A POISSON FORMULA FOR THE SPARSE RESULTANT

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Abstract. We present a Poisson formula for sparse resultants and a formula for the product of the roots of a family of Laurent polynomials, which are valid for arbitrary families of supports.

To obtain these formulae, we show that the sparse resultant associated to a family of supports can be identified with the resultant of a suitable multiprojective toric cycle in the sense of Rémond. This connection allows to study sparse resultants using multiprojective elimination theory and intersection theory of toric varieties.

1. Introduction

Sparse resultants are widely used in polynomial equation solving, a fact that has sparked a lot of interest in their computational and applied aspects, see for instance [CE00, Stu02, D'A02, JKSS04, CLO05, DE05, JMSW09]. They have also been studied from a more theoretical point of view because of their connections with combinatorics, toric geometry, residue theory, and hypergeometric functions [GKZ94, Stu94, CDS98, Kho99, CDS01, Est10].

Sparse elimination theory focuses on ideals and varieties defined by Laurent polynomials with given supports, in the sense that the exponents in their monomial expansion are *a priori* determined. The classical approach to this theory consists in regarding such Laurent polynomials as global sections of line bundles on a suitable projective toric variety. Using this interpretation, sparse elimination theory can be reduced to projective elimination theory. In particular, sparse resultants can be studied *via* the Chow form of this projective toric variety as it is done in [PS93, GKZ94, Stu94]. This approach works correctly when all considered line bundles are very ample, but might fail otherwise. In particular, important results obtained in this way, like the product formulae due to Pedersen and Sturmfels [PS93, Theorem 1.1 and Proposition 7.1], do not hold for families of Laurent polynomials with arbitrary supports.

In this paper, we define and study sparse resultants using the multiprojective elimination theory introduced by Rémond in [Rém01] and further developed in our joint paper with Krick [DKS13]. This approach gives a better framework to understand sparse elimination theory. In particular, it allows to understand precisely in which situations some classical formulae for sparse resultants hold, and how to modify them to work in general.

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In precise terms, let $M \simeq \mathbb{Z}^n$ be a lattice of rank $n \geq 0$ and $N = \mathrm{Hom}(M, \mathbb{Z})$ its dual lattice. Let $\mathbb{T}_M = \mathrm{Hom}(M, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^n$ be the associated algebraic torus over $\mathbb{C}$ and, for $a \in M$, we denote by $\chi^a \colon \mathbb{T}_M \to \mathbb{C}^\times$ the corresponding character.

Let $\mathcal{A}_i$, $i = 0, \dots, n$, be a family of $n+1$ nonempty finite subsets of M and put

$$\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_n).$$

For each i, consider a set of $\#\mathcal{A}_i$ variables $\boldsymbol{u}_i = \{u_{i,a}\}_{a \in \mathcal{A}_i}$ and let

$$F_i = \sum_{a \in \mathcal{A}_i} u_{i,a} \chi^a \in \mathbb{C}[\boldsymbol{u}_i][M] \tag{1.1}$$

be the general Laurent polynomial with support $\mathcal{A}_i$, where we denote by $\mathbb{C}[\boldsymbol{u}_i][M] \simeq \mathbb{C}[\boldsymbol{u}_i][t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ the group $\mathbb{C}[\boldsymbol{u}_i]$-algebra of M. Consider also the incidence variety given by

$$\Omega_{\boldsymbol{\mathcal{A}}} = \{(\xi, \boldsymbol{u}) \mid F_0(\boldsymbol{u}_0, \xi) = \dots = F_n(\boldsymbol{u}_n, \xi) = 0\} \subset \mathbb{T}_M \times \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}).$$

It is a subvariety of codimension $n+1$ defined over $\mathbb{Q}$.

We define the $\boldsymbol{\mathcal{A}}$-*resultant* or *sparse resultant*, denoted by $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$, as any primitive polynomial in $\mathbb{Z}[\boldsymbol{u}_0, \dots, \boldsymbol{u}_n]$ giving an equation for the direct image $\pi_* \Omega_{\boldsymbol{\mathcal{A}}}$ (Definition 2.1) where

$$\pi \colon \mathbb{T}_M \times \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i}) \to \prod_{i=0}^n \mathbb{P}(\mathbb{C}^{\mathcal{A}_i})$$

is the projection onto the second factor. It is well-defined up to a sign. This notion of sparse resultant coincides with the one proposed by Esterov in [Est10, Definition 3.1].

