THE GEOMETRY OF THE FLEX LOCUS OF A HYPERSURFACE

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Abstract. We give a formula in terms of multidimensional resultants for an equation for the flex locus of a projective hypersurface, generalizing a classical result of Salmon for surfaces in $\mathbb{P}^3$. Using this formula, we compute the dimension of this flex locus, and an upper bound for the degree of its defining equations. We also show that, when the hypersurface is generic, this bound is reached, and that the generic flex line is unique and has the expected order of contact with the hypersurface.

1. Introduction

A point of a projective variety is a flex point if there is a line with order of contact with the variety at this point higher than expected. It is a generalization of the notion of inflexion point of a curve. The study of the flex locus of curves and surfaces is a classical subject of geometry from the XIXth century, treated by Monge, Salmon and Cayley, among others. Currently, there is an increasing interest in this object in low dimensions, mainly due to its applications in incidence geometry [Tao14, Kat14, GK15, Kol15, EH16, SS18, GZ18].

In this text, we study the geometry of the flex locus of a hypersurface of a projective space of arbitrary dimension. Before explaining our results, we introduce some notation and summarize the previous. Let K be an algebraically closed field of characteristic zero, $\mathbb{P}^n$ the projective space over K of dimension $n \geq 1$, and V a hypersurface of $\mathbb{P}^n$ of degree $d \geq 1$. A point $p \in V$ is a *flex point* if there is a line with order of contact at least $n+1$ with the hypersurface V at the point p, and any such line is called a *flex line* (Definition 3.3). The *flex locus* of V is the set of all the flex points of V.

An important result in this context is the so-called Monge-Salmon-Cayley theorem for surfaces in $\mathbb{P}^3$, see for instance [Tao14, Kol15], generalized by Landsberg to the higher dimensional case [Lan99, Theorem 3]. It states that if the hypersurface V is irreducible, then it is ruled if and only if all of its points are flexes.

A hypersurface of degree less than n is necessarily ruled (Proposition 3.6) and its flex locus is the whole hypersurface. Hence, one restricts the study of the flex locus to the case $d \geq n$.

For a plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 2$, a point $p \in C$ is an inflexion point if and only if the determinant of the Hessian matrix of the defining polynomial of C vanishes at p. This implies that the flex locus of C is defined by a polynomial of degree $3d-6$. Hence if C contains no line, then it has at most $3d^2-6d$ inflexion points, by Bézout theorem.

For a surface $S \subset \mathbb{P}^3$ of degree $d \geq 3$, an old result of Salmon states that there is a homogeneous polynomial in $K[x_0, x_1, x_2, x_3]$ of degree $11d-24$ defining its flex locus [Sal65, Article 588, pages 277–278], see also [EH16, §11.2.1]. If S has no ruled

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component, then this result together with the Monge-Salmon-Cayley theorem and Bézout's theorem imply that the flex locus is a curve of S of degree at most $11d^2-24d$.

We first address the problem of computing the dimension, and the degree of both the defining equations and the flex locus. Let $\boldsymbol{x}=\{x_0,\dots,x_n\}$ be a set of $n+1$ variables and $f_V\in K[\boldsymbol{x}]$ a squarefree homogeneous polynomial defining V. Let t be another variable and $\boldsymbol{y}=\{y_0,\dots,y_n\}$ a further set of $n+1$ variables. Then we consider the family of bihomogeneous polynomials $f_{V,k}$, $k=0,\dots,d$, in $K[\boldsymbol{x},\boldsymbol{y}]$ determined by the expansion

$$f_V(\boldsymbol{x}+t\boldsymbol{y})=\sum_{k=0}^{d} f_{V,k}(\boldsymbol{x},\boldsymbol{y})\frac{t^k}{k!}.$$

Our first main result gives an equation for the flex locus of V in terms of multivariate resultants.

Theorem 1.1. *There is a homogeneous polynomial $\rho_V\in K[\boldsymbol{x}]$ with*

$$\deg(\rho_V)=d\sum_{k=1}^{n}\frac{n!}{k}-(n+1)!$$

defining the flex locus of V. It is uniquely determined modulo f_V by the condition

$$\mathrm{Res}^{\boldsymbol{y}}(f_{V,1}(\boldsymbol{x},\boldsymbol{y}),\dots,f_{V,n}(\boldsymbol{x},\boldsymbol{y}),\ell(\boldsymbol{y}))\equiv \ell^{n!}\rho_V \mod f_V,$$

for any linear form $\ell\in K[\boldsymbol{x}]$, where $\mathrm{Res}^{\boldsymbol{y}}$ denotes the resultant of $n+1$ homogeneous polynomials in the variables $\boldsymbol{y}$.

This result recovers the previous degree computations for the polynomial defining the flex locus of a plane curve or of a surface in $\mathbb{P}^3$. It also allows us to give a scheme structure to the flex locus: we define the *flex scheme* $\mathrm{Flex}(V)$ as the subscheme of $\mathbb{P}^n$ defined by the homogeneous polynomials f_V and ρ_V (Definition 3.11). This scheme does not depend on the choice f_V, unique up to a nonzero scalar factor, nor on that of ρ_V, unique modulo f_V. Thus, the flex locus of V is the reduced scheme associated to $\mathrm{Flex}(V)$.

The next corollary is a direct consequence of Theorem 1.1 and Landsberg's theorem generalizing the Monge-Salmon Cayley theorem [Lan99, Theorem 3].

Corollary 1.2. *If V has no ruled irreducible components, then $\mathrm{Flex}(V)$ is a complete intersection subscheme of $\mathbb{P}^n$ of dimension $n-2$ and of degree*

$$\deg(\mathrm{Flex}(V))=d^2\sum_{k=1}^{n}\frac{n!}{k}-d\,(n+1)!.$$

In particular, the flex locus of V is set-theoretically defined by equations of degree at most $\max(d,d\sum_{k=1}^{n}\frac{n!}{k}-(n+1)!)$, and its degree, as an algebraic set, is at most $d^2\sum_{k=1}^{n}\frac{n!}{k}-d(n+1)!$.

Set $\mathcal{L}_V$ for the union of the lines contained in V. When $d=n$, a flex line of V at a point $p\in V$ has order of contact at least $n+1$ at this point, and so it is necessarily contained in V by Bézout theorem. Hence in this case, $\mathcal{L}_V$ coincides with the flex locus of V.

Corollary 1.3. *Let V be a hypersurface of $\mathbb{P}^n$ of degree n without ruled irreducible component. Then $\mathcal{L}_V$ is a ruled subvariety of V of dimension $n-2$ and of degree at most*

$$n^3\,(n-1)!\sum_{k=2}^{n-1}\frac{1}{k}.$$

Our second main result ensures that the bound for the degree of the flex locus is sharp, and that other expected properties hold true in the generic case. These properties of generic hypersurfaces and flex are proven using resultant theory. These aspects were not considered in the original work of Salmon, and the obtained results are new in every dimension.

