV}_{\boldsymbol{w}^\perp}(\{\pi_{\boldsymbol{w}}(Q_\ell)\}_{\ell\neq j})&\le\mathrm{MV}_{\boldsymbol{w}^\perp}(\{\pi_{\boldsymbol{w}}(d_\ell\Delta^n+\boldsymbol{b}_\ell)\}_{\ell\neq j})\\&\le(n-1)!\Big(\prod_{\ell\neq j}d_\ell\Big)\operatorname{vol}_{\boldsymbol{w}^\perp}(\pi_{\boldsymbol{w}}(\Delta^n)).\end{aligned}\tag{3.16}$$

The projected simplex $\pi_{\boldsymbol{w}}(\Delta^n)$ can be covered by the union of the projection of its facets. One of the facets of $\Delta^n$ has $(n-1)$-dimensional volume equal to $\frac{\sqrt{n}}{(n-1)!}$, while the other $n$ facets have $(n-1)$-dimensional volume equal to $\frac{1}{(n-1)!}$. Since the volume cannot increase under orthogonal projections, we have that

$$\operatorname{vol}_{\boldsymbol{w}^\perp}(\pi_{\boldsymbol{w}}(\Delta^n))\le\frac{\sqrt{n}+n}{(n-1)!}.\tag{3.17}$$

In addition, $|\langle\boldsymbol{v},\boldsymbol{w}\rangle|\le\|\boldsymbol{v}\|_2$ since $\boldsymbol{w}\in S^{n-1}$. Then, the first part of the statement follows from (3.16), (3.17) and the definition of $\eta(\boldsymbol{f})$. The second part follows from the fact that, if the coefficients of the $f_i$'s are integers, then the relevant directional resultants are nonzero integers and so their absolute values are at least 1. □

Let us consider the statement of Proposition 3.15(3), in the classical dense case $Q_j=d_j\Delta^n$ for all $j$. In this situation, the only primitive vectors $\boldsymbol{v}$ to consider are $\boldsymbol{v}=\boldsymbol{e}_i$, $i=1,\dots,n$, and $\boldsymbol{v}=\boldsymbol{e}_0:=-\sum_{i=1}^n\boldsymbol{e}_i$. Given $d_j\ge1$, $j=1,\dots,n$, we write $\mathrm{Res}_{d_1,\dots,d_n}$ for the resultant of $n$ homogeneous polynomials in $n$ variables of respective degrees $d_1,\dots,d_n$, as defined in [CLO05]. Given a system of polynomials $f_1,\dots,f_n\in\mathbb{C}[x_1,\dots,x_n]$ with $\deg(f_j)\le d_j$ and $i=0,\dots,n$, the initial polynomials $f_1^{\boldsymbol{e}_i},\dots,f_n^{\boldsymbol{e}_i}$ form a system of $n$ polynomials of degrees $d_1,\dots,d_n$. In particular, we can evaluate $\mathrm{Res}_{d_1,\dots,d_n}$ at these polynomials. If we assume that these resultants are nonzero, we obtain

$$\begin{aligned}\eta(\boldsymbol{f})\le(n+\sqrt{n})\sum_{j=1}^n\frac{\log\|f_j\|_{\sup}}{d_j}+\frac{1}{2\prod_{j=1}^n d_j}\Bigg(&\sqrt{n}\log^+|\mathrm{Res}_{d_1,\dots,d_n}(f_1^{\boldsymbol{e}_0},\dots,f_n^{\boldsymbol{e}_0})^{-1}|\\&+\sum_{i=1}^n\log^+|\mathrm{Res}_{d_1,\dots,d_n}(f_1^{\boldsymbol{e}_i},\dots,f_n^{\boldsymbol{e}_i})^{-1}|\Bigg).\end{aligned}$$

In particular, if $f_1,\dots,f_n\in\mathbb{Z}[x_1,\dots,x_n]$, then

$$\eta(\boldsymbol{f})\le(n+\sqrt{n})\sum_{j=1}^n\frac{\log\|f_j\|_{\sup}}{d_j}.$$

## 4. Asymptotic equidistribution

We will apply here the results in the previous sections to study the asymptotic distribution of the roots of a sequence of systems of Laurent polynomials over $\mathbb{Z}$ and of random systems of Laurent polynomials over $\mathbb{C}$.

First, we will consider polynomials over $\mathbb{Z}$. Let $Q_i, \dots, Q_n \subset \mathbb{R}^n$ be a family of lattice polytopes such that $\mathrm{MV}_{\mathbb{R}^n}(Q_1, \dots, Q_n) \geq 1$. For each integer $\kappa \geq 1$ and $i = 1, \dots, n$, consider the finite subset of $\mathbb{Z}^n$ given by

$$\mathcal{A}_{\kappa,i} = \kappa Q_i \cap \mathbb{Z}^n. \tag{4.1}$$

**Proposition 4.2.** *For $\kappa \geq 1$ let $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ be a family of Laurent polynomials in $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that $\mathrm{supp}(f_{\kappa,i}) \subset \kappa Q_i$ and $\mathrm{Res}_{\mathcal{A}^{\boldsymbol{v}}_{\kappa,1},\dots,\mathcal{A}^{\boldsymbol{v}}_{\kappa,n}}(f^{\boldsymbol{v}}_{\kappa,1}, \dots, f^{\boldsymbol{v}}_{\kappa,n}) \neq 0$ for all $\boldsymbol{v} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$. Then there is a constant $c_1 > 0$ which does not depend on $\kappa$ such that*

$$\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)) \leq c_1 \left(\frac{\sum_{i=1}^n \log \|f_{\kappa,i}\|_{\sup}}{\kappa}\right)^{\frac{1}{3}} \left(1 + \log^+\left(\frac{\kappa}{\sum_{i=1}^n \log \|f_{\kappa,i}\|_{\sup}}\right)\right)^{\frac{2}{3}(n-1)}$$

