HEIGHTS OF TORIC VARIETIES, ENTROPY AND INTEGRATION OVER POLYTOPES

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Abstract. We present a dictionary between arithmetic geometry of toric varieties and convex analysis. This correspondence allows for effective computations of arithmetic invariants of these varieties. In particular, combined with a closed formula for the integration of a class of functions over polytopes, it gives a number of new values for the height (arithmetic analog of the degree) of toric varieties, with respect to interesting metrics arising from polytopes. In some cases these heights are interpreted as the average entropy of a family of random processes.

1. Introduction

Toric varieties form a remarkable class of algebraic varieties, endowed with an action of a torus having one Zariski dense open orbit. It is well known that their geometric properties can be described in terms of combinatorial objects such as fans and polytopes having the same dimension, say n, as the toric variety. For instance, the degree of a toric variety with respect to a nef toric divisor is $n!$ times the volume of the corresponding polytope.

In the book [4], we have extended this dictionary by linking the arithmetic geometry of toric varieties defined over a number field to convex analysis. Here, the arithmetic ingredients are given by (semipositive) metrics on the toric line bundle associated to a toric divisor. Each of these metrics correspond to a continuous concave function on the associated polytope, that we call the *local roof function*. These functions combine in a *global* roof function over the polytope. In this context, the arithmetic invariant analogous to the degree is the height which, similarly to the degree, can be expressed as $(n+1)!$ times the integral over the polytope of the global roof function.

For particular choices of metrics, these heights coincide with the average entropy of certain random processes associated to the polytope. Our toric "dictionary", combined with a closed formula for the integration of a class of functions over polytopes, allows to compute the values of these heights. All the results presented here can be found in more details in [4].

2. Heights and toric varieties

In this section we recall some of the basic constructions and results in [4]. Details and more information can be found in this reference. The reader can also consult [6, 5] for a background on the algebraic geometry of toric varieties.

Let $N \simeq \mathbb{Z}^n$ be a lattice of rank n and $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual lattice. Set $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n$ and $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$. We denote by $\langle x, u\rangle$ the pairing between $x \in M_\mathbb{R}$ and $u \in N_\mathbb{R}$.

To a *lattice fan* Σ on $N_\mathbb{R}$ we associate a *toric scheme* over the integers, denoted by X_Σ. This scheme is flat over $\mathrm{Spec}(\mathbb{Z})$ of relative dimension n. It is equipped with an action of the algebraic torus $\mathbb{T}_N := \mathrm{Spec}(\mathbb{Z}[M]) \simeq \mathbb{G}^n_{m,S}$ extending the natural action of $\mathbb{T}_N$ on itself. This action has a dense orbit, denoted X°_Σ and which is canonically isomorphic to $\mathbb{T}_N$. The scheme $X_{\Sigma,S}$ is projective whenever the fan Σ is complete and regular, and it is smooth whenever each cone of Σ is generated by a subset of a basis of N, *see* [5]. We will assume both properties from now on.

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A *virtual support function* is a continuous function $\Psi\colon N_{\mathbb{R}} \to \mathbb{R}$ whose restriction to each of the cones of Σ is an element of M. Such a function defines an invariant Cartier divisor D_Ψ of $X_{\Sigma,S}$ or, equivalently, an equivariant line bundle $L_{\Psi,S}$ together with an invariant rational section s_Ψ such that $\mathrm{div}(s_\Psi) = D_\Psi$, *see* [6, § 3.3 and 3.4]. The divisor D_Ψ is relatively ample if and only if Ψ is concave and restricts to different elements of M on each of the maximal cones of Σ. We will also suppose this from now on. Under this assumption, the polyhedron

$$\Delta_\Psi := \{x \in M_{\mathbb{R}} : \langle x, y\rangle \geq \Psi(y) \text{ for all } y \in N_{\mathbb{R}}\} \subset M_{\mathbb{R}}$$

is an n-dimensional polytope.