The informed reader should be aware that the $\boldsymbol{\mathcal{A}}$-resultant is usually defined as an irreducible polynomial in $\mathbb{Z}[\boldsymbol{u}]$ giving an equation for the Zariski closure $\overline{\pi(\Omega_{\boldsymbol{\mathcal{A}}})}$, if this is a hypersurface, and as 1 otherwise, as it is done in [GKZ94, Stu94]. In this paper, we call this object the $\boldsymbol{\mathcal{A}}$-*eliminant* or *sparse eliminant* instead, and we denote it by $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$. It follows from these definitions that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm \, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{d_{\boldsymbol{\mathcal{A}}}}, \tag{1.2}$$

with $d_{\boldsymbol{\mathcal{A}}}$ equal to the degree of the restriction of π to the incidence variety $\Omega_{\boldsymbol{\mathcal{A}}}$. This degree is not necessarily equal to 1 and so, in general, the sparse resultant and the sparse eliminant are different objects, see Example 3.14.

The definition of the sparse resultant in terms of a direct image rather than just a set-theoretical image, has better properties and produces more uniform statements. For instance, the partial degrees of the sparse resultant are given, for $i = 0, \dots, n$, by

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n),$$

where $\Delta_i \subset M_\mathbb{R}$ is the lattice polytope given as the convex hull of $\mathcal{A}_i$ and MV_M is the mixed volume function associated to the lattice M (Proposition 3.4). This equality holds for *any* family of supports, independently of their combinatorics.

One of our motivations comes from the need of a general Poisson formula for sparse resultants for our joint work with Galligo on the distribution of roots of families of Laurent polynomials [DGS13]. By a *Poisson formula* we mean an equality of the form

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(f_0, f_1, \dots, f_n) = Q(f_1, \dots, f_n) \cdot \prod_\xi f_0(\xi)^{m_\xi},$$

where $f_i \in \mathbb{C}[M]$ is a generic Laurent polynomial with support $\mathcal{A}_i$, $i = 0, \dots, n$, the product is over the roots ξ of $f_1, \dots, f_n$ in $\mathbb{T}_M$, m_ξ is the multiplicity of ξ, and $Q \in \mathbb{Q}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_n)^\times$ is a rational function to be determined. A formula of this type was stated by Pedersen and Sturmfels in [PS93] but it does not hold for arbitrary supports. An attempt to make it valid in full generality was made by Minimair in [Min03], but his approach has some inaccuracies.

The main result of this paper is the Poisson formula for the sparse resultant given below, which holds for *any* family of supports. We introduce some notation to state this properly.

Let $v \in N \setminus \{0\}$ and put $v^\perp \cap M \simeq \mathbb{Z}^{n-1}$ for its orthogonal lattice. For $i = 1, \dots, n$, we set $\mathcal{A}_{i,v}$ for the subset of points of $\mathcal{A}_i$ of minimal weight in the direction of v. This gives a family of n nonempty finite subsets of translates of the lattice $v^\perp \cap M$. We denote by $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}$ the corresponding sparse resultant, also called the sparse resultant of $\mathcal{A}_1, \dots, \mathcal{A}_n$ in the direction of v. Given Laurent polynomials $f_i \in \mathbb{C}[M]$ with support $\mathcal{A}_i$, $i = 1, \dots, n$, we denote by

$$\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v}, \dots, f_{n,v}) \in \mathbb{C}$$

the evaluation of this directional sparse resultant at the coefficients of the initial part of the f_i's in the direction of v, see Definition 4.1 for details. We also set $h_{\mathcal{A}_0}(v) = \min_{a \in \mathcal{A}_0} \langle v, a \rangle$ for the value at v of the support function of $\mathcal{A}_0$.

Theorem 1.1. *Let $\mathcal{A}_i \subset M$ be a nonempty finite subset and $f_i \in \mathbb{C}[M]$ a Laurent polynomial with support contained in $\mathcal{A}_i$, $i = 0, \dots, n$. Suppose that for all $v \in N \setminus \{0\}$ we have that $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v}, \dots, f_{n,v}) \neq 0$. Then*

$$\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\dots,\mathcal{A}_n}(f_0, f_1, \dots, f_n) = \pm \prod_v \mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v}, \dots, f_{n,v})^{-h_{\mathcal{A}_0}(v)} \cdot \prod_\xi f_0(\xi)^{m_\xi},$$

the first product being over the primitive vectors $v \in N$ and the second over the roots ξ of $f_1, \dots, f_n$ in $\mathbb{T}_M$, and where m_ξ denotes the multiplicity of ξ.

Both products in the above formula are finite. Indeed, $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}} \neq 1$ only if v is an inner normal to a facet of the Minkowski sum $\sum_{i=1}^n \Delta_i$. Moreover, by Bernstein theorem [Ber75, Theorem B], the hypothesis that no directional sparse resultant vanishes implies that the set of roots of the family f_i, $i = 1, \dots, n$, is finite.