*and, for any $0 < \varepsilon < 1$,*

$$\Delta_{\mathrm{rad}}(Z(\boldsymbol{f}_\kappa), \varepsilon) \leq c_1 \frac{\sum_{i=1}^n \log \|f_{\kappa,i}\|_{\sup}}{\varepsilon \kappa}.$$

*Proof.* This follows easily from Theorem 1.4 and Proposition 3.15(3). □

*Proof of Theorem 1.7.* Following the notation in the statement of Theorem 1.7, $\nu_{Z(\boldsymbol{f}_\kappa)}$ is the discrete measure associated to $Z(\boldsymbol{f}_\kappa)$ and $\nu_{\mathrm{Haar}}$ is the measure on $(\mathbb{C}^\times)^n$ induced by the Haar probability measure on $(S^1)^n$.

We have to show that, for every continuous function with compact support $h$,

$$\lim_{\kappa \to \infty} \int_{(\mathbb{C}^\times)^n} h \, \mathrm{d}\nu_{Z(\boldsymbol{f}_\kappa)} = \int_{(\mathbb{C}^\times)^n} h \, \mathrm{d}\nu_{\mathrm{Haar}}.$$

It is enough to prove the statement for the characteristic function $h_U$ of the open sets of the form

$$U = \{(z_1, \dots, z_n) \in (\mathbb{C}^\times)^n \mid r_{1,j} < |z_j| < r_{2,j}, \ \alpha_j < \arg(z_j) < \beta_j \text{ for all } j\}, \tag{4.3}$$

with $0 \leq r_{1,j} < r_{2,j} \leq \infty$, $r_{i,j} \neq 1$ and $-\pi < \alpha_j < \beta_j \leq \pi$, since any continuous function with compact support can be uniformly approximated by a linear combinations of the aforementioned characteristic functions.

Consider first the case where $U \cap (S^1)^n = \emptyset$. Due to the conditions imposed on the numbers $r_{i,j}$, there exists $\varepsilon > 0$ such that $U$ is disjoint with the set

$$\{\boldsymbol{\xi} \in \mathbb{C}^n \mid 1 - \varepsilon < |\xi_j| < (1 - \varepsilon)^{-1} \text{ for all } j\}.$$

Hence,

$$\int_{(\mathbb{C}^\times)^n} h_U \, \mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} = \frac{\deg(Z(\boldsymbol{f}_\kappa)|_U)}{\kappa^n \, \mathrm{MV}(\boldsymbol{Q})} \leq \Delta_{\mathrm{rad}}(\boldsymbol{f}_\kappa, \varepsilon),$$

where $\mathrm{MV}(\boldsymbol{Q})$ denotes the mixed volume of the polytopes $Q_1, \dots, Q_n \subset \mathbb{R}^n$ and $Z(\boldsymbol{f}_\kappa)|_U = \sum_{\boldsymbol{\xi} \in |Z(\boldsymbol{f}_\kappa)| \cap U} m_{\boldsymbol{\xi}} [\boldsymbol{\xi}]$. Proposition 4.2 implies that this integral goes to 0 for $\kappa \to \infty$, which proves the statement in this case, since $\int_{\mathbb{C}^n} h_U \, \mathrm{d}\nu_{\mathrm{Haar}} = 0$.

Consider now the case where $U \cap (S^1)^n \neq \emptyset$. Set $\overline{U} = \{\boldsymbol{z} \mid \alpha_j \leq \arg(z_j) \leq \beta_j \text{ for all } j\}$. Then

$$\int_{(\mathbb{C}^\times)^n} h_U \,\mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} - \int_{(\mathbb{C}^\times)^n} h_U \,\mathrm{d}\nu_{\mathrm{Haar}} = \int_{(\mathbb{C}^\times)^n} h_U \,\mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi}$$
$$= \int_{(\mathbb{C}^\times)^n} \Big(h_{\overline{U}} - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi}\Big) \mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} - \int_{(\mathbb{C}^\times)^n} h_{\overline{U}\setminus U} \,\mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)}.$$

We have

$$\int_{(\mathbb{C}^\times)^n} \Big| h_{\overline{U}} - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi}\Big| \mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} \leq \Big| \frac{\deg(Z(\boldsymbol{f}_\kappa)_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\kappa^n \operatorname{MV}(\boldsymbol{Q})} - \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi}\Big| \leq \Delta_{\mathrm{ang}}(\boldsymbol{f}_\kappa).$$