Let $X_\Sigma(\mathbb{C})$ and $\mathbb{T}_N(\mathbb{C})$ respectively denote the analytification of the scheme X_Σ and of the algebraic torus $\mathbb{T}$. Also let $\mathbb{S} := \{t \in \mathbb{T}_N(\mathbb{C}) \mid |t| = 1\}$ be the *compact subtorus* of $\mathbb{T}_N(\mathbb{C})$. There is a map $\mathrm{val}\colon X_\Sigma^\circ(\mathbb{C}) \to N_{\mathbb{R}}$, defined, in a given splitting $X_\Sigma^\circ(\mathbb{C}) = \mathbb{T}_N(\mathbb{C}) \simeq (\mathbb{C}^\times)^n$, by

$$\mathrm{val}(x_1, \dots, x_n) = (-\log |x_1|_v, \dots, -\log |x_n|_v).$$

This map does not depend on the choice of the splitting and the compact torus $\mathbb{S}$ coincides with its fiber over the point $0 \in N_{\mathbb{R}}$.

We furthermore consider a semipositive *toric metric* $\|\cdot\|$ on the analytification of the line bundle L_Ψ, that is, a semipositive metric which is invariant under the action of $\mathbb{S}$. We denote by $\overline{L}_\Psi$ the line bundle metrized in this way. Such a toric metrized line bundle defines a continuous function $\psi_{\overline{L}}\colon N_{\mathbb{R}} \to \mathbb{R}$ given, for $p \in \mathbb{T}(\mathbb{C})$, by

$$\psi_{\overline{L}}(\mathrm{val}(p)) = \log \|s_\Psi(p)\|_v.$$

This function is concave. We can then consider its *Legendre-Fenchel dual* $\psi_{\overline{L}}^\vee\colon M_{\mathbb{R}} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\psi_{\overline{L}}^\vee(x) = \inf_{u \in N_{\mathbb{R}}} (\langle u, x\rangle - \psi_{\overline{L}}(u)).$$

The *stability set* of a concave function is the set of points where its Legendre-Fenchel dual is $> -\infty$. It turns out that the stability set of $\psi_{\overline{L}}$ coincides with the polytope Δ_Ψ and that the function $\psi_{\overline{L}}^\vee$ is continuous and concave on Δ_Ψ. The *roof function* of $\overline{L}$, denote $\vartheta_{\overline{L}}$, is defined as the restriction of $\psi_{\overline{L}}^\vee$ to the polytope Δ_Ψ.

Given a flat projective and smooth scheme X over $\mathrm{Spec}(\mathbb{Z})$ equipped with a semipositive metrized line bundle $\overline{L}$, we can define using arithmetic intersection theory a *height function*, denoted $\mathrm{h}_{\overline{L}}$, for subschemes of X_Σ, *see* [2, 8, 10]. It is the arithmetic analogue of the notion of degree of subvarieties.

One of the main results in [4] is that the height of a toric scheme with respect to a toric semipositive metrized line bundle can be expressed as the integral of the associated roof function [4, Theorem 5.2.5]. In precise terms,

$$\mathrm{h}_{\overline{L}_\Psi}(X_\Sigma) = (n+1)! \int_{\Delta_\Psi} \vartheta_{\overline{L}} \, \mathrm{d}\,\mathrm{vol}_M, \tag{1}$$

where vol_M is the Haar measure on $M_{\mathbb{R}}$ normalized so that the lattice M has covolume 1.

3. Metrics from polytopes and entropy

In some cases, the height of a toric variety with respect to a toric semipositive metrized line bundle has an interpretation in terms of the average entropy of a family of random processes.

Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope of dimension n and Γ an arbitrary polytope containing it. For a point x in the interior of Δ, we denote by Π_x the partition of Γ consisting of the cones $\eta_{x,F}$ of vertex x and base the relative interior of each proper face F of Γ. We consider Γ as a probability space endowed with the uniform probability distribution. Let β_x be the random variable that maps a point $y \in \Gamma$ to the base F of the unique cone $\eta_{x,F}$ that contains y. Clearly, the probability that a given face F is returned is the ratio of the volume of the cone based on F to the volume of Γ. We have

$$\mathrm{vol}_n(\eta_{x,F}) = n^{-1} \mathrm{dist}(x, F)\, \mathrm{vol}_{n-1}(F),$$

where vol_n and vol_{n-1} respectively denote the Euclidean n-th and $(n-1)$-th Euclidean volume of convex subsets of $\mathbb{R}^n$, and $\mathrm{dist}(x,F)$ denotes the distance of the point x to the face F. Hence,

$$P(\beta_x = F) = \begin{cases} \dfrac{\mathrm{dist}(x,F)\,\mathrm{vol}_{n-1}(F)}{n\,\mathrm{vol}_n(\Gamma)} & \text{if } \dim(F) = n-1, \\ 0 & \text{if } \dim(F) \leq n-2. \end{cases} \tag{2}$$

The *entropy* of the random variable β_x is

$$\mathcal{E}(x) = -\sum_F P(\beta_x = F)\log(P(\beta_x = F)),$$

where the sum is over the facets F of Γ.

From the polytope Γ, we can construct a concave function on itself as follows. For each facet F of Γ, we denote by $u'_F \in \mathbb{R}^n$ the inner normal vector to F of Euclidean norm $(n-1)!\,\mathrm{vol}_{n-1}(F)$. Set $\lambda(F) = \inf_{x\in\Gamma}\langle x, u'_F\rangle$ and consider the affine polynomial defined as

$$\ell_F(x) = \langle x, u'_F\rangle - \lambda(F). \tag{3}$$

Hence,

$$\Gamma = \{x \in M_{\mathbb{R}} \mid \ell_F(x) \geq 0 \text{ for every facet } F\}.$$

In particular, ℓ_F is nonnegative on Γ, and we can consider the function $\vartheta_\Gamma\colon \Gamma \to \mathbb{R}$ defined by

$$\vartheta_\Gamma(x) = -\frac{1}{2}\sum_F \ell_F(x)\log(\ell_F(x)).$$

By [4, Lemma 6.2.1], this function is concave.

Notation 1. Let Σ_Δ and Ψ_Δ be the fan and the support function on $N_{\mathbb{R}} = \mathbb{R}^n$ induced by the polytope Δ, *see* [4, Example 2.5.13]. Let X_{Σ_Δ} and L_{Ψ_Δ} be the corresponding toric scheme and line bundle. The restriction of the function ϑ_Γ above to Δ is a continuous concave function and so, by [4, Theorem 4.8.1], it corresponds to a semipositive toric metric on L_{Ψ_Δ}. We denote this metric by $\|\cdot\|_{\Delta,\Gamma}$ and we write $\overline{L}_{\Psi_\Delta}$ for the line bundle L_{Ψ_Δ} equipped with this toric metric.

Example 1. Let $\Delta^n = \{(x_1,\dots,x_n) \mid x_i \geq 0, \sum_i x_i \leq 1\}$ be the standard simplex of $\mathbb{R}^n$ and consider the case when $\Gamma = \Delta = \Delta^n$. The corresponding concave function on Δ^n is given by

$$\vartheta_{\Delta^n}(x_1,\dots,x_n) = -\frac{1}{2}\Big(1-\sum_i x_i\Big)\log\Big(1-\sum_i x_i\Big) - \frac{1}{2}\sum_{i=1}^n x_i\log(x_i).$$

From [4, Example 2.4.3 and 4.3.9(1)], we deduce that the corresponding toric metric is the Fubini-Study metric on $\mathcal{O}(1)$, the universal line bundle on the projective space $\mathbb{P}^n$.

Remark 1. This kind of metrics are interesting for the Kähler geometry of toric varieties. Given a Delzant polytope $\Delta \subset \mathbb{R}^n$, Guillemin has constructed a "canonical" Kähler structure on the associated symplectic toric variety, *see* [7] for details. Following Guillemin, this canonical Kähler structure is codified by a convex function on the polytope, dubbed the "symplectic potential".