Theorem 1.4. *Let V be a generic hypersurface of $\mathbb{P}^n$ of degree $d \geq n$. Then*

(1) $\mathrm{Flex}(V)$ *is a reduced subscheme (that is, a subvariety) of V of dimension $n-2$;*
(2) *for a generic flex point p of V, there is a unique flex line containing it. If $d = n$, then this line is contained in V, whereas if $d > n$, then its order of contact with V at p is exactly $n+1$.*

For a cubic surface S in $\mathbb{P}^3$, Salmon's degree bound is $11\cdot 3-24 = 9$. If S is smooth, it contains 27 lines and their union is the complete intersection of S with a surface of degree 9. The next result gives an analogous result for generic hypersurfaces of $\mathbb{P}^n$ of degree n. It is a direct consequence of Theorem 1.4 and Corollary 1.3.

Corollary 1.5. *Let V be a generic hypersurface of $\mathbb{P}^n$ of degree n. Then $\mathcal{L}_V$ is a ruled subvariety of V of dimension $n-2$ of degree equal to*

$$n^3\,(n-1)!\sum_{k=2}^{n-1}\frac{1}{k},$$

complete intersection of V with a hypersurface of degree $n^2\,(n-1)!\,\sum_{k=2}^{n-1}\frac{1}{k}$.

Salmon's theorem for surfaces has been revisited several times. In particular, in the recent book [EH16], the authors reprove it by performing suitable computations in the Chow ring of a Grassmaniann. Our proof of this result proceeds by identifying lines with points of $\mathbb{P}^n \times \mathbb{P}^n$ outside the diagonal, in the spirit of Salmon's original approach; see [Sal65, Articles 473 and 588, pages 94–95 and 277–278] and Remark 3.13. Although this seems less natural from the point of view of intersection theory, it nevertheless allows us to find explicit equations for the flex locus using resultants and to prove it in a more general setting; see Theorem 1.1.

The paper is organized in the following way. In Section 2 we review the definition and properties of multidimensional resultants that will be used in the sequel. The proof of Theorem 1.1 is given in Section 3, whereas in Section 4 and Section 5 we show that the flex subscheme is generically reduced and that the generic flex line is unique and has the expected order of contact, thus proving Theorem 1.4.

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2. Preliminaries on resultants

The resultant of a family of homogeneous multivariate polynomials plays a central role throughout this text. Therefore, in this section we briefly review this notion and some of its basic properties. We refer to [CLO05, Jou91, GKZ94] for the proofs and more details.

We denote by $\mathbb{N}$ the set of nonnegative integers and by K an algebraically closed field of characteristic zero. Boldface symbols indicate finite sets or sequences, where the type and number should be clear from the context. For instance, for $n \in \mathbb{N}$ we denote by $\boldsymbol{y}$ the set of variables $\{y_0, \dots, y_n\}$, so that $K[\boldsymbol{y}] = K[y_0, \dots, y_n]$.

Let $\boldsymbol{d} = (d_0, \dots, d_n) \in \mathbb{N}^{n+1}$. For $i = 0, \dots, n$, we consider the general homogeneous polynomial of degree d_i in the variables $\boldsymbol{y}$ given by

$$F_i = \sum_{|\boldsymbol{a}|=d_i} c_{i,\boldsymbol{a}} \boldsymbol{y}^{\boldsymbol{a}},$$

the sum being over the vectors of $\boldsymbol{a} = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$ of length $|\boldsymbol{a}| = \sum_{j=0}^{n} a_j = d_i$, and where each $c_{i,\boldsymbol{a}}$ is a variable and $\boldsymbol{y}^{\boldsymbol{a}}$ stands for the monomial $\prod_{j=0}^{n} y_j^{a_j}$.

For each i, set $\boldsymbol{c}_i = \{c_{i,\boldsymbol{a}} \mid \boldsymbol{a} \in \mathbb{N}^{n+1}, |\boldsymbol{a}| = d_i\}$ for the set of $\binom{d_i+n}{n}$ variables corresponding to the coefficients of F_i, and $A = \mathbb{Z}[\boldsymbol{c}_0, \dots, \boldsymbol{c}_n]$ for the universal ring of coefficients. As usual, given $P \in A$ and a system of homogeneous polynomials $g_i \in K[\boldsymbol{y}]$ of degree d_i, $i = 0, \dots, n$, we write

$$P(g_0, \dots, g_n) \in K$$

for the evaluation of P in the coefficients of the g_i's.

Denote by I and by $\mathfrak{m}$ the ideals of $A[\boldsymbol{y}]$ respectively defined by $F_0, \dots, F_n$ and by $y_0, \dots, y_n$. The *elimination ideal* of the system $\boldsymbol{F} = (F_0, \dots, F_n)$ is the ideal of A defined by

$$E_{\boldsymbol{d}} = \{P \in A \mid \exists k \in \mathbb{N} \text{ with } P\,\mathfrak{m}^k \subset I\}.$$

It is a principal ideal, and the *resultant* of $\boldsymbol{F}$, denoted by $\mathrm{Res}_{\boldsymbol{d}}$, is defined as its unique generator satisfying the additional condition

$$\mathrm{Res}_{\boldsymbol{d}}(y_0^{d_0}, \dots, y_n^{d_n}) = 1.$$

It is an irreducible polynomial in the ring A that is homogeneous of degree $\prod_{j \neq i} d_j$ in each set of variables $\boldsymbol{c}_i$, $i = 0, \dots, n$.

The resultant also verifies the following formula for the descent of dimension [Jou91, Lemme 4.8.9 and §5.7].

Proposition 2.1. *With notation as above,*

$$\mathrm{Res}_{(d_0,\dots,d_{n-1},d_n)}(F_0, \dots, F_{n-1}, y_n^{d_n}) = \mathrm{Res}_{(d_0,\dots,d_{n-1})}^{d_n},$$

where $\mathrm{Res}_{(d_0,\dots,d_{n-1})}$ *denotes the resultant of* n *general homogeneous polynomials in* $A[y_0, \dots, y_{n-1}]$ *of respective degrees* $d_0, \dots, d_{n-1}$.

The resultant satisfies the *Poisson formula* that we state below, see [Jou91, Proposition 2.7] or [CLO05, Theorem 3.4, Chapter 3] for its proof.

Proposition 2.2. *Let $g_0, g_0' \in K[\boldsymbol{y}]$ be homogeneous polynomials of degree d_0, and $g_1, \dots, g_n \in K[\boldsymbol{y}]$ homogeneous polynomials of respective degrees $d_1, \dots, d_n$ with a finite number of common zeros in $\mathbb{P}^n$. For each common zero $\eta \in \mathbb{P}^n$ of $g_1, \dots, g_n$, let m_η denote its multiplicity. Then*

$$\operatorname{Res}_{\boldsymbol{d}}(g_0, g_1, \dots, g_n) \prod_{\eta} g_0'(\eta)^{m_\eta} = \operatorname{Res}_{\boldsymbol{d}}(g_0', g_1, \dots, g_n) \prod_{\eta} g_0(\eta)^{m_\eta},$$

both products being over the set of common zeros of $g_1, \dots, g_n$ in $\mathbb{P}^n$.

A fundamental property of resultants is that their vanishing characterizes the systems of $n+1$ homogeneous polynomials in $n+1$ variables that are degenerate, in the sense that their zero set in $\mathbb{P}^n$ is nonempty. Precisely, a system of homogeneous polynomials $g_0, \dots, g_n \in K[\boldsymbol{y}]$ of respective degrees $d_0, \dots, d_n$, has a common zero in $\mathbb{P}^n$ if and only if $\operatorname{Res}_{\boldsymbol{d}}(g_0, \dots, , g_n) = 0$.