Again, Proposition 4.2 implies that this integral goes to 0 for $\kappa \to \infty$. On the other hand, $\overline{U} \setminus U$ is a union of a finite number of subsets $U_l$ of the form (4.3) such that $U_l \cap (S^1)^n = \emptyset$ for all $l$. By the previous considerations, $\int_{(\mathbb{C}^\times)^n} h_{U_l} \mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} \to_\kappa 0$ and so $\int_{(\mathbb{C}^\times)^n} h_{\overline{U}\setminus U} \,\mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} \to_\kappa 0$. Hence

$$\lim_{\kappa\to\infty} \int_{(\mathbb{C}^\times)^n} h_U \,\mathrm{d}\delta_{Z(\boldsymbol{f}_\kappa)} = \prod_{j=1}^n \frac{\beta_j - \alpha_j}{2\pi} = \int_{\mathbb{C}^n} h_U \,\mathrm{d}\nu_{\mathrm{Haar}} = 0,$$

which concludes the proof. □

We will now consider random systems of Laurent polynomials with complex coefficients. To explain and prove our results, we have to consider metrics and measures on projective spaces over $\mathbb{C}$. Let $N \geq 1$ and consider the standard Riemannian structure on $\mathbb{C}^N$ induced by the Euclidean norm $\|\cdot\|_2$. Let $S^{2N-1} = \{\boldsymbol{z} \in \mathbb{C}^N \mid \|\boldsymbol{z}\|_2 = 1\}$ be the unit sphere with the induced Riemannian structure. The map $S^{2N-1} \to \mathbb{P}(\mathbb{C}^N)$ given by $(z_0, \dots, z_{N-1}) \mapsto (z_0 : \cdots : z_{N-1})$ gives a principal bundle with fiber $S^1$. The Fubini-Study metric on $\mathbb{P}(\mathbb{C}^N)$ is defined as the unique Riemannian structure such that this map is a Riemannian submersion, see [KN69] for details.

The geodesics of $\mathbb{P}(\mathbb{C}^N)$ coincide with lines. Hence, we can define a distance between two points $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ as the length of the line segment joining them, and we will denote it by $\mathrm{dist}_{\mathrm{FS}}(\boldsymbol{z}_1, \boldsymbol{z}_2)$. However, it will be more convenient to consider the distance function $\mathrm{dist} := \sin(\mathrm{dist}_{\mathrm{FS}})$. This function can be computed with the formula

$$\mathrm{dist}(\boldsymbol{z}_1, \boldsymbol{z}_2) = \sqrt{1 - \Big(\frac{|\langle \widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2\rangle|}{\|\widetilde{\boldsymbol{z}}_1\|_2 \|\widetilde{\boldsymbol{z}}_2\|_2}\Big)^2} \tag{4.4}$$

for any choice of representatives $\widetilde{\boldsymbol{z}}_i \in \mathbb{C}^N \setminus \{\boldsymbol{0}\}$, $i = 1, 2$.

**Lemma 4.5.** *Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{P}(\mathbb{C}^N)$ with $\widetilde{\boldsymbol{z}}_i \in \mathbb{C}^N$, $i = 1, 2$, representatives of these points such that $\|\widetilde{\boldsymbol{z}}_2\|_2 = 1$. Then* $\mathrm{dist}(\boldsymbol{z}_1, \boldsymbol{z}_2) \leq \|\widetilde{\boldsymbol{z}}_2 - \widetilde{\boldsymbol{z}}_1\|_2$.

*Proof.* We have that

$$\|\widetilde{\boldsymbol{z}}_2 - \widetilde{\boldsymbol{z}}_1\|_2^2 = \langle \widetilde{\boldsymbol{z}}_2 - \widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2 - \widetilde{\boldsymbol{z}}_1\rangle = 1 + \|\widetilde{\boldsymbol{z}}_1\|_2^2 - 2\operatorname{Re}(\langle \widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2\rangle) \geq 1 + \|\widetilde{\boldsymbol{z}}_1\|_2^2 - 2|\langle \widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2\rangle|.$$

On the other hand, the formula (4.4) gives $\mathrm{dist}(\boldsymbol{z}_1, \boldsymbol{z}_2)^2 = 1 - \big(\frac{|\langle \widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2\rangle|}{\|\widetilde{\boldsymbol{z}}_1\|_2}\big)^2$. Hence,

$$\|\widetilde{\boldsymbol{z}}_2 - \widetilde{\boldsymbol{z}}_1\|_2^2 - \mathrm{dist}(\boldsymbol{z}_1, \boldsymbol{z}_2)^2 = \Big(\|\widetilde{\boldsymbol{z}}_1\|_2 - \frac{|\langle \widetilde{\boldsymbol{z}}_1, \widetilde{\boldsymbol{z}}_2\rangle|}{\|\widetilde{\boldsymbol{z}}_1\|_2}\Big)^2 \geq 0,$$

which proves the statement. □

We will need the following Łojasiewicz inequality for a hypersurface of a complex projective space. For a homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_{N-1}]$ of degree $d$, and a point $\boldsymbol{z} \in \mathbb{P}(\mathbb{C}^N)$, the value

$$\frac{|f(\boldsymbol{z})|}{\|\boldsymbol{z}\|_2^d}$$

is well-defined. For a subset $E \subset \mathbb{P}(\mathbb{C}^N)$, we write $\mathrm{dist}(\boldsymbol{z}, E)$ for the distance between $\boldsymbol{z}$ and $E$.