Example 1.2. Let $M = \mathbb{Z}^2$ and consider the family of nonempty finite subsets of $\mathbb{Z}^2$

$$\mathcal{A}_0 = \mathcal{A}_1 = \{(0,0), (-1,0), (0,-1)\}, \quad \mathcal{A}_2 = \{(0,0), (1,0), (0,1), (0,2)\}.$$

Consider also a family of generic Laurent polynomials in $\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ supported in these subsets, that is

$$f_i = \alpha_{i,0} + \alpha_{i,1}\, t_1^{-1} + \alpha_{i,2}\, t_2^{-1}, i = 0, 1, \quad f_2 = \alpha_{2,0} + \alpha_{2,1}\, t_1 + \alpha_{2,2}\, t_2 + \alpha_{2,3}\, t_2^2.$$

with $\alpha_{i,j} \in \mathbb{C}$.

The resultant $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2}$ is a polynomial in two sets of 3 variables and a set of 4 variables. It is multihomogeneous of multidegree $(3, 3, 1)$ and has 24 terms.

Considering the Minkowski sum $\Delta_1 + \Delta_2$ we obtain that, in this case, the only nontrivial directional sparse resultants are those corresponding to the vectors $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-2, -1)$, and $(0, -1)$. Computing them together with their corresponding exponents in the Poisson formula, Theorem 1.1 shows that

$$\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2}(f_0,f_1,f_2)=\pm\alpha_{1,2}\,\alpha_{1,1}^2\,\alpha_{2,0}\prod_{i=1}^{3}f_0(\xi_i).$$

where the ξ_i's are the solutions of the system of equations $f_1=f_2=0$.

Each of the supports generates the lattice $\mathbb{Z}^2$, and so $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}=\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$. However, the formula in [PS93, Theorem 1.1] gives in this case an exponent 1 to the coefficient $\alpha_{1,1}$, instead of 2. Hence, this formula does not work in this case. Minimair's reformulation of the Pedersen-Sturmfels formula in [Min03, Theorem 8] gives an expression for the exponent of $\alpha_{1,1}$ that evaluates to $\frac{0}{0}$, and so it also fails in this case.

As a by-product of our approach, we obtain a formula for the product of the roots of a family of Laurent polynomials. For a nonzero complex number $\gamma\in\mathbb{C}^\times$ and $v\in N$, we consider the point in the torus $\gamma^v\in\mathbb{T}_M$ given by the homomorphism $M\to\mathbb{C}^\times$, $a\mapsto\langle a,v\rangle$.

Corollary 1.3. *Let $\mathcal{A}_i\subset M$ be a nonempty finite subset and $f_i\in\mathbb{C}[M]$ a Laurent polynomial with support contained in $\mathcal{A}_i$, $i=1,\dots,n$. Suppose that for all $v\in N\setminus\{0\}$ we have that $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v},\dots,f_{n,v})\neq 0$. Then*

$$\prod_{\xi}\xi^{m_\xi}=\pm\prod_{v}\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v},\dots,f_{n,v})^{v},$$

the first product being over the roots ξ of $f_1,\dots,f_n$ in $\mathbb{T}_M$ and the second over the primitive vectors $v\in N$, and where m_ξ denotes the multiplicity of ξ. Equivalently, for $a\in M$,

$$\prod_{\xi}\chi^a(\xi)^{m_\xi}=\pm\prod_{v}\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v},\dots,f_{n,v})^{\langle a,v\rangle}.$$

This result makes explicit both the scalar factor and the exponents in Khovanskii's formula in [Kho99, §6, Theorem 1].

As a consequence of the Poisson formula in Theorem 1.1, we obtain an extension to the sparse setting of the "hidden variable" technique for solving polynomial equations, which is crucial for computational purposes [CLO05, §3.5], see Theorem 1.4 below.

To do this, let $n\geq 1$ and set $M=\mathbb{Z}^n$ and, for $i=1,\dots,n$, consider the general Laurent polynomials $F_i\in\mathbb{Z}[\boldsymbol{u}_i][t_1^{\pm1},\dots,t_n^{\pm1}]$ with support $\mathcal{A}_i$ as in (1.1). Each F_i can be alternatively considered as a Laurent polynomial in the variables $\boldsymbol{t}':=\{t_1,\dots,t_{n-1}\}$ and coefficients in the ring $\mathbb{Z}[\boldsymbol{u}_i][{t_n}^{\pm1}]$. In this case, we denote it by $F_i(\boldsymbol{t}')$. The support of this Laurent polynomial is the nonempty finite subset $\varpi(\mathcal{A}_i)\subset\mathbb{Z}^{n-1}$, where $\varpi\colon\mathbb{R}^n\to\mathbb{R}^{n-1}$ denotes the projection onto the first $n-1$ coordinates of $\mathbb{R}^n$. We then set

$$\mathrm{Res}^{t_n}_{\mathcal{A}_1,\dots,\mathcal{A}_n}=\mathrm{Res}_{\varpi(\mathcal{A}_1),\dots,\varpi(\mathcal{A}_n)}(F_1(\boldsymbol{t}'),\dots,F_n(\boldsymbol{t}'))\in\mathbb{C}[\boldsymbol{u}_1,\dots,\boldsymbol{u}_n][t_n^{\pm1}].\tag{1.3}$$

In other words, we "hide" the variable t_n among the coefficients of the F_i's and we consider the corresponding sparse resultant.