The following result gives a criterion to decide if a such degenerate system has a unique zero and, if it does, allows to compute it, see [Jou91, Lemma 4.6.1] or [JKSS04, Corollary 4.7] for its proof.

Proposition 2.3. *Let $g_0, \dots, g_n \in K[\boldsymbol{y}]$ be homogeneous polynomials of respective degrees $d_0, \dots, d_n$. Suppose that $\operatorname{Res}_{\boldsymbol{d}}(g_0, \dots, , g_n) = 0$ and that there is $0 \le i_0 \le n$ and $\boldsymbol{a}_0 \in \mathbb{N}^{n+1}$ with $|\boldsymbol{a}_0| = d_{i_0}$ such that*

$$\frac{\partial \operatorname{Res}_{\boldsymbol{d}}}{\partial c_{i_0, \boldsymbol{a}_0}}(g_0, \dots, , g_n) \neq 0.$$

Then the zero set of $g_0, \dots, g_n$ in $\mathbb{P}^n$ consists of a single point η, and for $i = 0, \dots, n$,

$$(\eta^{\boldsymbol{a}})_{|\boldsymbol{a}|=d_i} = \left(\frac{\partial \operatorname{Res}_{\boldsymbol{d}}}{\partial c_{i,\boldsymbol{a}}}(g_0, \dots, , g_n) \right)_{|\boldsymbol{a}|=d_i} \in \mathbb{P}^{\binom{d_i+n}{n}}$$

where the coordinates of these projective points are indexed by the vectors $\boldsymbol{a} \in \mathbb{N}^{n+1}$ with $|\boldsymbol{a}| = d_i$.

3. The equation of the flex locus

In this section, we obtain an explicit equation for the flex locus of a projective hypersurface by means of resultants. Using this equation, we define the flex scheme and we compute its dimension, the degree of its defining equations and its degree, thus giving the proof of Theorem 1.1.

Definition 3.1. Let V be a subvariety of $\mathbb{P}^n$ and p a point V. For a line L of $\mathbb{P}^n$ containing p, its *order of contact* with V at p is defined as

$$\operatorname{ord}_p(V, L) = \dim_K(\mathcal{O}_{L,p}/\iota^* \mathcal{I}_V),$$

where $\mathcal{O}_{L,p}$ is the local ring of L at p, $\mathcal{I}_V$ the ideal sheaf of V, and $\iota \colon L \hookrightarrow \mathbb{P}^n$ the inclusion map.

The order of contact of a line is either a positive integer or $+\infty$. We have that $\operatorname{ord}_p(V, L) = 1$ if and only if L intersects V transversally at p, and $\operatorname{ord}_p(V, L) = +\infty$ if and only if L is contained in V.

For the rest of this section, we assume that V is a (non necessarily irreducible) hypersurface of degree $d \ge 1$. Fix then a defining polynomial f_V of V, that is, a homogeneous polynomial in $K[\boldsymbol{x}] = K[x_0, \dots, x_n]$ of minimal degree such that V

coincides with $Z(f_V)$, the set of zeros of f_V in $\mathbb{P}^n$. Such a polynomial is squarefree, and unique up to a nonzero scalar factor.

The next lemma translates the notion of order of contact with the hypersurface V into algebraic terms. Given a variable t, we denote by val_t the t-adic valuation in the local ring $K[t]_{(0)} \simeq \mathcal{O}_{\mathbb{A}^1,0}$: for $h \in K[t]_{(0)}$ written as h_1/h_2 with $h_1, h_2 \in K[t]$ and h_2 not vanishing at the point 0, $\mathrm{val}_t(h)$ is defined as the least exponent appearing in the nonzero monomials of the polynomial h_1.

Lemma 3.2. *Let $p \in V$ and L a line of $\mathbb{P}^n$ containing p. Let $\varphi\colon \mathbb{A}^1 \to \mathbb{P}^n$ be an affine map parameterizing a neighborhood of p in L, and such that $\varphi(0) = p$. Write $\varphi = (\ell_0, \dots, \ell_n)$ with $\ell_i \in K[t]$ an affine polynomial, $i = 0, \dots, n$. Then*

$$\mathrm{ord}_p(V, L) = \mathrm{val}_t(f_V(\ell_0, \dots, \ell_n)).$$

Proof. Up to a reordering of the homogeneous coordinates of $\mathbb{P}^n$, we can suppose that $\ell_0(0) \neq 0$. Let $\widetilde{f}_V$ denote the dehomogenization of f_V in the chart $(x_0 \neq 0) \simeq \mathbb{A}^n$. Using the relation

$$f_V = x_0^d \, \widetilde{f}_V\Big(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\Big)$$

and the fact that $\mathrm{val}_t(\ell_0) = 0$, we obtain

$$\mathrm{val}_t(f_V(\ell_0, \dots, \ell_n)) = \mathrm{val}_t\Big(\widetilde{f}_V\Big(\frac{\ell_1}{\ell_0}, \dots, \frac{\ell_n}{\ell_0}\Big)\Big) = \mathrm{ord}_p(V, L),$$

where the second equality follows from the definition of the order of contact, and the fact that $\widetilde{f}_V$ is a local equation for the germ of hypersurface (V, p) and φ a parametrization of the germ of line (L, p). □

Definition 3.3. Let $p \in V$. The *order of osculation* of V at p is defined as

$$\mu_p(V) = \sup_L \mathrm{ord}_p(V, L),$$

where the supremum is taken over the lines L of $\mathbb{P}^n$ containing p. The point p is a *flex point* of V whenever $\mu_p(V) \geq n + 1$. A line L with order of contact with V at p at least $n + 1$ is called a *flex line.*

Consider again the group of variables $\boldsymbol{y} = \{y_0, \dots, y_n\}$ and a further variable t, and let $f_{V,k}$, $k = 0, \dots, d$, be the family of polynomials in $K[\boldsymbol{x}, \boldsymbol{y}]$ determined by the expansion

$$f_V(\boldsymbol{x} + t\boldsymbol{y}) = \sum_{k=0}^{d} f_{V,k}(\boldsymbol{x}, \boldsymbol{y}) \frac{t^k}{k!}. \tag{3.1}$$

For $k = 0, \dots, d$,

$$f_{V,k}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{0 \leq i_1, \dots, i_k \leq n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(\boldsymbol{x})\, y_{i_1} \cdots y_{i_k}.$$

In particular, $f_{V,k}$ is bihomogeneous of bidegree $(d - k, k)$.

For a point $p \in \mathbb{P}^n$ and each $k \in \mathbb{N}$, consider the subvariety of $\mathbb{P}^n$ defined as

$$Z_p^k = \{q \in \mathbb{P}^n \mid f_{V,1}(p, q) = \cdots = f_{V,k}(p, q) = 0\}.$$

The next lemma shows that the order of osculation of V at p can be read from the dimensions of these subvarieties.