**Lemma 4.6.** *Let $f \in \mathbb{C}[x_0, \dots, x_{N-1}]$ be a homogeneous polynomial of degree $d \geq 0$ and $\boldsymbol{z} \in \mathbb{P}(\mathbb{C}^N)$. Then*

$$\frac{|f(\boldsymbol{z})|}{\|\boldsymbol{z}\|_2^d} \geq \Big( \sup_{\boldsymbol{x} \in \mathbb{P}(\mathbb{C}^N)} \frac{|f(\boldsymbol{x})|}{\|\boldsymbol{x}\|_2^d} \Big) \mathrm{dist}(\boldsymbol{z}, V(f))^d.$$

*Proof.* Let $\boldsymbol{z}, \boldsymbol{x} \in \mathbb{P}(\mathbb{C}^N)$ such that $\boldsymbol{x} \notin V(f)$. Let $\widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{x}}$ be representatives of these points in the sphere $S^{2N-1}$ and set $f_{\widetilde{\boldsymbol{x}}}(t) = f(\widetilde{\boldsymbol{z}} + t\widetilde{\boldsymbol{x}}) \in \mathbb{C}[t]$. This is a univariate polynomial of degree $d$ with leading coefficient $f(\widetilde{\boldsymbol{x}})$. Then, there exist $\xi_j \in \mathbb{C}$, $j = 1, \dots, d$, such that $f_{\widetilde{\boldsymbol{x}}} = f(\widetilde{\boldsymbol{x}}) \prod_j (t - \xi_j)$ and so

$$|f(\widetilde{\boldsymbol{z}})| = |f_{\widetilde{\boldsymbol{x}}}(0)| = |f(\widetilde{\boldsymbol{x}})| \prod_j |\xi_j|.$$

For each $j$, we have that $\widetilde{\boldsymbol{z}} + \xi_j \widetilde{\boldsymbol{x}} \in V(f)$. Using Lemma 4.5, we deduce that

$$|\xi_j| = \|(\widetilde{\boldsymbol{z}} + \xi_j \widetilde{\boldsymbol{x}}) - \widetilde{\boldsymbol{z}}\|_2 \geq \mathrm{dist}(\boldsymbol{z}, \widetilde{\boldsymbol{z}} + \xi_j \widetilde{\boldsymbol{x}}) \geq \mathrm{dist}(\boldsymbol{z}, V(f)).$$

We deduce that

$$\frac{|f(\boldsymbol{z})|}{\|\boldsymbol{z}\|_2^d} = |f(\widetilde{\boldsymbol{z}})| \geq |f(\widetilde{\boldsymbol{x}})| \, \mathrm{dist}(\boldsymbol{z}, V(f))^d = \frac{|f(\boldsymbol{x})|}{\|\boldsymbol{x}\|_2^d} \mathrm{dist}(\boldsymbol{z}, V(f))^d.$$

Since this holds for all $\boldsymbol{x} \notin V(f)$, the result follows. □

Let $\mu_{\mathrm{FS}}$ denote the measure on $\mathbb{P}(\mathbb{C}^N)$ induced by the Fubini-Study metric. Then $\mu_{\mathrm{FS}}(\mathbb{P}(\mathbb{C}^N)) = \frac{\pi^{N-1}}{(N-1)!}$. We will consider the normalized measure given by

$$\mu = \frac{(N-1)!}{\pi^{N-1}} \mu_{\mathrm{FS}}.$$

A result of Beltrán and Pardo [BP07, Theorem 1] shows that, for a hypersurface $H \subset \mathbb{P}(\mathbb{C}^N)$ of degree $d$, the normalized measure of the tube around $H$ of radius $\rho \geq 0$ is bounded from above by

$$15d(N-1)^2\rho^2.$$

Applying this result, we deduce the following bound for the volume of the set where a polynomial can take small values. For $\delta > 0$ and a homogeneous polynomial $f \in \mathbb{C}[\boldsymbol{x}]$, we consider the subset of $\mathbb{P}(\mathbb{C}^N)$ given by

$$V(f)_\delta = \Big\{ \boldsymbol{z} \in \mathbb{P}(\mathbb{C}^N) \Big| \frac{|f(\boldsymbol{z})|}{\|\boldsymbol{z}\|_2^d} < \delta \Big\}. \tag{4.7}$$

**Proposition 4.8.** *Let $\delta > 0$ and $f \in \mathbb{C}[x_0, \dots, x_{N-1}]$ a homogeneous polynomial of degree $d \geq 1$. Then*

$$\mu(V(f)_\delta) \leq 15dN^3\Big(\frac{\delta}{\|f\|_{\sup}}\Big)^{\frac{2}{d}}.$$

*In particular, if $f \in \mathbb{Z}[x_0, \dots, x_{N-1}]$, then $\mu(V(f)_\delta) \leq 15dN^3\delta^{\frac{2}{d}}$.*

*Proof.* Let $\boldsymbol{z} \in V(f)_\delta$. Using Lemma 4.6, we deduce that

$$\delta > \bigg(\sup_{\boldsymbol{x}} \frac{|f(\boldsymbol{x})|}{\|\boldsymbol{x}\|_2^d}\bigg) \operatorname{dist}(\boldsymbol{z}, V(f))^d \geq \frac{\|f\|_{\sup}}{N^{\frac{d}{2}}} \operatorname{dist}(\boldsymbol{z}, V(f))^d.$$

Hence,

$$\operatorname{dist}(\boldsymbol{z}, V(f)) < N^{\frac{1}{2}}\bigg(\frac{\delta}{\|f\|_{\sup}}\bigg)^{\frac{1}{d}}$$

and so $V(f)_\delta$ is contained in the tube around $V(f)$ of radius $N^{\frac{1}{2}}\big(\frac{\delta}{\|f\|_{\sup}}\big)^{\frac{1}{d}}$. The first part of the result follows then from the Beltrán–Pardo bound for the volume of this tube. The second part follows from the fact that $\|f\|_{\sup} \geq |f| \geq 1$, because of the inequality (3.6) and the fact that the coefficients of $f$ are integer numbers. □

Let us keep the preceding notation and set $\boldsymbol{\mathcal{A}}_\kappa = (\mathcal{A}_{\kappa,1}, \dots, \mathcal{A}_{\kappa,n})$ with $\mathcal{A}_{\kappa,i}$ as in (4.1). Each point of the projective space $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ can be identified with a system $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ of Laurent polynomials such that $\operatorname{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $i = 1, \dots, n$, modulo a multiplicative scalar. The associated cycle $Z(\boldsymbol{f}_\kappa)$ is well-defined, since it does not depend on this multiplicative scalar.