With the notation above, when $\Gamma = \Delta$ and u'_F is a primitive vector in N for every facet F, the function $-\vartheta_\Gamma$ coincides with this symplectic potential, *see* [7, Appendix 2, (3.9)]. In this case, the metric $\|\cdot\|_{\Delta,\Gamma}$ on the line bundle L_{Ψ_Δ} is smooth and positive, and its Chern form gives this canonical Kähler form.

The following result shows that the average entropy of the random variables β_x, $x \in \Delta$, with respect to the uniform distribution on Δ can be expressed in terms of the height of the toric variety X_{Σ_Δ} with respect to $\overline{L}$.

Theorem 1. *With the above notation,*

$$\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta \mathcal{E}\,\mathrm{d}\,\mathrm{vol}_n = \frac{1}{n!\,\mathrm{vol}_n(\Gamma)}\left(\frac{2\,\mathrm{h}_{\overline{L}}(X_{\Sigma_\Delta})}{(n+1)\deg_L(X_{\Sigma_\Delta})} - \lambda(\Gamma)\log(n!\,\mathrm{vol}_n(\Gamma))\right)$$

with $\lambda(\Gamma)=\sum_F \lambda(F)$*, the sum being over the facets* F *of* Γ*. In particular, if* $\Gamma=\Delta$*,*

$$\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta \mathcal{E}\,\mathrm{d}\,\mathrm{vol}_n=\frac{2\,\mathrm{h}_{\overline{L}}(X_{\Sigma_\Delta})}{(n+1)\deg_L(X_{\Sigma_\Delta})^2}-\lambda(\Gamma)\frac{\log(\deg_L(X_{\Sigma_\Delta}))}{\deg_L(X_{\Sigma_\Delta})}.$$

Proof. By [9, Lemma 5.1.1], the vectors u'_F satisfy the Minkowski condition $\sum_F u'_F=0$. Hence

$$\sum_F \ell_F=-\sum_F \lambda(F)=-\lambda(\Gamma).$$

Let x be a point in the interior of Δ and F a facet of Γ. We deduce from (2) that $P(\beta_x=F)=\ell_F(x)/(n!\,\mathrm{vol}_n(\Gamma))$. Hence,

$$\begin{aligned}\mathcal{E}(x)&=-\sum_F\frac{\ell_F(x)}{n!\,\mathrm{vol}_n(\Gamma)}\log\Big(\frac{\ell_F(x)}{n!\,\mathrm{vol}_n(\Gamma)}\Big)\\&=\frac{1}{n!\,\mathrm{vol}_n(\Gamma)}\bigg(-\sum_F\ell_F(x)\log(\ell_F(x))-\lambda(\Gamma)\log(n!\,\mathrm{vol}_n(\Gamma))\bigg)\\&=\frac{1}{n!\,\mathrm{vol}_n(\Gamma)}\bigg(2\vartheta_\Gamma(x)-\lambda(\Gamma)\log(n!\,\mathrm{vol}_n(\Gamma))\bigg).\end{aligned}$$

The result then follows from the expression for the height of X_{Σ_Δ} in (1) and the analogous expression for its degree in [6, page 111, Corollary]. □

Example 2. The Fubini-Study metric of $\mathcal{O}(1)$ corresponds to the case when Γ and Δ are the standard simplex Δ^n. In that case, the average entropy of the random variables β_x, $x\in\Delta$, is

$$\frac{1}{n!}\int_{\Delta^n}\mathcal{E}\,\mathrm{d}\,\mathrm{vol}_n=\frac{2\,\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)}{(n+1)}.$$

4. Integration on polytopes

In this section, we present a closed formula for the integral over a polytope of a function of one variable composed with a linear form, extending in this direction Brion's formula for the case of a simplex [3], see Proposition 1 and Corollary 2 below. This formula allow us to compute the height of toric varieties with respect to the metrics arising from polytopes as in §3.