The following result shows that the roots of this Laurent polynomial coincide with the t_n-coordinate of the roots of the family f_i, $i=1,\dots,n$, and that their corresponding multiplicities are preserved. It generalizes and precises [CLO05, Proposition 5.15], which is stated for generic families of dense polynomial equations.

Theorem 1.4. *Let $\mathcal{A}_i \subset \mathbb{Z}^n$ be a nonempty finite subset and $f_i \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ a Laurent polynomial with support contained in $\mathcal{A}_i$, $i = 1, \dots, n$. Suppose that for all $v \in \mathbb{Z}^n \setminus \{0\}$ we have that $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(f_{1,v}, \dots, f_{n,v}) \neq 0$. Then there exist $\lambda \in \mathbb{C}^\times$ and $d \in \mathbb{Z}$ such that*

$$\mathrm{Res}^{t_n}_{\mathcal{A}_1,\dots,\mathcal{A}_n}(f_1, \dots, f_n) = \lambda\, t_n^d \prod_{\xi} (t_n - \xi_n)^{m_\xi}, \tag{1.4}$$

the product being being over the roots $\xi = (\xi_1, \dots, \xi_n)$ of $f_1, \dots, f_n$ in $(\mathbb{C}^\times)^n$, and where m_ξ denotes the multiplicity of ξ.

Indeed, the exponent d in (1.4) can be made explicit in terms of "mixed integrals" in the sense of [PS08, Definition 1.1] or, equivalently, "shadow mixed volumes" as in [Est08, Definition 1.7], see Remark 4.8 for further details.

In addition, we also obtain a product formula for the addition of supports (Corollary 4.6) and we extend the height bound for the sparse resultant in [Som04, Theorem 1.1] to arbitrary collections of supports (Proposition 3.15).

The exponent $d_{\boldsymbol{\mathcal{A}}}$ in (1.2) can be expressed in combinatorial terms (Proposition 3.13). Hence, all formulae and properties for the sparse resultant can be restated for sparse eliminants at the cost of paying attention to the relative position of the supports with respect to the lattice generated by an essential subfamily, see §3 for details.

Our approach is based on multiprojective elimination theory. Let $Z_{\boldsymbol{\mathcal{A}}}$ be the multiprojective toric cycle associated to the family $\boldsymbol{\mathcal{A}}$, and denote by $|Z_{\boldsymbol{\mathcal{A}}}|$ its supporting subvariety. In Proposition 3.2 we show that

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(|Z_{\boldsymbol{\mathcal{A}}}|), \quad \mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}}),$$

were $\mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}$ and $\mathrm{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}$ respectively denote the eliminant and the resultant associated to the vectors $\boldsymbol{e}_i$, $i = 0, \dots, n$, in the standard basis of $\mathbb{Z}^{n+1}$, see §2.2 for details. Both eliminants and resultants play an important role in this theory, but it is well known that multiprojective resultants are the central objects because they reflect better the geometric operations at an algebraic level.

Our proof of Theorem 1.1 is based on the standard properties of multiprojective resultants and on tools from toric geometry, together with the classical Bernstein's theorem and its refinement for valued fields due to Smirnov [Smi96].

We remark that the formula in [PS93, Theorem 1.1] is stated for general Laurent polynomials and that it amounts to an equality modulo an unspecified scalar factor in $\mathbb{Q}^\times$. In Theorem 4.2, we extend this product formula to an arbitrary family of supports and we precise the value of this scalar factor up to a sign. Theorem 1.1 follows from this result after showing that the formula in Theorem 4.2 can be evaluated into a particular family of Laurent polynomials exactly when no directional sparse resultant vanishes.

The paper is organized as follows: in §2 we introduce some notation and show a number of preliminary results concerning intersection theory on multiprojective spaces, toric varieties and cycles, and root counting on algebraic tori. In §3 we show that the sparse eliminant and the sparse resultant respectively coincide with the eliminant and the resultant of a multiprojective toric cycle and, using this interpretation, we derive some of their basic properties from the corresponding ones for general eliminants and resultants. In §4 we prove the Poisson formula for the sparse resultant and we derive some of its consequences. In §5 we give some more examples, compare our results with previous ones, and establish sufficient conditions for these previous results to hold.