Lemma 3.4. *Let $p \in V$.*

(1) *For each $k \in \mathbb{N}$, the subvariety $Z_p^k \subset \mathbb{P}^n$ is a cone centered at p, union of the lines having order of contact with V at p greater than k.*
(2) *The order of osculation of V at p is the least $k \in \mathbb{N}$ such that $Z_p^k = \{p\}$.*

Proof. Fix $k \in \mathbb{N}$ and choose a representative $\boldsymbol{p} \in K^{n+1} \setminus \{\boldsymbol{0}\}$ of the point $p \in \mathbb{P}^n$. We have that $f_V(\boldsymbol{p} + t\boldsymbol{p}) = (1+t)^d f_V(\boldsymbol{p})$ and so, for all $j \in \mathbb{N}$,

$$f_{V,j}(\boldsymbol{p}, \boldsymbol{p}) = d(d-1)\cdots(d-j) f_V(\boldsymbol{p}) = 0.$$

Hence $p \in Z_p^k$.

Let $q \in \mathbb{P}^n$ be a point different from p and L the unique line containing p and q. This line is parametrized by the affine map $\varphi \colon \mathbb{A}^1 \to \mathbb{P}^n$ defined by $\varphi(t) = \boldsymbol{p} + \boldsymbol{q}\, t$ for any choice of representatives $\boldsymbol{p}, \boldsymbol{q} \in K^{n+1} \setminus \{\boldsymbol{0}\}$ of p and q. Lemma 3.2 combined with the expansion (3.1) implies that the condition $q \in Z_p^k$ is equivalent to $\mathrm{ord}_p(V, L) > k$. Hence q lies in Z_p^k if and only if the line L is contained in this subvariety and has order of contact with V at p greater than k, which proves (1).

By definition, the order of osculation of V at p is the least $k \in \mathbb{N}$ such that there is no line L with $\mathrm{ord}_p(V, L) > k$. Hence (2) is a consequence of (1). □

The following corollary follows directly from Lemma 3.4 and the definition of flex points.

Corollary 3.5. *A point $p \in V$ is a flex point if and only if $Z_p^n \neq \{p\}$ or, equivalently, if and only if $\dim(Z_p^n) \geq 1$.*

The next two propositions are classical, and they are also consequences of Lemma 3.4. We include their proofs for lack of a suitable reference.

Proposition 3.6. *Let $p \in V$. Then either $n \leq \mu_p(V) \leq d$ or there is a line containing p and contained in V. In particular, every hypersurface of degree at most $n-1$ is ruled.*

Proof. For $k \in \mathbb{N}$, the subvariety Z_p^k is defined by k equations. If this subvariety consists of the single point p, then this number of equations k has to be at least n, by Krull's Hauptidealsatz. Lemma 3.4(2) then gives the lower bound $\mu_p(V) \geq n$.

On the other hand, if $k > d$ then $Z_p^k = Z_p^n$ because $f_{V,j} = 0$ for all $j > d$. Hence the Z_p^k's form a sequence of subvarieties that is decreasing with respect to the inclusion, and constant for $k \geq d$. By Lemma 3.4(2), if $Z_p^d = \{p\}$ then $\mu_p(V) \leq d$. Else, by Lemma 3.4(1), each line contained in Z_p^d has an order of contact that is arbitrarily large. By Lemma 3.2, such a line is necessarily contained in V.

To conclude, we observe that the last statement is a direct consequence of the first one. □

Proposition 3.7. *Every singular point of V is a flex point.*

Proof. A point $p \in V$ is singular if and only if $f_{V,1}(p, \boldsymbol{y}) = 0$. Hence Z_p^n is defined by $n-1$ equations. Since this subvariety contains p, it is nonempty and so, by Krull's Hauptidealsatz, its dimension is at least 1. By Corollary 3.5, this point is necessarily a flex. □

For a homogeneous polynomial $g \in K[\boldsymbol{x}]$ of degree $e \geq 1$, we set

$$R_{V,g} = \mathrm{Res}^{\boldsymbol{y}}_{(1,\dots,n,e)}(f_{V,1}(\boldsymbol{x}, \boldsymbol{y}), \dots, f_{V,n}(\boldsymbol{x}, \boldsymbol{y}), g(\boldsymbol{y})) \in K[\boldsymbol{x}], \tag{3.2}$$

where $\mathrm{Res}^{\boldsymbol{y}}_{(1,\dots,n,e)}$ denotes the resultant of $n+1$ homogeneous polynomials in the variables $\boldsymbol{y}$ of respective degrees $1,\dots,n,e$.

Proposition 3.8. *Let $g \in K[\boldsymbol{x}]$ be a homogeneous polynomial of degree $e \geq 1$. Then $R_{V,g}$ defines the flex locus of V in the open subset $\mathbb{P}^n \setminus Z(g)$.*

Proof. Let $p \in V$ such that $g(p) \neq 0$. If $R_{V,g}(p) = 0$ then $Z(g)$ intersects Z_p^n, by the vanishing property of the resultant. Since $p \notin Z(g)$, this implies that $Z_p^n \neq \{p\}$. By Corollary 3.5, p is a flex point. Conversely, suppose that p is a flex point. Since g is not a constant, $Z(g)$ is a hypersurface and, by Corollary 3.5, $\dim(Z_p^n) \geq 1$. Hence $Z(g)$ does intersect Z_p^n and so $R_{V,g}(p) = 0$, as stated. □

The polynomial $R_{V,g}$ gives an equation for the flex locus of V outside the hypersurface $Z(g)$, but might vanish at points in $Z(g)$ that are not flexes. The next result, corresponding to Theorem 1.1 in the introduction, shows that this equation can be replaced by another one defining the flex locus of V in the whole of the projective space.

Theorem 3.9. *There exists a homogeneous polynomial $\rho_V \in K[\boldsymbol{x}]$ with*

$$\deg(\rho_V) = d \sum_{k=1}^{n} \frac{n!}{k} - (n+1)!$$

defining the flex locus of V. It is uniquely determined modulo f_V by the condition that, for any linear form $\ell \in K[\boldsymbol{x}]$,

$$R_{V,\ell} \equiv \ell^{n!} \rho_V \mod f_V. \tag{3.3}$$

To prove it, we need the following auxiliary result.

Lemma 3.10. *Let $g, h \in K[\boldsymbol{x}]$ be two homogeneous polynomials of the same positive degree. Then*

$$h^{n!} R_{V,g} \equiv g^{n!} R_{V,h} \mod f_V.$$

Proof. Let $p \in V$ and $\boldsymbol{p} \in K^{n+1} \setminus \{\boldsymbol{0}\}$ a representative of this point. If p is not a flex, then $Z_p^n = \{p\}$ by Corollary 3.5. By Bézout's theorem, the intersection multiplicity of $f_{V,1}(\boldsymbol{p},\boldsymbol{y}), \dots, f_{V,n}(\boldsymbol{p},\boldsymbol{y})$ at p is $n!$ and hence, by the Poisson formula (Proposition 2.2),

$$h(\boldsymbol{p})^{n!} R_{V,g}(\boldsymbol{p}) = g(\boldsymbol{p})^{n!} R_{V,h}(\boldsymbol{p}). \tag{3.4}$$