Set $\mu_\kappa$ for the normalized Fubini-Study measure on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ and let $\lambda_\kappa \colon \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}) \to \mathbb{R}_{\geq 0}$ be a probability density function, that is, a $\mu_\kappa$-measurable function with

$$\int_{\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})} \lambda_\kappa \, \mathrm{d}\mu_\kappa = 1.$$

Let $\boldsymbol{f}_\kappa$ be a random system of Laurent polynomials with $\operatorname{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $i = 1, \dots, n$, distributed according to the probability law given by $\lambda_\kappa$ with respect to $\mu_\kappa$. We can then consider the angle discrepancy of $Z(\boldsymbol{f}_\kappa)$ and, for $0 < \varepsilon < 1$, the radius discrepancy of $Z(\boldsymbol{f}_\kappa)$ with respect to $\varepsilon$, as random variables on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$. We denote by $\mathbb{E}(\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)); \lambda_\kappa)$ and $\mathbb{E}(\Delta_{\mathrm{rad}}(Z(\boldsymbol{f}_\kappa), \varepsilon); \lambda_\kappa)$ the expected value of these random variables.

**Theorem 4.9.** *For $\kappa \geq 1$, let $\lambda_\kappa$ be a probability density function on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ and $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ a random system of Laurent polynomials with $\operatorname{supp}(f_{\kappa,i}) \subset \kappa Q_i$, $i = 1, \dots, n$, distributed according to the probability law given by $\lambda_\kappa$ with respect to $\mu_\kappa$. Assume that the sequence $(\lambda_\kappa)_{\kappa \geq 1}$ is uniformly bounded. Then there is a constant $c_2 > 0$ which does not depend on $\kappa$ such that*

$$\mathbb{E}(\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)); \lambda_\kappa) \leq c_2 \, \frac{\log(\kappa+1)^{\frac{2}{3}n - \frac{1}{3}}}{\kappa^{\frac{1}{3}}}$$

*and, for any $0 < \varepsilon < 1$,*

$$\mathbb{E}(\Delta_{\mathrm{rad}}(Z(\boldsymbol{f}_\kappa), \varepsilon); \lambda_\kappa) \leq c_2 \, \frac{\log(\kappa+1)}{\varepsilon\kappa}.$$

*In particular,*

$$\lim_{\kappa \to \infty} \mathbb{E}(\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)); \lambda_\kappa) = 0 \quad \text{and} \quad \lim_{\kappa \to \infty} \mathbb{E}(\Delta_{\mathrm{rad}}(Z(\boldsymbol{f}_\kappa), \varepsilon); \lambda_\kappa) = 0.$$

*Proof.* We first estimate the expected value of the angle discrepancy, which is given by the formula

$$\mathbb{E}(\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)); \lambda_\kappa) = \int_{\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})} \Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa))\lambda_\kappa(\boldsymbol{f}) \,\mathrm{d}\mu_\kappa.$$

Consider the Minkowski sum $Q = \sum_{i=1}^n Q_i$, which is a lattice polytope on $\mathbb{R}^n$ of dimension $n$ because of the assumption that the mixed volume of $Q_1, \dots, Q_n$ is positive. For each primitive vector $\boldsymbol{v} \in \mathbb{Z}^n$ which is an inner normal to a facet of $Q$, consider the directional resultant

$$R_{\kappa,\boldsymbol{v}} = \mathrm{Res}_{\mathcal{A}_{\kappa,1}^{\boldsymbol{v}},\dots,\mathcal{A}_{\kappa,n}^{\boldsymbol{v}}} \in \mathbb{Z}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_n],$$

where $\boldsymbol{u}_i$ is a group of $\#\mathcal{A}_{\kappa,i}$ variables. Proposition 3.15(2) implies that its total degree is bounded by $\deg(R_{\kappa,\boldsymbol{v}}) = c_{\boldsymbol{v}}\kappa^{n-1}$ for a constant $c_{\boldsymbol{v}}$ independent of $\kappa$. Its total number of variables is $\#\boldsymbol{\mathcal{A}}_\kappa = \sum_i \mathcal{A}_{\kappa,i} = \sum_i \kappa Q_i \cap \mathbb{Z}^n$. This number can be bounded by $c_3\kappa^n$ for a constant $c_3$ independent of $\kappa$.