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2. Preliminaries

All along this text, bold symbols indicate finite sets or sequences of objects, where the type and number should be clear from the context. For instance, $\boldsymbol{x}$ might denote the set of variables $\{x_1, \ldots, x_n\}$ so that, if K is a field, then $K[\boldsymbol{x}] = K[x_1, \ldots, x_n]$.

We denote by $\mathbb{N}$ the set of nonnegative integers. Given a vector $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, we set $|\boldsymbol{b}| = \sum_{i=1}^n b_i$ for its length.

2.1. **Cycles on multiprojective spaces.** In this subsection, we give the notation and basic facts on intersection theory of multiprojective spaces. Most of the material is taken from [DKS13, §1.1 and 1.2].

Let K be a field and $\mathbb{K}$ an algebraically closed field containing K. For instance, K and $\mathbb{K}$ might be taken as $\mathbb{Q}$ and $\mathbb{C}$, respectively. For $m \geq 0$ and $\boldsymbol{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$, we consider the multiprojective space over K given by

$$\mathbb{P}_K^{\boldsymbol{n}} = \mathbb{P}_K^{n_0} \times \cdots \times \mathbb{P}_K^{n_m}. \tag{2.1}$$

For $i = 0, \ldots, m$, let $\boldsymbol{x}_i = \{x_{i,0}, \ldots, x_{i,n_i}\}$ be a set of $n_i + 1$ variables and put $\boldsymbol{x} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m\}$. The multihomogeneous coordinate ring of $\mathbb{P}_K^{\boldsymbol{n}}$ is then given by $K[\boldsymbol{x}] = K[\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m]$. It is multigraded by declaring $\mathbf{deg}(x_{i,j}) = \boldsymbol{e}_i \in \mathbb{N}^{m+1}$, the $(i+1)$-th vector of the standard basis of $\mathbb{R}^{m+1}$. For $\boldsymbol{d} = (d_0, \ldots, d_m) \in \mathbb{N}^{m+1}$, we denote by $K[\boldsymbol{x}]_{\boldsymbol{d}}$ its component of multidegree $\boldsymbol{d}$.

Set

$$\mathbb{N}_{d_i}^{n_i+1} = \{\boldsymbol{a}_i \in \mathbb{N}^{n_i+1} \mid |\boldsymbol{a}_i| = d_i\} \quad \text{and} \quad \mathbb{N}_{\boldsymbol{d}}^{\boldsymbol{n}+\boldsymbol{1}} = \prod_{i=0}^m \mathbb{N}_{d_i}^{n_i+1}. \tag{2.2}$$

With this notation, a multihomogeneous polynomial $f \in K[\boldsymbol{x}]_{\boldsymbol{d}}$ writes down as

$$f = \sum_{\boldsymbol{a} \in \mathbb{N}_{\boldsymbol{d}}^{\boldsymbol{n}+\boldsymbol{1}}} \alpha_{\boldsymbol{a}} \, \boldsymbol{x}^{\boldsymbol{a}}$$

where, for each index $\boldsymbol{a} \in \mathbb{N}_{\boldsymbol{d}}^{\boldsymbol{n}+\boldsymbol{1}}$, $\alpha_{\boldsymbol{a}}$ denotes an element of K and $\boldsymbol{x}^{\boldsymbol{a}} = \prod_{i,j} x_{i,j}^{a_{i,j}}$.

A *cycle* on $\mathbb{P}_K^{\boldsymbol{n}}$ is a $\mathbb{Z}$-linear combination

$$X = \sum_V m_V V \tag{2.3}$$

where the sum is over the irreducible subvarieties V of $\mathbb{P}_K^{\boldsymbol{n}}$ and $m_V = 0$ for all but a finite number of V. The subvarieties V such that $m_V \neq 0$ are called the *irreducible components* of X. The *support* of X, denoted by $|X|$, is the union of its irreducible components. We also denote by $X_{\mathbb{K}}$ the cycle on $\mathbb{P}_{\mathbb{K}}^{\boldsymbol{n}}$ obtained from X by the base change $K \hookrightarrow \mathbb{K}$, that is

$$X_{\mathbb{K}} = \sum_V m_V (V \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\mathbb{K})).$$

A cycle is *equidimensional* or of *pure dimension* if all its irreducible components are of the same dimension. For $r = 0, \ldots, |\boldsymbol{n}|$, we denote by $Z_r(\mathbb{P}_K^{\boldsymbol{n}})$ the group of cycles on $\mathbb{P}^{\boldsymbol{n}}$ of pure dimension r.

