A NOTE ON THE DETERMINANT 308 IN PROSKURYAKOV'S LINEAR ALGEBRA BOOK

ANTONIO J. DI SCALA AND MARTÍN SOMBRA

Abstract. We put in evidence and correct a mistake in the formula for the determinant 308 in Proskuryakov's linear algebra book. We apply this formula to reprove the well-known fact that the Fubini-Study metric on the complex projective space is Einstein.

This short note is motivated by a mistake in the formula for the interesting determinant 308 in Proskuriakov's classical book of linear algebra problems. We checked several of its many editions including the some of first ones and of the more recents [Pro67, Pro05] as well as the translations [Pro78a, Pro78b], and noticed that the mistake has not been corrected.

Problem 308 asks to compute the determinant

$$P308 = \det \begin{bmatrix} x_1 & a_1b_2 & a_1b_3 & \cdots & a_1b_n \\ a_2b_1 & x_2 & a_2b_3 & \cdots & a_2b_n \\ a_3b_1 & a_3b_2 & x_3 & \cdots & a_3b_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & a_nb_3 & \cdots & x_n \end{bmatrix}. \tag{1}$$

The correct expression for this determinant is

$$P308 = \Big(\prod_{k=1}^{n}(x_k - a_kb_k)\Big)\Big(1 + \sum_{k=1}^{n} \frac{a_kb_k}{x_k - a_kb_k}\Big), \tag{2}$$

which in Proskuryakov's book appears with denominators x_k instead of $x_k - a_kb_k$, see for instance [Pro78b, page 321].

Indeed, this formula is a consequence of the more general one for the determinant of a sum of matrices [Mar75, pages 162-163], as it is also hinted in [Pro78b, pages 40-41]. For convenience, we give here a self-contained proof based on the multilinearity of the determinant function.

Proof of Formula (2). Denote by M the $n \times n$ matrix in (1), which can be written as the sum of a diagonal and a rank 1 matrix as

$$M = \operatorname{diag}(x_1 - a_1b_1, \dots, x_n - a_nb_n) + a \cdot b^T$$

for the n vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Considering the determinant as a function of the columns of the matrix, we have that

$$P308 = \det(M) = \det((x_1 - a_1b_1)\, e_1 + b_1a, \dots, (x_n - a_nb_n)\, e_n + b_na),$$

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where e_i denotes the standard n vector $(0,\dots,0,\overset{i}{1},0,\dots,0)$. By the multilinearity of the determinant function and the fact that it vanishes when the vectors are linearly dependent, we have that

$$\begin{aligned} P308 = &\det((x_1 - a_1 b_1)\, e_1, \dots, (x_n - a_n b_n)\, e_n) \\ &+ \sum_{k=1}^{n} \det(b_1 a, \dots, b_{k-1} a, (x_k - a_k b_k)\, e_k, b_{k+1} a, \dots, b_n a) \\ = &\prod_{k=1}^{n} (x_k - a_k b_k) + \sum_{i=1}^{n} a_k b_k \prod_{l \neq k} (x_l - a_l b_l), \end{aligned}$$

which gives the intended formula □

As an application, we compute the Ricci form of the Fubini-Study metric on the n-dimensional complex projective space $\mathbb{P}^n$. In Riemannian geometry, this computation is usually done using the invariance of this metric with respect to the action of the unitary group as in [Mor07, §13.3]. By contrast, Formula (2) allows to do it in a direct way.

Let $Z_0, \dots, Z_n$ be the homogeneous coordinates of this projective space and for each $k \in \{0, \dots, n\}$ consider the open chart $U_k = (Z_k \neq 0) \simeq \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. The *Fubini-Study form* ω_{FS} is the Kähler form on $\mathbb{P}^n$ given in these coordinates by

$$\omega_{\mathrm{FS}} := \mathrm{i} \partial \overline{\partial} \log(1 + \|z\|^2)$$

where $\partial, \overline{\partial}$ are the Dolbeault operators and $\|z\| = (|z_1|^2 + \dots + |z_n|^2)^{1/2}$. The corresponding Hermitian matrix with respect to the frame $\frac{\partial}{\partial z_i}$, $i = 1, \dots, n$ writes down as

$$\begin{aligned} H = &\left[\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log(1 + \|z\|^2) \right]_{i,j} \\ = &\frac{1}{(1 + \|z\|^2)} \begin{bmatrix} 1 + \|z\|^2 - \overline{z}_1\, z_1 & -\overline{z}_1\, z_2 & \cdots & -\overline{z}_1\, z_n \\ -\overline{z}_2\, z_1 & 1 + \|z\|^2 - \overline{z}_2\, z_2 & \cdots & -\overline{z}_2\, z_n \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{z}_n\, z_1 & -\overline{z}_n\, z_2 & \cdots & 1 + \|z\|^2 - \overline{z}_n\, z_n \end{bmatrix} \end{aligned}$$

and by [Mor07, Formula (12.6)], the associated Ricci form is then given by

$$\rho_{\mathrm{FS}} := -\mathrm{i} \partial \overline{\partial} \log(\det(H)).$$

Notice that $\det(H)$ is a special case of $P308$ with

$$x_i = \frac{1 + \|z\|^2 - |z_i|^2}{(1 + \|z\|^2)^2}, \quad a_i = \frac{-\overline{z}_i}{(1 + \|z\|^2)^2}, \quad b_i = \frac{z_i}{(1 + \|z\|^2)^2} \qquad \text{for } i = 1, \dots, n.$$

Now a straightforward application of Formula (2) gives $\det(H) = (1 + \|z\|^2)^{-n-1}$. This implies that

$$\rho_{\mathrm{FS}} = -\mathrm{i} \partial \overline{\partial} \log((1 + \|z\|^2)^{-n-1}) = (r + 1)\, \omega_{\mathrm{FS}},$$

showing that the Fubini-Study metric is Einstein with $r + 1$ as Einstein constant.

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Dipartimento di Scienze Matematiche, Politecnico di Torino. Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Email address: antonio.discala@polito.it

Institució Catalana de Recerca i Estudis Avançats (ICREA). Passeig Lluís Companys 23, 08010 Barcelona, Spain

Departament de Matemàtiques i Informàtica, Universitat de Barcelona. Gran Via 585, 08007 Barcelona, Spain

Email address: sombra@ub.edu