We consider the vector space $\mathbb{R}^n$ with its usual scalar product, that we denote $\langle\cdot,\cdot\rangle$, and its Lebesgue measure, that we denote vol_n. We also consider a polytope $\Delta\subset\mathbb{R}^n$ of dimension n.

Definition 1. Let $u\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$, the *aggregate of* Δ *in* the affine subset

$$L_{u,\lambda}:=\{x\in\mathbb{R}^n\mid\langle x,u\rangle=\lambda\}$$

is the union of all the faces of Δ contained in $L_{u,\lambda}$. An *aggregate* V *of* Δ *in the direction* u is an aggregate in $L_{u,\lambda}$ for some $\lambda\in\mathbb{R}$.

We denote by $\dim(V)$ the maximal dimension of a face of Δ contained in V. In particular, $\dim(\emptyset)=-1$.

We write $\Delta(u)$ for the set of non-empty aggregates of Δ in the direction u. In particular, $\Delta(0)=\{\Delta\}$. Note that, if $V\in\Delta(u)$ and x is a point in the affine space spanned by V, then the value $\langle x,u\rangle$ is independent of x. We denote this common value by $\langle V,u\rangle$.

For any two aggregates $V_1,V_2\in\Delta(u)$, we have $V_1=V_2$ if and only if $\langle V_1,u\rangle=\langle V_2,u\rangle$.

Example 3.

(1) Every facet of a polytope is an aggregate in the direction orthogonal to the facet.
(2) If u is general enough, the set $\Delta(u)$ agrees with the set of vertices of Δ.
(3) Let $\Delta=\{(x,y)\in\mathbb{R}^2\mid 0\le x,y\le 1\}$ be the unit square and $u=(1,1)$. Then the set of aggregates $\Delta(u)$ contains three elements: $\{(0,0)\}$, $\{(1,0),(0,1)\}$ and $\{(1,1)\}$.

In each facet F of Δ we choose a point m_F. Let L_F be the linear hyperplane defined by F and π_F the orthogonal projection of $\mathbb{R}^n$ onto L_F. Then, $F - m_F$ is a polytope in L_F of full dimension $n-1$. To ease the notation, we identify $F - m_F$ with F. Observe that, with this identification, for $V \in \Delta(u)$, the intersection $V \cap F$ is an aggregate of F in the direction $\pi_F(u)$. We also denote by u_F the inner normal vector to F of norm 1.

Definition 2. Let $u \in \mathbb{R}^n$ be a vector. For each aggregate V in the direction of u, we define the coefficients $C_k(\Delta, u, V)$, $k \in \mathbb{N}$, recursively. If $u = 0$, then V is either $\emptyset$ or Δ. For both cases, we set

$$C_k(\Delta, 0, V) = \begin{cases} \mathrm{vol}_n(V) & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

If $u \neq 0$, we set

$$C_k(\Delta, u, V) = -\sum_F \frac{\langle u_F, u\rangle}{\|u\|^2} C_k(F, \pi_F(u), V \cap F),$$

where the sum is over the facets F of Δ. This recursive formula implies that $C_k(\Delta, u, V) = 0$ for all $k > \dim(V)$.

Let $\mathcal{C}^n(\mathbb{R})$ be the space of functions of one real variable which are n-times continuously differentiable. For $f \in \mathcal{C}^n(\mathbb{R})$ and $0 \leq k \leq n$, we write $f^{(k)}$ for the k-th derivative of f. We want to give a formula that, for $f \in \mathcal{C}^n(\mathbb{R})$, computes

$$\int_\Delta f^{(n)}(\langle x, u\rangle)\, \mathrm{d}\,\mathrm{vol}_n(x)$$

in terms of the values of the function $x \mapsto f(\langle x, u\rangle)$ at the vertices of Δ. However, when u is orthogonal to some faces of Δ of positive dimension, such a formula necessarily depends on the values of the derivatives of f.