Given a multihomogeneous ideal $I \subset K[\boldsymbol{x}]$, we denote by $V(I)$ the subvariety of $\mathbb{P}^{\boldsymbol{n}}_K$ defined by I. For each minimal prime ideal P of I, we denote by m_P its *multiplicity*, defined as the length of the $K[\boldsymbol{x}]$-module $(K[\boldsymbol{x}]/I)_P$. We then set

$$Z(I) = \sum_P m_P V(P)$$

for the cycle on $\mathbb{P}^{\boldsymbol{n}}_K$ defined by I.

We denote by $\mathrm{Div}(\mathbb{P}^{\boldsymbol{n}}_K)$ the group of Cartier divisors on $\mathbb{P}^{\boldsymbol{n}}_K$. Given a multihomogeneous rational function $f \in K(\boldsymbol{x})^\times$, we denote by

$$\mathrm{div}(f) \in \mathrm{Div}(\mathbb{P}^{\boldsymbol{n}}_K)$$

the associated Cartier divisor. Using [Har77, Propositions II.6.2 and II.6.11] and the fact that the ring $K[\boldsymbol{x}]$ is factorial, we can verify that that every Cartier divisor on $\mathbb{P}^{\boldsymbol{n}}_K$ is of this form.

Let X be a cycle of pure dimension r and $D \in \mathrm{Div}(\mathbb{P}^{\boldsymbol{n}}_K)$ a Cartier divisor intersecting X properly. We denote by $X \cdot D$ the intersection product of X and D, with intersection multiplicities as in [Har77, §I.1.7, page 53], see also [DKS13, Definition 1.3]. It is a cycle of pure dimension $r-1$.

Let $X \in Z_r(\mathbb{P}^{\boldsymbol{n}}_K)$ and $\boldsymbol{b} \in \mathbb{N}^m_r$ a vector of length r. For $i = 0, \dots, m$, we denote by $H_{i,j} \in \mathrm{Div}\left(\mathbb{P}^{\boldsymbol{n}}_{\mathbb{K}}\right)$, $j = 1, \dots, b_i$, the inverse image under the projection $\mathbb{P}^{\boldsymbol{n}}_{\mathbb{K}} \to \mathbb{P}^{n_i}_{\mathbb{K}}$ of a family of b_i generic hyperplanes of $\mathbb{P}^{n_i}_{\mathbb{K}}$. The *degree* of X of index $\boldsymbol{b}$, denoted by $\deg_{\boldsymbol{b}}(X)$, is defined as the degree of the 0-dimensional cycle

$$X_{\mathbb{K}} \cdot \prod_{i=0}^{m} \prod_{j=1}^{b_i} H_{i,j}.$$

The Chow ring of $\mathbb{P}^{\boldsymbol{n}}_K$, denoted by $A^*(\mathbb{P}^{\boldsymbol{n}}_K)$, can be written down as

$$A^*(\mathbb{P}^{\boldsymbol{n}}_K) = \mathbb{Z}[\theta_0, \dots, \theta_m]/(\theta_0^{n_0+1}, \dots, \theta_m^{n_m+1}) \tag{2.4}$$

where θ_i denotes the class of the inverse image under the projection $\mathbb{P}^{\boldsymbol{n}}_K \to \mathbb{P}^{n_i}_K$ of a hyperplane of $\mathbb{P}^{n_i}_K$ [Ful98, Example 8.4.2].

Given $X \in Z_r(\mathbb{P}^{\boldsymbol{n}}_K)$, its class in the Chow ring is

$$[X] = \sum_{\boldsymbol{b}} \deg_{\boldsymbol{b}}(X)\, \theta_0^{n_0-b_0} \cdots \theta_m^{n_m-b_m}, \tag{2.5}$$

the sum being over all $\boldsymbol{b} \in \mathbb{N}^{m+1}_r$ such that $b_i \leq n_i$ for all i. It is a homogeneous element of $A^*(\mathbb{P}^{\boldsymbol{n}}_K)$ of degree $|\boldsymbol{n}| - r$ containing the information of all the mixed degrees of X. In the particular case when $X = Z(f)$ with $f \in K[\boldsymbol{x}]_{\boldsymbol{d}}$, we have that

$$[Z(f)] = \sum_{i=0}^{m} d_i \theta_i.$$

Let $X \in Z_r(\mathbb{P}^{\boldsymbol{n}}_K)$ and $f \in K[\boldsymbol{x}]$ a multihomogeneous polynomial such that X and $\mathrm{div}(f)$ intersect properly. The multiprojective Bézout theorem says that

$$[X \cdot \mathrm{div}(f)] = [X] \cdot [Z(f)], \tag{2.6}$$

see for instance [DKS13, Theorem 1.11].