On the other hand, if p is flex then Z_p^n has positive dimension, again by Corollary 3.5. This implies that the system $f_{V,1}(\boldsymbol{p},\boldsymbol{y}), \dots, f_{V,n}(\boldsymbol{p},\boldsymbol{y}), G(\boldsymbol{y})$ has a common zero and so $R_{V,g}(p) = 0$ and, similarly $R_{V,h}(p) = 0$. Hence (3.4) reduces to $0 = 0$ in this case. Thus the equality (3.4) holds for every point of V, which implies the statement. □

Proof of Theorem 3.9. Let $\boldsymbol{u} = \{u_0, \dots, u_n\}$ and $\boldsymbol{v} = \{v_0, \dots, v_n\}$ be two sets of $n+1$ variables and consider the linear forms

$$\ell_{\boldsymbol{u}} = \sum_{i=0}^{n} u_i x_i \quad \text{and} \quad \ell_{\boldsymbol{v}} = \sum_{i=0}^{n} v_i x_i.$$

By Lemma 3.10, for every choice of $\boldsymbol{\alpha}, \boldsymbol{\beta} \in K^{n+1} \setminus \{\boldsymbol{0}\}$,

$$R_{V,\ell_{\boldsymbol{\alpha}}(\boldsymbol{x})}\, \ell_{\boldsymbol{\beta}}(\boldsymbol{x})^{n!} \equiv R_{F,\ell_{\boldsymbol{\beta}}(\boldsymbol{x})}\, \ell_{\boldsymbol{\alpha}}(\boldsymbol{x})^{n!} \mod f_V.$$

We deduce that there is a trihomogeneous polynomial $s \in K[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}]$ such that

$$R_{V,\ell_{\boldsymbol{u}}} \ell_{\boldsymbol{v}}^{n!} - R_{V,\ell_{\boldsymbol{v}}} \ell_{\boldsymbol{u}}^{n!} + s f_V = 0. \tag{3.5}$$

The polynomials $\ell_{\boldsymbol{u}}^{n!}, \ell_{\boldsymbol{v}}^{n!}, F$ form a regular sequence in $K[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}]$, and hence the syzygy (3.5) is necessarily a Koszul syzygy. Hence there are trihomogeneous polynomials $\rho_V, \sigma \in K[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}]$ such that

$$R_{V,\ell_{\boldsymbol{u}}} = \ell_{\boldsymbol{u}}^{n!} \rho_V + f_V \sigma.$$

Since $\deg_{\boldsymbol{u}}(R_{V,\ell_{\boldsymbol{u}}}) = n!$ and $\deg_{\boldsymbol{v}}(R_{V,\ell_{\boldsymbol{u}}}) = 0$, we deduce that $\rho_V \in K[\boldsymbol{x}]$. The equality (3.3) is obtained by specializing the variables $\boldsymbol{u}$ into the coefficients of the linear form ℓ.

By this equality (3.3) and Proposition 3.8, ρ_V defines the flex locus of V in the open subset $\mathbb{P}^n \setminus Z(\ell)$. Varying ℓ, we deduce that ρ_V defines the flex locus in the whole of V.

The resultant $\mathrm{Res}^{\boldsymbol{y}}_{(1,\dots,n,1)}$ is a multihomogeneous polynomial and, for $i = 0, \dots, n-1$, its degree in the set of variables $\boldsymbol{c}_i$ corresponding to the coefficients of the ith polynomial is $n!/(i+1)$. Hence

$$\deg_{\boldsymbol{x}}(R_{V,\ell}) = \sum_{i=0}^{n-1} \deg_{\boldsymbol{x}}(f_{V,i+1}) \deg_{\boldsymbol{c}_i}\left(\mathrm{Res}^{\boldsymbol{y}}_{(1,\dots,n,1)}\right) = \sum_{k=1}^{n} (d-k)\frac{n!}{k} = d\sum_{k=1}^{n} \frac{n!}{k} - n \cdot n!$$

Hence $\deg_{\boldsymbol{x}}(\rho_V) = \deg_{\boldsymbol{x}}(R_{V,\ell}) - n! = d \sum_{k=1}^{n} \frac{n!}{k} - (n+1)!$, as stated.

The uniqueness of the polynomial ρ_V satisfying (3.3) follows by considering any linear form ℓ that is not a zero divisor modulo f_V, completing the proof. □

Definition 3.11. The *flex scheme* of V, denoted by $\mathrm{Flex}(V)$, is the subscheme of $\mathbb{P}^n$ defined by the homogeneous polynomials f_V and ρ_V.

This scheme does not depend on the choice of f_V, unique up to a nonzero scalar factor, nor on that of ρ_V, unique modulo f_V. By Theorem 3.9, its support $|\mathrm{Flex}(V)|$, that is, its set of closed points, coincides with the flex locus of V.

Example 3.12. Let C be a plane curve of degree $d \geq 2$, and $f_C \in K[x_0, x_1, x_2]$ its defining polynomial. A computation using the Euler identities shows that, for any linear form ℓ,

$$-(d-1)^2 \mathrm{Res}^{\boldsymbol{y}}_{(1,2,1)}(f_{C,1}(\boldsymbol{x}, \boldsymbol{y}), f_{C,2}(\boldsymbol{x}, \boldsymbol{y}), \ell(\boldsymbol{y})) \equiv \ell^2 \det(\mathrm{H}(f_C)) \mod f_C, \tag{3.6}$$

where $\mathrm{H}(f_C)$ stands for the Hessian matrix of f_C. Thus we recover from Theorem 3.9 the well-known fact that a point $p \in C$ is an inflexion point if and only the determinant of the Hessian matrix of f_C vanishes at p, see for instance [BK86, §7.3, Theorem 1].

Giving a closed form for a canonical representative for ρ_V modulo f_V seems to be a challenge on its own. In the case of curves, we have just seen in Example 3.12 that such a representative is given by the determinant of the Hessian matrix of f_V. For $n = 3$, Salmon also obtained a representative of this polynomial as a determinantal closed formula in terms of covariants, based on an approach by Clebsch [Sal65, Articles 589 to 597]. It would be interesting to generalize these formulae to higher dimensions.

For a surface S in $\mathbb{P}^3$, Theorem 3.9 shows that the flex locus of S is defined by an equation of degree

$$\deg(\rho_S) = d\sum_{k=1}^{3} \frac{3!}{k} - (3+1)! = 11d - 24,$$

recovering the result of Salmon.

Remark 3.13. In the book [Sal65], Salmon studied the flex locus of the surface S by means of elimination theory. His Article 473 in pages 94–95 of *loc. cit.* gives a general method to compute, for three surfaces depending on parameters and satisfying a certain intersection theoretic condition, the degree of the condition so that these surfaces contain a common line. His Article 588 in pages 277–278 of *loc. cit.* applies this degree computation to the three surfaces that arise in the study of the flex locus. In our notation, these three surfaces are those defining the variety Z_p^3 for a point $p \in S$.

4. The flex subscheme of a generic hypersurface

In this section, we show that for a generic hypersurface of $\mathbb{P}^n$ of degree $d \geq n$, the bounds for the flex locus in Corollary 1.2 are sharp, or equivalently that the flex scheme is reduced, and hence that it is equal to the flex locus. The next result corresponds to Theorem 1.4(1) in the introduction.