Set $\delta_{\kappa,\boldsymbol{v}} = \kappa^{-2n\deg(R_{\kappa,\boldsymbol{v}})}$. Consider the subset $V(R_{\kappa,\boldsymbol{v}})_{\delta_{\kappa,\boldsymbol{v}}} \subset \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ as defined in (4.7) and put

$$U_\kappa = \bigcup_{\boldsymbol{v}} V(R_{\kappa,\boldsymbol{v}})_{\delta_{\kappa,\boldsymbol{v}}},$$

the union being over all primitive inner normal vectors to facets of $Q$. Using the fact that $0 \le \Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)) \le 1$ and the hypothesis that the functions $\lambda_\kappa$ are uniformly bounded, we deduce that

$$0 \le \int_{U_\kappa} \Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa))\lambda_\kappa(\boldsymbol{f}) \,\mathrm{d}\mu_\kappa \le \Big(\sup_{\boldsymbol{f}_\kappa} \lambda_\kappa(\boldsymbol{f}_\kappa)\Big)\mu_d(U_\kappa) \le c_4\mu_\kappa(U_\kappa)$$

for a constant $c_4$ independent of $\kappa$. By Proposition 4.8,

$$\mu_\kappa(U_\kappa) \le \sum_{\boldsymbol{v}} \mu_\kappa(V(R_{\kappa,\boldsymbol{v}})_{\delta_{\kappa,\boldsymbol{v}}}) \le \sum_{\boldsymbol{v}} 15\deg(R_{\kappa,\boldsymbol{v}})\big(\#\boldsymbol{\mathcal{A}}_\kappa\big)^3 \delta_{\kappa,\boldsymbol{v}}^{\frac{2}{\deg(R_{\kappa,\boldsymbol{v}})}} \le 15\Big(\sum_{\boldsymbol{v}} c_{\boldsymbol{v}}\kappa^{n-1}\Big)(c_3\kappa^n)^3\kappa^{-4n} = c_5\kappa^{-1}, \tag{4.10}$$

with $c_5 = 15\big(\sum_{\boldsymbol{v}} c_{\boldsymbol{v}}\big)c_3^3$. Hence, $\mu_\kappa(U_\kappa) \to_\kappa 0$ and so $\int_{U_\kappa} \Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa))\lambda_\kappa(\boldsymbol{f}_\kappa) \,\mathrm{d}\mu_\kappa \to_\kappa 0$ as well.

Let $\boldsymbol{f}_\kappa \in \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}) \setminus U_\kappa$ and choose a representative $\widetilde{\boldsymbol{f}}_\kappa = (\widetilde{f}_{\kappa,1}, \dots, \widetilde{f}_{\kappa,n}) \in \mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa} \setminus \{\boldsymbol{0}\}$ with $\|\widetilde{\boldsymbol{f}}_\kappa\|_2 = 1$. By Proposition 3.15(2), $\eta(\boldsymbol{f}_\kappa) = \eta(\widetilde{\boldsymbol{f}}_\kappa)$. Note that the Minkowski sum $\sum_i \kappa Q_i$ coincides with $\kappa Q$. Hence, the only non trivial directional resultants of the family of finite sets $\mathcal{A}_{\kappa,1}, \dots, \mathcal{A}_{\kappa,n}$ are those of the form $R_{\kappa,\boldsymbol{v}}$ as considered above. As before, let $\Delta^n \subset \mathbb{R}^n$ be the standard $n$-simplex. Choose $e \ge 1$ and $\boldsymbol{b}_i \in \mathbb{Z}^n$ such that $Q_i \subset e\Delta^n + \boldsymbol{b}_i$ for all $i$. Hence, $\kappa\mathcal{A}_{\kappa,i} \subset \kappa e Q_i + \kappa\boldsymbol{b}_i$ for all $i$. Proposition 3.15(3) implies that there is a constant $c_6$ independent of $\kappa$ such that

$$\begin{aligned}
\eta(\boldsymbol{f}_\kappa) &\le \frac{1}{\kappa^n\,\mathrm{MV}(\boldsymbol{Q})}\Big((\kappa e)^{n-1}(n+\sqrt{n})\sum_{i=1}^n \log\|\widetilde{f}_{\kappa,i}\|_{\sup} + \frac{1}{2}\sum_{\boldsymbol{v}} \|\boldsymbol{v}\|_2 \log^+ |R_{\kappa,\boldsymbol{v}}(\boldsymbol{f}_\kappa^{\boldsymbol{v}})^{-1}|\Big)\\
&\le \frac{1}{\kappa^n\,\mathrm{MV}(\boldsymbol{Q})}\Big((\kappa e)^{n-1}(n+\sqrt{n})\sum_{i=1}^n \log(\#\mathcal{A}_{\kappa,i}) + n\sum_{\boldsymbol{v}} \|\boldsymbol{v}\|_2 \deg(R_{\kappa,\boldsymbol{v}})\log(\kappa)\Big)\\
&\le c_6\frac{\log(\kappa+1)}{\kappa},
\end{aligned}$$

the second and fourth sums being over the primitive inner normals $\boldsymbol{v}$ to the facets of $Q$. Here, we used the fact that $\|\widetilde{f}_{\kappa,i}\|_{\sup} \leq \#\mathcal{A}_{\kappa,i}$ for $\widetilde{\boldsymbol{f}}_{\kappa}$ in the unit sphere of $\mathbb{C}^{\boldsymbol{\mathcal{A}}_{\kappa}}$, the definition of the set $U_{\kappa}$, the bound $\#\mathcal{A}_{\kappa,i} \leq \#\boldsymbol{\mathcal{A}}_{\kappa} \leq c_3\kappa^n$ and the inequality $\deg(R_{\kappa,\boldsymbol{v}}) \leq c_{\boldsymbol{v}}\kappa^{n-1}$ that we explained before.