Definition 2.1. Let $\varphi\colon \mathbb{P}_K^{\boldsymbol{n}_1} \to \mathbb{P}_K^{\boldsymbol{n}_2}$ be a morphism and V an irreducible subvariety of $\mathbb{P}_K^{\boldsymbol{n}_1}$ of dimension r. The *degree* of φ on V is defined as

$$\deg(\varphi|_V) = \begin{cases} [K(V) : K(\varphi(V))] & \text{if } \dim(\varphi(V)) = r, \\ 0 & \text{if } \dim(\varphi(V)) < r. \end{cases}$$

The *direct image* under φ of V is defined as $\varphi_* V = \deg(\varphi|_V)\, \varphi(V)$. It is a cycle of dimension r. This notion extends by linearity to equidimensional cycles and induces a linear map

$$\varphi_*\colon Z_r(\mathbb{P}_K^{\boldsymbol{n}_1}) \longrightarrow Z_r(\mathbb{P}_K^{\boldsymbol{n}_2}).$$

Let H be a hypersuface of $\mathbb{P}_K^{\boldsymbol{n}_2}$ not containing the image of φ. The *inverse image* of H under φ is defined as the hypersurface $\varphi^* H = \varphi^{-1}(H)$. This notion extends by linearity to a $\mathbb{Z}$-linear map

$$\varphi^* : \mathrm{Div}(\mathbb{P}_K^{\boldsymbol{n}_2}) \dashrightarrow \mathrm{Div}(\mathbb{P}_K^{\boldsymbol{n}_1}),$$

well-defined for Cartier divisors whose support does not contain the image of φ.

Direct images of cycles, inverse images of Cartier divisors and intersection products are related by the *projection formula* [Ser65, Chapter V, §C.7, formula (11)]: let $\varphi : \mathbb{P}_K^{\boldsymbol{n}_1} \to \mathbb{P}_K^{\boldsymbol{n}_2}$ be a morphism, X an equidimensional cycle on $\mathbb{P}_K^{\boldsymbol{n}_1}$ and D a Cartier divisor on $\mathbb{P}_K^{\boldsymbol{n}_2}$ intersecting $\varphi_* X$ properly. Then

$$\varphi_* X \cdot D = \varphi_*(X \cdot \varphi^* D). \tag{2.7}$$

Lemma 2.2. *Let $0 \le q \le m$ and denote by $\mathrm{pr}\colon \mathbb{P}_K^{\boldsymbol{n}} \to \prod_{i=0}^q \mathbb{P}_K^{n_i}$ the projection onto the first $q+1$ factors of $\mathbb{P}_K^{\boldsymbol{n}}$. Let $X \in Z_r(\mathbb{P}_K)$ and $\boldsymbol{b} \in \mathbb{N}_r^{q+1}$. Then*

$$\deg_{\boldsymbol{b}}(\mathrm{pr}_* X) = \deg_{\boldsymbol{b},\boldsymbol{0}}(X).$$

Proof. We suppose without loss of generality that K is algebraically closed. We proceed by induction on the dimension of X. For $r = 0$ we have that $X = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} \boldsymbol{\xi}$ with $\boldsymbol{\xi} \in \mathbb{P}_K^{\boldsymbol{n}}$ and $m_{\boldsymbol{\xi}} \in \mathbb{Z}$, and $\boldsymbol{b} = \boldsymbol{0} \in \mathbb{N}^{q+1}$. Then $\mathrm{pr}_* X = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}}\, \mathrm{pr}_* \boldsymbol{\xi}$ and so

$$\deg_{\boldsymbol{0}}(\mathrm{pr}_* X) = \sum_{\boldsymbol{\xi}} m_{\boldsymbol{\xi}} = \deg_{\boldsymbol{0},\boldsymbol{0}}(X).$$

Now let $r \ge 1$. Choose $0 \le i_0 \le q$ such that $b_{i_0} \ge 1$ and let $H \in \mathrm{Div}\left(\prod_{i=0}^q \mathbb{P}_K^{n_i}\right)$ be the inverse image of a generic hyperplane of $\mathbb{P}_K^{n_{i_0}}$ under the projection of $\prod_{i=0}^q \mathbb{P}_K^{n_i}$ onto the i_0-th factor. This Cartier divisor intersects $\mathrm{pr}_* X$ properly and, by (2.7),

$$\mathrm{pr}_* X \cdot H = \mathrm{pr}_*(X \cdot \mathrm{pr}^* H).$$

Using this, together with the multiprojective Bézout theorem in (2.6) and the inductive hypothesis, we deduce that

$$\begin{aligned} \deg_{\boldsymbol{b}}(\mathrm{pr}_*(X)) = \deg_{\boldsymbol{b}-\boldsymbol{e}_{i_0}}(\mathrm{pr}_* X \cdot H) &= \deg_{\boldsymbol{b}-\boldsymbol{e}_{i_0}}(\mathrm{pr}_*(X \cdot \mathrm{pr}^* H)) \\ &= \deg_{\boldsymbol{0},\boldsymbol{b}-\boldsymbol{e}_{i_0}}(X \cdot \mathrm{pr}^* H) = \deg_{\boldsymbol{0},\boldsymbol{b}}(X), \end{aligned}$$

which proves the statement. □

We refer to [DKS13, §1.2] for other properties of mixed degrees of cycles, including their behavior with respect to linear projections, products and ruled joins.