Theorem 4.1. *Let $d \geq n$ and $f \in K[\boldsymbol{x}]$ a generic homogeneous polynomial of degree d. Then* $\mathrm{Flex}(Z(f))$ *is a reduced subscheme of $Z(f)$ of dimension $n-2$. In particular*

1. *$Z(f)$ has no ruled components;*
2. *the flex locus of $Z(f)$ is the complete intersection of two hypersurfaces of respective degrees d and $d\sum_{k=1}^{n} \frac{n!}{k} - (n+1))!$;*
3. *the degree of the flex locus of $Z(f)$ is equal to $d^2 \sum_{k=1}^{n} \frac{n!}{k} - d(n+1)!$.*

Let $d \geq n$ and consider the general polynomial of degree d in the variables $\boldsymbol{x}$

$$F = \sum_{|\boldsymbol{a}|=d} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}},$$

the sum being over the vectors $\boldsymbol{a} \in \mathbb{N}^{n+1}$ of length d. Put $\boldsymbol{c} = \{c_\alpha\}_{|\alpha|=d}$ for the set of $\binom{n+d}{n}$ variables corresponding to the coefficients of F. Thus, F is an irreducible polynomial in $K[\boldsymbol{c}, \boldsymbol{x}]$, bihomogeneous of bidegree $(1, d)$.

The polynomials $F_k \in K[\boldsymbol{c}, \boldsymbol{x}, \boldsymbol{y}]$, $k = 0, \dots, d$, are determined by the expansion

$$F(\boldsymbol{x} + t\boldsymbol{y}) = \sum_{k=0}^{d} F_k(\boldsymbol{x}, \boldsymbol{y}) \frac{t^k}{k!}. \tag{4.1}$$

Following (3.2), for a linear form $\ell \in K[\boldsymbol{x}]$ we set

$$R_{F,\ell} := R_{Z(F),\ell} = \mathrm{Res}^{\boldsymbol{y}}_{1,\dots,n,1}(F_1(\boldsymbol{x}, \boldsymbol{y}), \dots, F_n(\boldsymbol{x}, \boldsymbol{y}), \ell(\boldsymbol{y})).$$

It is a bihomogeneous polynomial in $K[\boldsymbol{c}, \boldsymbol{x}]$ with bidegree given by

$$\deg_{\boldsymbol{c}}(R_{F,\ell}) = \sum_{k=1}^{n} \frac{n!}{k} \quad \text{and} \quad \deg_{\boldsymbol{x}}(R_{F,\ell}) = d \sum_{k=1}^{n} \frac{n!}{k} - n \cdot n!. \tag{4.2}$$

We first prove the existence of a universal polynomial Φ_d in $K[\boldsymbol{c}, \boldsymbol{x}]$ with the property that, for any hypersurface V of $\mathbb{P}^n$ of degree d, its flex polynomial ρ_V can be obtained as the evaluation of Φ_d at the coefficients of a defining polynomial of V.

Proposition 4.2. *There is a bihomogeneous polynomial $\Phi_d \in K[\boldsymbol{c}, \boldsymbol{x}]$ with*

$$\deg_{\boldsymbol{c}}(\Phi_d) = \sum_{k=1}^{n} \frac{n!}{k} \quad \textit{and} \quad \deg_{\boldsymbol{x}}(\Phi_d) = d \sum_{k=1}^{n} \frac{n!}{k} - (n+1)! \tag{4.3}$$

such that, for any squarefree homogeneous polynomial $f \in K[\boldsymbol{x}]$ of degree d,

$$\rho_{Z(f)}(\boldsymbol{x}) = \Phi_d(f, \boldsymbol{x}). \tag{4.4}$$

It is uniquely determined modulo F in the ring $K[\boldsymbol{c}, \boldsymbol{x}]$ by the condition that, for any linear form $\ell \in K[\boldsymbol{x}]$,

$$R_{F,\ell} \equiv \ell^{n!}\Phi_d \mod F. \tag{4.5}$$

Proof. Adapting the proof of Theorem 3.9 to the present situation, we can show the existence of a bihomogeneous polynomial $\Phi_d \in K[\boldsymbol{c}, \boldsymbol{x}]$ satisfying the congruence (4.5) for any linear form $\ell \in K[\boldsymbol{x}]$. The formulae (4.3) for the degrees of Φ_d in the variables $\boldsymbol{c}$ and $\boldsymbol{x}$ follow from this congruence and the corresponding formulae for $R_{F,\ell}$ in (4.2).

For a squarefree homogeneous polynomial $f \in K[\boldsymbol{x}]$ of degree d, the congruence (4.5) can be evaluated into the coefficients of f, specializing to

$$R_{Z(f),\ell} \equiv \ell^{n!}\Phi_d(f) \mod f.$$

The equality (4.4) then follows from the unicity of $\rho_{Z(f)}$ modulo f. □

Remark 4.3. By the definition of the resultant, as recalled in Section 2, the universal polynomial Φ_d can be chosen as a primitive polynomial with integer coefficients, that is as an irreducible polynomial in $\mathbb{Z}[\boldsymbol{c}, \boldsymbol{x}]$.

Lemma 4.4. *The polynomial $R_{F,y_0}(1, 0, \dots, 0)$ is irreducible in $K[\boldsymbol{c}]$.*

Proof. Set for short $R = R_{F,y_0}$. By Proposition 2.1,

$$R = \mathrm{Res}^{\boldsymbol{y}'}_{1,\dots,n}(F_1(\boldsymbol{x}, 0, \boldsymbol{y}'), \dots, F_n(\boldsymbol{x}, 0, \boldsymbol{y}')) \tag{4.6}$$

where $\boldsymbol{y}'$ denotes the set of variables $\{y_1, \dots, y_n\}$. Hence

$$R(1, 0, \dots, 0) = \mathrm{Res}^{\boldsymbol{y}'}_{1,\dots,n}(F_1((1, 0, \dots, 0), (0, \boldsymbol{y}')), \dots, F_n((1, 0, \dots, 0), (0, \boldsymbol{y}')))$$

and, for $j = 0, \dots, d$,

$$F_j((1, 0, \dots, 0), (0, \boldsymbol{y}')) = j! \sum_{\boldsymbol{a}'} c_{d-j,\boldsymbol{a}'}\, y_1^{a_1'} \dots y_n^{a_n'} \in K[\boldsymbol{c}, \boldsymbol{y}'], \tag{4.7}$$

the sum being over the vectors $\boldsymbol{a}' \in \mathbb{N}^n$ of length j. We deduce that $R(1, 0, \dots, 0)$ coincides, up to a nonzero scalar, with the resultant of n generic polynomials in n variables of degrees $1, 2, \dots, n$. In particular, it is irreducible. □

Lemma 4.5. *The polynomial R_{F,y_0} does not depend on the variable $c_{d,0,\dots,0}$, and it is irreducible in $K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}$.*

Proof. The first statement follows from the formula (4.6) and the fact that the polynomials $F_k(\boldsymbol{x}, 0, \boldsymbol{y}')$, $k = 1, \dots, n$, do not depend on $c_{d,0,\dots,0}$ and so neither does R, which gives the first statement.