Proposition 1. ([4, Proposition 6.1.4]) Let $\Delta \subset \mathbb{R}^n$ be a polytope of dimension n and $u \in \mathbb{R}^n$. Then, for any $f \in \mathcal{C}^n(\mathbb{R})$,

$$\int_\Delta f^{(n)}(\langle x, u\rangle)\, \mathrm{d}\,\mathrm{vol}_n(x) = \sum_{V \in \Delta(u)} \sum_{k \geq 0} C_k(\Delta, u, V) f^{(k)}(\langle V, u\rangle).$$

The coefficients $C_k(\Delta, u, V)$ are uniquely determined by this identity.

Corollary 1. *Let $\Delta \subset \mathbb{R}^n$ be a polytope of dimension n and $u \in \mathbb{R}^n$. Then,*

$$\sum_{V \in \Delta(u)} \sum_{k=0}^{\min\{i, \dim(V)\}} C_k(\Delta, u, V) \frac{\langle V, u\rangle^{i-k}}{(i-k)!} = \begin{cases} 0 & \text{for } i = 0, \dots, n-1, \\ \mathrm{vol}_n(\Delta) & \text{for } i = n. \end{cases}$$

Proof. This follows from proposition 1 applied to the functions $f(z) = z^i/i!$. □

The following result gives the basic properties of the coefficients associated to the aggregates of a polytope.

Proposition 2. ([4, Proposition 6.1.6]) Let $\Delta \subset \mathbb{R}^n$ be a polytope of dimension n and $u \in \mathbb{R}^n$. Let $V \in \Delta(u)$ and $k \geq 0$.

(1) The coefficient $C_k(\Delta, u, V)$ is homogeneous of weight $k - n$ in the sense that, for $\lambda \in \mathbb{R}^\times$,

$$C_k(\Delta, \lambda u, V) = \lambda^{k-n} C_k(\Delta, u, V).$$

(2) The coefficients $C_k(\Delta, u, V)$ satisfy the vector relation

$$C_k(\Delta, u, V) \cdot u = -\sum_F C_k(F, \pi_F(u), V \cap F) \cdot u_F,$$

where the sum is over the facets F of Δ.

(3) Let $\Delta_1, \Delta_2 \subset \mathbb{R}^n$ be two polytopes of dimension n intersecting along a common facet and such that $\Delta = \Delta_1 \cup \Delta_2$. Then $V \cap \Delta_i = \emptyset$ or $V \cap \Delta_i \in \Delta_i(u)$ and

$$C_k(\Delta, u, V) = C_k(\Delta_1, u, V \cap \Delta_1) + C_k(\Delta_2, u, V \cap \Delta_2).$$

In case Δ is a simplex, the linear system given by Corollary 1 has as many unknowns as equations. In this case, the coefficients corresponding to an aggregate in a given direction are determined by this linear system. The following result gives a closed formula for those coefficients.

Proposition 3. ([4, Proposition 6.1.7]) Let $\Delta \subset \mathbb{R}^n$ be a simplex and $u \in \mathbb{R}^n$. Write $d_W = \dim(W)$ for $W \in \Delta(u)$. Then, for $V \in \Delta(u)$ and $0 \le k \le \dim(V)$,

$$C_k(\Delta, u, V) = (-1)^{d_V - k}\frac{n!}{k!}\operatorname{vol}_n(\Delta) \sum_{\substack{\eta \in \mathbb{N}^{\Delta(u)\setminus\{V\}} \\ |\eta| = d_V - k}} \prod_{W \in \Delta(u)\setminus\{V\}} \frac{\binom{d_W + \eta_W}{d_W}}{\langle V - W, u\rangle^{d_W + \eta_W + 1}}.$$

Remark 2. We can rewrite the formula in Proposition 3 in terms of vertices instead of aggregates as follows:

$$C_k(\Delta, u, V) = (-1)^{d_V - k}\frac{n!}{k!}\operatorname{vol}_n(\Delta) \sum_{|\beta| = d_V - k} \prod_{\nu \notin V} \langle V - \nu, u\rangle^{-\beta_\nu - 1}, \tag{4}$$

where the product is over the vertices ν of Δ not lying in V and the sum is over the tuples β of non negative integers of length $d_V - k$, indexed by those same vertices of Δ that are not in V, that is, $\beta \in \mathbb{N}^{n-d_V}$ and $|\beta| = d_V - k$.