Using Theorem 1.4 and the fact that the function $t^{\frac{1}{3}}\log\left(\frac{\alpha}{t}\right)^{\frac{n-1}{3}}$ is increasing for small values of $t > 0$, we deduce that, for $\boldsymbol{f}_{\kappa} \in \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_{\kappa}}) \setminus U_{\kappa}$,

$$\begin{aligned}\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_{\kappa})) &\leq c_7\eta(\boldsymbol{f}_{\kappa})^{\frac{1}{3}}\log\left(\frac{c_8}{\eta(\boldsymbol{f}_{\kappa})}\right)^{\frac{2}{3}(n-1)}\\ &\leq c_9\Big(\frac{\log(\kappa+1)}{\kappa}\Big)^{\frac{1}{3}}\log\left(\frac{\kappa}{\log(\kappa+1)}\right)^{\frac{2}{3}(n-1)} \leq c_{10}\,\frac{\log(\kappa+1)^{\frac{2}{3}n-\frac{1}{3}}}{\kappa^{\frac{1}{3}}}\end{aligned}$$

for suitable constants $c_7$, $c_8$, $c_9$ and $c_{10}$. This proves the first part of the statement. For the radius discrepancy, we proceed in a similar way: given $\varepsilon > 0$, we write $\mathbb{E}(\Delta_{\mathrm{rad}}(\boldsymbol{f}_{\kappa},\varepsilon);\lambda_{\kappa})$ as an integral, which we split into two parts. We bound the first using that $0 \leq \Delta_{\mathrm{rad}}(\boldsymbol{f}_{\kappa},\varepsilon) \leq 1$ and the estimate (4.10), while the second integral can be bounded by applying Theorem 1.4. □

*Proof of Theorem 1.8.* The proof is similar to the one given for Theorem 1.7. Write $\nu_{\kappa} = \frac{\mathbb{E}(Z(\boldsymbol{f}_{\kappa});\lambda_{\kappa})}{\kappa^n\,\mathrm{MV}(\boldsymbol{Q})}$ for short, where $\mathbb{E}(Z(\boldsymbol{f}_{\kappa});\lambda_{\kappa})$ is the expected zero density measure of $\boldsymbol{f}_{\kappa}$ . To prove the statement, it is enough to show that, for all subsets $U$ as in (4.3),

$$\lim_{\kappa\to\infty}\nu_{\kappa}(U) = \nu_{\mathrm{Haar}}(U\cap(S^1)^n) = \begin{cases}\prod_{j=1}^n \frac{\beta_j-\alpha_j}{2\pi} & \text{if } U\cap(S^1)^n \neq \emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

If $U\cap(S^1)^n = \emptyset$, then there exists $\varepsilon > 0$ such that

$$\deg(Z(\boldsymbol{f}_{\kappa})|_U) \leq \deg(Z(\boldsymbol{f}_{\kappa}))\Delta_{\mathrm{rad}}(\boldsymbol{f}_{\kappa},\varepsilon) \leq \kappa^n\,\mathrm{MV}(\boldsymbol{Q})\Delta_{\mathrm{rad}}(\boldsymbol{f}_{\kappa},\varepsilon)$$

and so $\nu_{\kappa}(U) \leq \mathbb{E}(\Delta_{\mathrm{rad}}(\boldsymbol{f}_{\kappa},\varepsilon);\lambda_{\kappa})$. Theorem 4.9 then implies that $\lim_{\kappa\to\infty}\nu_{\kappa}(U) = 0 = \nu_{\mathrm{Haar}}(U)$.

If $U\cap(S^1)^n \neq \emptyset$, we set $\overline{U} = \{\boldsymbol{z} \mid \alpha_j \leq \arg(z_j) \leq \beta_j \text{ for all } j\}$, and then we have

$$\nu_{\kappa}(U) - \prod_{j=1}^n\frac{\beta_j-\alpha_j}{2\pi} = \Big(\nu_{\kappa}(\overline{U}) - \prod_{j=1}^n\frac{\beta_j-\alpha_j}{2\pi}\Big) - \nu_{\kappa}(\overline{U}\setminus U).$$

Set $R_{\kappa} = \prod_{\boldsymbol{v}} R_{\kappa,\boldsymbol{v}}$ for the product of the directional resultants of $\mathcal{A}_1,\dots,\mathcal{A}_n$. Then

$$\begin{aligned}\Big|\nu_{\kappa}(\overline{U}) - \prod_{j=1}^n\frac{\beta_j-\alpha_j}{2\pi}\Big| &= \int_{\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_{\kappa}})\setminus V(R_{\kappa})}\Big|\frac{\deg(Z(\boldsymbol{f}_{\kappa})_{\boldsymbol{\alpha},\boldsymbol{\beta}})}{\kappa^n\,\mathrm{MV}(\boldsymbol{Q})} - \prod_{j=1}^n\frac{\beta_j-\alpha_j}{2\pi}\Big|\lambda_{\kappa}(\boldsymbol{f}_{\kappa})\,\mathrm{d}\mu_{\kappa}\\ &\leq \int_{\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_{\kappa}})}\Delta_{\mathrm{ang}}(\boldsymbol{f}_{\kappa})\lambda_{\kappa}(\boldsymbol{f}_{\kappa})\,\mathrm{d}\mu_{\kappa}.\end{aligned}$$