To prove the second one, set again $R = R_{F,y_0}$ and consider a factorization

$$R = Q_1 Q_2$$

with $Q_1, Q_2 \in K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}$. Since R is a bihomogeneous polynomial in $K[\boldsymbol{c}, \boldsymbol{x}]$, we can assume that its factors are also of this kind. By Lemma 4.4, $R(1, 0, \dots, 0)$ is an irreducible polynomial in $K[\boldsymbol{c}]$ and so one of these factors, say Q_1, has degree 0 in the variables $\boldsymbol{c}$ or equivalently, does not depend on the coefficients of F.

For each choice of $p \in \mathbb{P}^n \setminus Z(x_0)$, we can construct a squarefree homogeneous polynomial f of degree d such that p is not a flex point of $Z(f)$, and a linear form

ℓ such that $\ell(p) \neq 0$. Proposition 3.8 then implies that $R(p) \neq 0$ and, *a fortiori*, $Q_1(p) \neq 0$. Hence Q_1 is a unit of $K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}$ and R is irreducible, concluding the proof. □

Lemma 4.6. *The ideal $(F, \Phi_d) \subset K[\boldsymbol{c}, \boldsymbol{x}]$ is of height 2, and x_0 is not a zero divisor modulo this ideal.*

Proof. Set again $R = R_{F,y_0}$ for short. By Lemma 4.5, this polynomial does not depend on the variable $c_{d,0,\dots,0}$. Hence, it is coprime with F, as F is irreducible. By Proposition 4.2, $R \equiv x_0^{n!} \Phi_d \mod F$, and so Φ_d is also coprime with F, giving the first statement.

For the second statement, set $F' = F(0, x_1, \dots, x_n)$ and $\Phi'_d = \Phi_d(0, x_1, \dots, x_n)$, so that

$$F \equiv F' \quad \text{and} \quad \Phi_d \equiv \Phi'_d \mod x_0.$$

Again by Proposition 4.2,

$$R_{F,y_n}(0, x_1, \dots, x_n) \equiv x_n^{n!} \Phi'_d \mod F'.$$

With the same arguments as for the previous case, we deduce that F' and Φ'_d are coprime. Hence x_0, F, P is a regular sequence in $K[\boldsymbol{c}, \boldsymbol{x}]$.

Since F, Φ_d is a regular sequence in $K[\boldsymbol{c}, \boldsymbol{x}]$ and this ring is Cohen-Macaulay, the associated primes of the ideal (F, Φ_d) are of height 2. Since x_0, F, Φ_d is also a regular sequence, x_0 does not lie in any of these associated primes and so this variable is not a zero divisor modulo (F, Φ_d), as stated. □

Lemma 4.7. *The ideal $(F, \Phi_d) \subset K[\boldsymbol{c}, \boldsymbol{x}]$ is prime.*

Proof. By Lemma 4.6, x_0 is not a zero divisor modulo (F, Φ_d) and so the morphism

$$K[\boldsymbol{c}, \boldsymbol{x}]/(F, \Phi_d) \longrightarrow K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}/(F, \Phi_d)$$

is an inclusion. Hence, it is enough to prove that the ideal $(F, \Phi_d) \subset K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}$ is prime. Thanks to (4.5) applied with $\ell = x_0$, we obtain an isomorphism

$$K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}/(F, \Phi_d) \longrightarrow K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}/(F, R_{F,y_0})$$

and we are reduced to show that $(F, R_{F,y_0}) \subset K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}$ is prime.

Set $\boldsymbol{c}' = \boldsymbol{c} \setminus \{c_{d,0,\dots,0}\}$ and write $F = c_{d,0,\dots,0}\, x_0^d + \widetilde{F}$ with $\widetilde{F} \in K[\boldsymbol{c}', \boldsymbol{x}]$. As R_{F,y_0} does not depend on $c_{d,0,\dots,0}$ by Lemma 4.5, we get a well-defined isomorphism

$$K[\boldsymbol{c}', \boldsymbol{x}]_{x_0}/(R_{F,y_0}) \longrightarrow K[\boldsymbol{c}, \boldsymbol{x}]_{x_0}/(F, R_{F,y_0}).$$

By Lemma 4.5 again, R_{F,y_0} is irreducible in $K[\boldsymbol{c}', \boldsymbol{x}]_{x_0}$, and the statement follows. □

Proof of Theorem 4.1. Setting $N = \binom{d+n}{n} - 1$, let Y be the subscheme of $\mathbb{P}^N \times \mathbb{P}^n$ defined by F and Φ_d. By Lemmas 4.6 and 4.7, this is an irreducible variety of dimension $N + n - 2$. Let

$$\pi \colon Y \longrightarrow \mathbb{P}^N$$

the map induced by the projection onto the second factor.

For a generic choice of $\boldsymbol{\alpha} \in \mathbb{P}^N$, the homogeneous polynomial $F(\boldsymbol{\alpha}, \boldsymbol{x}) \in K[\boldsymbol{x}]$ is squarefree and, by Proposition 4.2 and Theorem 3.9, its fiber $\pi^{-1}(\boldsymbol{\alpha})$ identifies with the flex locus of the hypersurface of $\mathbb{P}^n$ defined by this polynomial. The same result implies that the dimension of this flex locus is either $n - 1$ or $n - 2$. Since Y has dimension $N + n - 2$, the theorem of dimension of fibers implies that $\pi^{-1}(\boldsymbol{\alpha})$ has dimension $n - 2$.

Finally, the fact that Y is a variety and Bertini's theorem [Jou83, Théorème 6.3(3)] imply that this fiber is reduced, completing the proof. □

5. Generic flex points

For a squarefree homogeneous polynomial $f \in K[\boldsymbol{x}]$ of degree $d \geq n$ and a flex point p of the hypersurface $Z(f)$, we consider the following properties:

(1) there is a unique flex line of $Z(f)$ at p;
(2) for a flex line L of $Z(f)$ at p, if $d = n$, then L is contained in $Z(f)$ whereas if $d > n$, then the order of contact of L with $Z(f)$ at p is equal to $n + 1$.

In this section we prove the next result, corresponding to Theorem 1.4(2) stated in the introduction.

Theorem 5.1. *Let $f \in K[\boldsymbol{x}]$ be a generic homogeneous polynomial of degree $d \geq n$, and p a generic point of* $\mathrm{Flex}(Z(f))$. *Then (f, p) satisfies the conditions* (1) *and* (2).

We begin with some notation and preliminary results. For $d \geq 0$, set $N = \binom{d+n}{n} - 1$ and let $\mathbb{P}^N$ be the projective space of nonzero homogeneous forms of degree d modulo scalar factors. For $k = 0, \dots, d$, we introduce the incidence subvariety

$$\begin{aligned}\Gamma_k &= Z(F_0, \dots, F_k) \\ &= \{((c_{\boldsymbol{a}})_{|\boldsymbol{a}|=d}, p, q) \mid F_i(p, q) = 0 \text{ for } i = 0, \dots, n\} \subset \mathbb{P}^N \times (\mathbb{P}^n \setminus Z(x_0)) \times Z(x_0),\end{aligned}$$

with $Z(x_0)$ the hyperplane at infinity of $\mathbb{P}^n$ and F_i as in (4.1).