Example 4. Let $\Delta \subset \mathbb{R}^n$ be a simplex and $u \in \mathbb{R}^n$. If a vertex ν_0 of Δ is an aggregate in the direction of u, then formula (4) reduces to

$$C_0(\Delta, u, \nu_0) = n!\operatorname{vol}_n(\Delta) \prod_{\nu \neq \nu_0} \langle \nu_0 - \nu, u\rangle^{-1}, \tag{5}$$

where the product runs over all vertices of Δ different from ν_0. Suppose that the simplex is presented as the intersection of $n+1$ halfspaces as

$$\Delta = \bigcap_{i=0}^{n} \{x \in \mathbb{R}^n \mid \langle x, u_i\rangle - \lambda_i \ge 0\}$$

with $u_i \in \mathbb{R}^n \setminus \{0\}$ and $\lambda_i \in \mathbb{R}$. Up to a reordering, we can assume that u_0 is an inner normal vector to the unique face of Δ not containing ν_0. We denote by ε the sign of $(-1)^n \det(u_1, \dots, u_n)$. Then the above coefficient can be alternatively written as

$$C_0(\Delta, u, \nu_0) = \frac{\varepsilon \det(u_1, \dots, u_n)^{n-1}}{\prod_{i=1}^{n} \det(u_1, \dots, u_{i-1}, u, u_{i+1}, \dots, u_n)}.$$

From the equation (5), we obtain the following extension of Brion's "short formula" for the case of a simplex [3, Théorème 3.2], see also [1].

Corollary 2. *Let $\Delta \subset \mathbb{R}^n$ be a simplex of dimension n that is the convex hull of points ν_i, $i = 0, \dots, n$, and let $u \in \mathbb{R}^n$ such that $\langle \nu_i, u\rangle \neq \langle \nu_j, u\rangle$ for $i \neq j$. Then, for any $f \in \mathcal{C}^n(\mathbb{R})$,*

$$\int_\Delta f^{(n)}(\langle x, u\rangle)\, \mathrm{d}\operatorname{vol}_n(x) = n!\operatorname{vol}_n(\Delta) \sum_{i=0}^{n} \frac{f(\langle \nu_i, u\rangle)}{\prod_{j \neq i} \langle \nu_i - \nu_j, u\rangle}.$$

Proof. This follows from Proposition 1 and formula (5). □

The following result gives the value of the integral over a simplex of a function of the form $\ell(x)\log(\ell(x))$, where ℓ is an affine function.

Proposition 4. Let $\Delta \subset \mathbb{R}^n$ be a simplex of dimension n and $\ell\colon \mathbb{R}^n \to \mathbb{R}$ an affine function which is non-negative on Δ. Write $\ell(x) = \langle x, u\rangle - \lambda$ for some vector u and constant λ. Then

$$\frac{1}{\operatorname{vol}_n(\Delta)} \int_\Delta \ell(x)\log(\ell(x))\, \mathrm{d}\operatorname{vol}_n(x) = \sum_{V \in \Delta(u)} \sum_{\beta'} \binom{n}{n - |\beta'|} \frac{\ell(V)\left(\log(\ell(V)) - \sum_{j=2}^{|\beta'|+1} \frac{1}{j}\right)}{(|\beta'| + 1)\prod_{\nu \notin V}\left(-\left(\frac{\ell(\nu)}{\ell(V)} - 1\right)^{\beta'_\nu}\right)},$$

where the second sum runs over $\beta' \in (\mathbb{N}^\times)^{n-\dim(V)}$ with $|\beta'| \le n$ and the product is over the $n - \dim(V)$ vertices ν of Δ not in V.