We have that $\overline{U}\setminus U$ is a union of a finite number of subsets $U_l$ of the form (4.3) such that $U_l\cap(S^1)^n = \emptyset$ for all $l$. By the previous considerations, $\lim_{\kappa\to\infty}\nu_{\kappa}(U_l) = 0$ and so $\lim_{\kappa\to\infty}\nu_d(\overline{U}\setminus U) = 0$. Theorem 4.9 then implies that

$$\lim_{\kappa\to\infty}\nu_{\kappa}(U) = \lim_{\kappa\to\infty}\nu_{\kappa}(\overline{U}) = \prod_{j=1}^n\frac{\beta_j-\alpha_j}{2\pi} = \nu_{\mathrm{Haar}}(U).$$

□

**Remark 4.11.** It is not clear to us whether the upper bound in Proposition 4.8 for the volume of the set $V(f)_\delta$ is sharp or not. It would be interesting to clarify this point, as a qualititive improvement on this bound might enlarge the range of applicability of theorems 4.9 and 1.8.

**Remark 4.12.** In some situations, it might be interesting to consider probability distributions on the complex linear space $\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}$ rather than on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$. For a point $\boldsymbol{f}_\kappa \in \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$, the associated cycle $Z(\boldsymbol{f}_\kappa)$ does not depend on the choice of a representative in $\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}$ for this point and, *a fortiori*, the same holds for the angle and radius discrepancies of $Z(\boldsymbol{f}_\kappa)$. Hence, one might consider random variables on $\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}$ arising from this cycle as random variables on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$, by applying Federer's coarea formula (see for instance [BP07, Theorem 20]).

In precise terms, the normal Jacobian of the map $\varpi\colon \mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa} \setminus \{\boldsymbol{0}\} \to \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ with respect to the Euclidean structure on $\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa} \setminus \{\boldsymbol{0}\}$ and the Fubini-Study one on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ is given, for $\boldsymbol{g} \in \mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa} \setminus \{\boldsymbol{0}\}$, by

$$\mathrm{NJ}_{\boldsymbol{g}}\, \varpi = \|\boldsymbol{g}\|^{-2N_\kappa}$$

with $N_\kappa = \#\boldsymbol{\mathcal{A}}_\kappa - 1$. Given a probability density function $\lambda_\kappa\colon \mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa} \to \mathbb{R}$, one might derive a corresponding probability density function on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa})$ by integrating along the fibers of $\varpi$ as

$$\lambda_\kappa(\boldsymbol{f}_\kappa) = \frac{\pi^{N_\kappa}}{N_\kappa!} \int_{\varpi^{-1}(\boldsymbol{f}_\kappa)} \Lambda_\kappa(\boldsymbol{g}) \|\boldsymbol{g}\|_2^{2N_\kappa}\, \mathrm{d}\varpi^{-1}(\boldsymbol{f}_\kappa), \tag{4.13}$$

where $\mathrm{d}\varpi^{-1}(\boldsymbol{f}_\kappa)$ is the volume form of the fiber $\varpi^{-1}(\boldsymbol{f}_\kappa)$. The probability distribution given by $\Lambda_\kappa$ of, for instance, the angle discrepancy, can then be computed, for any Borel subset $I \subset [0,1]$, as

$$\mathrm{Prob}(\Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)) \in I; \Lambda_\kappa) = \int_{\Delta_{\mathrm{ang}}^{-1}(I)} \lambda_\kappa(\boldsymbol{f}_\kappa)\, \mathrm{d}\mu_\kappa,$$

with $\Delta_{\mathrm{ang}}^{-1}(I) = \{\boldsymbol{f}_\kappa \in \mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}) \mid \Delta_{\mathrm{ang}}(Z(\boldsymbol{f}_\kappa)) \in I\}$. This is a consequence of the coarea formula.

**Example 4.14.** Let $\boldsymbol{f}_\kappa = (f_{\kappa,1}, \dots, f_{\kappa,n})$ be a random system of Laurent polynomials with $\mathrm{supp}(f_{\kappa,i}) \subset \kappa Q_i$ whose coefficients $\{f_{\kappa,i,\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathcal{A}_{\kappa,i}}$ are independent complex Gaussian random variables with mean 0 and variance 1. This is a probability distribution on $\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}$ whose density function is defined, for $\boldsymbol{f}_\kappa \in \mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa}$, as

$$\Lambda_\kappa(\boldsymbol{f}_\kappa) = \prod_{i=1}^{n} \prod_{\boldsymbol{a}\in\mathcal{A}_{\kappa,i}} \frac{1}{\pi}\, \mathrm{e}^{-|f_{\kappa,i,\boldsymbol{a}}|^2} = \frac{1}{\pi^{\#\boldsymbol{\mathcal{A}}_\kappa}}\, \mathrm{e}^{-\|\boldsymbol{f}_\kappa\|_2^2}.$$

The random cycle $Z(\boldsymbol{f}_\kappa)$ might be described by a probability distribution on $\mathbb{P}(\mathbb{C}^{\boldsymbol{\mathcal{A}}_\kappa-})$. The corresponding density function is the constant function $\lambda_\kappa = 1$. This can be seen by computing the integral along the fibers (4.13), or simply by observing that $\Lambda_\kappa$ is a function of the radius $\|\boldsymbol{f}_\kappa\|_2$.

Theorem 1.8 implies then that the sequence of roots of $\boldsymbol{f}_\kappa$ converge weakly to the equidistribution on $(S^1)^n$ when $\kappa \to \infty$. In this way, we recover a result of Bloom and Shiffman [BS07, Example 3.5].

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