Lemma 5.2. *The subvariety Γ_k is irreducible and has dimension $N + 2n - k$.*

Proof. Consider the surjective map $\mathrm{pr}_1 \colon \Gamma_k \to (\mathbb{P}^n \setminus Z(x_0)) \times Z(x_0)$ induced by the projection onto the last two factors. To study the fibers of this map over a point (p, q), we can reduce to the case $p = (1 : 0 : \cdots : 0)$, by applying a suitable linear change of coordinates.

For a point $q \in Z(x_0)$, the identities in (4.7) imply that $F_j((1, 0, \dots, 0), q)$, $j = 0, \dots, k$, are nonzero linear forms in the variables $\boldsymbol{c}$ depending on disjoint subsets of variables, and so they are independent. Hence the fiber $\mathrm{pr}_1^{-1}((1, 0, \dots, 0), q)$ is a linear space of dimension $N - k$, and a similar statement holds for any pair of points (p, q). Thus Γ_k is a geometric vector bundle of dimension $N - k - 1$ over the base space $(\mathbb{P}^n \setminus Z(x_0)) \times Z(x_0)$. Since this base is irreducible and has dimension $2n - 1$, the subvariety is also irreducible and has dimension $N + 2n - k$, as claimed. □

For the special case $k = n$, the subvariety Γ_n consists of the triples (f, p, q) where f is a homogeneous polynomial of degree d, $p \in \mathbb{P}^n \setminus Z(x_0)$ is a flex point of the hypersurface $Z(f)$, and $q \in Z(x_0)$ determines a flex line containing p. Let $\Omega \subset \mathbb{P}^N \times (\mathbb{P}^n \setminus Z(x_0))$ denotes the set of pairs (f, p) where $p \in \mathbb{P}^n \setminus Z(x_0)$ is a flex point of $Z(f)$, and

$$\pi \colon \Gamma_n \longrightarrow \Omega \tag{5.1}$$

the map induced by the projection of $\mathbb{P}^N \times (\mathbb{P}^n \setminus Z(x_0)) \times Z(x_0)$ onto its first two factors.

Proposition 5.3. *The map π is birational.*

Proof. Since π is the restriction to the irreducible subvariety Γ_n of the proper map $\mathbb{P}^N \times (\mathbb{P}^n \setminus Z(x_0)) \times Z(x_0) \longrightarrow \mathbb{P}^N \times (\mathbb{P}^n \setminus Z(x_0))$, its image Ω is also an irreducible subvariety. Indeed, by Proposition 3.8 and Proposition 4.2, it is the subvariety of $\mathbb{P}^N \times (\mathbb{P}^n \setminus Z(x_0))$ defined by the polynomials F and R_{F,y_0}.

The inversion property of the resultant (Proposition 2.3), implies that the map π is invertible on the open subset of points $(f,p)\in\Omega$ where

$$\frac{\partial \operatorname{Res}^{\boldsymbol{y}}_{(1,\dots,n,1)}}{\partial c_{i_0,\boldsymbol{a}_0}}(f_1(\boldsymbol{p},\boldsymbol{y}),\dots,f_n(\boldsymbol{p},\boldsymbol{y}),y_0)\neq 0 \tag{5.2}$$

for a representative $\boldsymbol{p}\in K^{n+1}\setminus\{\boldsymbol{0}\}$ of p and a pair of indices $0\leq i_0\leq n-1$ and $\boldsymbol{a}_0\in\mathbb{N}^{n+1}$ with $|\boldsymbol{a}_0|=i_0+1$.

We next want to prove that this open subset is nonempty. To this end, it is enough to show that there is a point (f,p_0) in Ω with $p_0=(1:0\cdots:0)\in\mathbb{P}^n\setminus Z(x_0)$ satisfying at least one of the inequations (5.2). We have that $F(1,0,\dots,0)=c_{d,0,\dots,0}$ and, by Lemma 4.5, the polynomial R_{F,y_0} does not depend on this variable. Thus $(f,p_0)\in\Omega$ if and only if it satisfies the independent conditions

$$c_{d,0,\dots,0}=0 \quad\text{and}\quad R_{Z(f),y_0}(1,0,\dots,0)=0. \tag{5.3}$$

By (4.7), each of the evaluations in (5.2) for the point (f,p_0) coincide, up to a fixed (that is, neither depending on i_0 nor on $\boldsymbol{a}_0$) nonzero scalar factor, with

$$\frac{\partial R_{Z(f),y_0}}{\partial c_{\boldsymbol{b}_0}}(1,0,\dots,0)$$

for a vector $\boldsymbol{b}_0\in\mathbb{N}^{n+1}$ with $|\boldsymbol{b}_0|=d$. Hence, the condition that (f,p_0) satisfies (5.2) is equivalent to

$$\nabla R_{F,y_0}(1,0,\dots,0)(f)\neq\boldsymbol{0}, \tag{5.4}$$

where $\nabla R_{F,y_0}$ denotes the gradient operator. By Lemma 4.4, R_{F,y_0} is an irreducible polynomial, and *a fortiori* squarefree. Hence, the condition (5.4) is verified for a generic f satisfying (5.3).

We deduce that the map π is invertible on a nonempty open subset of Ω. Since Ω is irreducible, such an open subset is dense, and so π is birational. □

Proof of Theorem 5.1. By Proposition 5.3, there are dense open subsets $U\subset\Gamma_n$ and $W\subset\Omega$ such that the restriction of the map π in (5.1) to these subsets is an isomorphism. In particular, for each $(f,p)\in W$ there is a unique flex line containing the point p.

If $d=n$, then any such flex line has order of contact at least $n+1$ at the point p, and so it is necessarily contained in $Z(f)$.

If $d>n$ then Γ_{n+1} is a proper subvariety of Γ_n by Lemma 5.2. By Lemma 3.4(1), for each $(f,p,q)\in U\setminus\Gamma_{n+1}$, the line containing p and q has order of contact equal to $n+1$. Hence, every pair (f,p) in the dense open subset $W':=W\setminus\pi(\Gamma_{n+1})$ of Ω satisfies both conditions (1) and (2).

Set $Z=\Omega\setminus W'$ and consider the map $\varpi\colon Z\to\mathbb{P}^N$ defined by $(f,p)\mapsto f$. If this map is not dominant, then for $f\in\mathbb{P}^N\setminus\varpi(Z)$ we have that $\{f\}\times\operatorname{Flex}(Z(f))$ is disjoint from Z, giving the statement in this case.

Otherwise, by the theorem of dimension of fibers [Sha94, §1.6, Theorem 7], there is a dense open subset $T\subset\mathbb{P}^N$ such that, for $f\in T$,

$$\dim(\varpi^{-1}(f))=\dim(Z)-\dim(\mathbb{P}^N)<n-2.$$

On the other hand, $\dim(\operatorname{Flex}(Z(f)))$ is either $n-1$ or $n-2$. Hence for all $f\in T$, no component of $\{f\}\times\operatorname{Flex}(Z(f))$ can be contained in Z.

In both cases, there is a dense open subset T of $\mathbb{P}^N$ such that, for each $f \in T$, we have that f is squarefree and there is a dense open subset U_f of the flex locus of $Z(f)$ such that for each $p \in U_f$, the pair (f, p) satisfies the conditions (1) and (2), completing the proof. □

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