If $\ell(x)$ is the defining equation of a hyperplane containing a facet F of Δ, then

$$\frac{1}{\mathrm{vol}_n(\Delta)}\int_\Delta \ell(x)\log(\ell(x))\,\mathrm{d}x = \frac{\ell(\nu_F)}{n+1}\Big(\log(\ell(\nu_F)) - \sum_{j=2}^{n+1}\frac{1}{j}\Big),$$

where ν_F denotes the unique vertex of Δ not contained in F.

Proof. This follows from the formulae (1) and (4) with the function $f^{(n)}(z) = (z-\lambda)\log(z-\lambda)$, a $(n-k)$-th primitive of which is

$$f^{(k)}(z) = \frac{(z-\lambda)^{n-k+1}}{(n-k+1)!}\left(\log(z-\lambda) - \sum_{j=2}^{n-k+1}\frac{1}{j}\right).$$

□

We obtain the following formula for the height of a toric variety with respect to the toric metrics considered in §3, in terms of the coefficients $C_k(\Delta, u_i, V)$.

Theorem 2. *Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope of dimension n and Γ an arbitrary polytope containing it. Let X_{Σ_Δ} and $\overline{L} = \overline{L}_{\Psi_\Delta}$ the toric be as in Notation 1, and ℓ_F and u'_F the affine polynomial and the inner normal vector associated to a facet F of Γ as in (3) . Then*

$$\mathrm{h}_{\overline{L}}(X_{\Sigma_\Delta}) = \frac{(n+1)!}{2}\sum_F \sum_{V\in\Delta(u'_F)} \sum_{k=0}^{\dim(V)} C_k(\Delta, u'_F, V)\frac{\ell_F(V)^{n-k+1}}{(n-k+1)!}\left(\sum_{j=2}^{n-k+1}\frac{1}{j} - \log(\ell_F(V))\right),$$

the first sum being over the facets F of Γ. Suppose furthermore that $\Delta \subset \mathbb{R}^n$ is a simplex. Then

$$\mathrm{h}_{\overline{L}}(X_{\Sigma_\Delta}) = \frac{n!}{2}\,\mathrm{vol}_M(\Delta)\sum_F \ell_F(\nu_F)\Big(\sum_{j=2}^{n+1}\frac{1}{j} - \log(\ell_F(\nu_F))\Big), \tag{6}$$

where ν_F is the unique vertex of Δ not contained in the facet F.

Proof. The first statement follows readily from the formula (1) and Proposition 1 applied to the functions

$$f_i(z) = \left(\log(z-\lambda_i) - \sum_{j=2}^{n+1}\frac{1}{j}\right)(z-\lambda_i)^{n+1}/(n+1)!.$$

The second statement follows similarly from Proposition 4. □

Example 5. Let $\mathcal{O}(1)$ be the universal line bundle of $\mathbb{P}^n$. As we have seen in Example 1, the Fubini-Study metric of $\mathcal{O}(1)$ corresponds to the case of the standard simplex Δ^n. Hence we recover from (6) the well known expression for the height of $\mathbb{P}^n$ with respect to the Fubini-Study metric in [2, Lemma 3.3.1]:

$$\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n) = \frac{n+1}{2}\sum_{j=2}^{n+1}\frac{1}{j} = \sum_{h=1}^{n}\sum_{j=1}^{h}\frac{1}{2j}.$$

Hence, in this case the average entropy of the random variables β_x, $x \in \Delta^n$, is

$$\frac{1}{n!}\int_{\Delta^n}\mathcal{E}\,\mathrm{d}\,\mathrm{vol}_n = \frac{2\,\mathrm{h}_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n)}{(n+1)} = \sum_{j=2}^{n+1}\frac{1}{j}.$$

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