2.2. **Eliminants and resultants of multiprojective cycles.** In this subsection, we recall the notions and basic properties of eliminants of varieties and resultants of cycles following Rémond [Rém01] and our joint paper with Krick [DKS13]. We also give an alternative definition of these objects with a more geometric flavor, and show that both coincide (Proposition 2.5).

Let A be a factorial ring with field of fractions K. Let $\boldsymbol{n} = (n_0, \dots, n_m) \in \mathbb{N}^{m+1}$ and let $\mathbb{P}_K^{\boldsymbol{n}}$ be the corresponding multiprojective space as in (2.1). Given $r \geq 0$ and a family of vectors $\boldsymbol{d} = (\boldsymbol{d}_0, \dots, \boldsymbol{d}_r) \in (\mathbb{N}^{m+1} \setminus \{\boldsymbol{0}\})^{r+1}$, we set

$$N_i = \#\mathbb{N}_{d_i}^{n+1} - 1 = \prod_{j=0}^{m} \binom{d_{i,j} + n_j}{n_j} - 1, \quad i = 0, \dots, r,$$

with $\mathbb{N}_{d_i}^{n+1}$ as in (2.2). We will work in the multiprojective space $\mathbb{P}_K^{\boldsymbol{N}} = \prod_{i=0}^{r} \mathbb{P}_K^{N_i}$ with $\boldsymbol{N} = (N_0, \dots, N_r) \in \mathbb{N}^{r+1}$. For each i we consider a set of $N_i + 1$ variables $\boldsymbol{u}_i = \{u_{i,\boldsymbol{a}}\}_{\boldsymbol{a} \in \mathbb{N}_{\boldsymbol{d}}^{\boldsymbol{n}+\boldsymbol{1}}}$. The coordinates of $\mathbb{P}_K^{N_i}$ are indexed by the elements of $\mathbb{N}_{d_i}^{n+1}$, and so $K[\boldsymbol{u}_i]$ is the homogeneous coordinate ring of $\mathbb{P}_K^{N_i}$. Hence, if we set $\boldsymbol{u} = \{\boldsymbol{u}_0, \dots, \boldsymbol{u}_r\}$, then $K[\boldsymbol{u}]$ is the multihomogeneous coordinate ring of $\mathbb{P}_K^{\boldsymbol{N}}$.

Consider the general multihomogeneous polynomial of multidegree $\boldsymbol{d}_i$ given by

$$F_i = \sum_{\boldsymbol{a} \in \mathbb{N}_{\boldsymbol{d}_i}^{\boldsymbol{n}+\boldsymbol{1}}} u_{i,\boldsymbol{a}}\, \boldsymbol{x}^{\boldsymbol{a}} \in K[\boldsymbol{u}_i][\boldsymbol{x}], \tag{2.8}$$

and denote by $\mathrm{div}(F_i)$ the Cartier divisor on $\mathbb{P}_K^{\boldsymbol{n}} \times \mathbb{P}_K^{\boldsymbol{N}}$ it defines. Given $X \in Z_r(\mathbb{P}_K^{\boldsymbol{n}})$, the family of Cartier divisors $\mathrm{div}(F_i)$, $i = 0, \dots, r$, intersects $X \times \mathbb{P}_K^{\boldsymbol{N}}$ properly. We then set

$$\Omega_{X,\boldsymbol{d}} = (X \times \mathbb{P}_K^{\boldsymbol{N}}) \cdot \prod_{i=0}^{r} \mathrm{div}(F_i), \tag{2.9}$$

which is a cycle on $\mathbb{P}_K^{\boldsymbol{n}} \times \mathbb{P}_K^{\boldsymbol{N}}$ of pure codimension $|\boldsymbol{n}| + 1$. When $X = V$ is an irreducible subvariety, it coincides with the incidence variety of V and F_i's. Consider also the morphism given by the projection onto the second factor

$$\rho \colon \mathbb{P}_K^{\boldsymbol{n}} \times \mathbb{P}_K^{\boldsymbol{N}} \longrightarrow \mathbb{P}_K^{\boldsymbol{N}}. \tag{2.10}$$

Definition 2.3. Let $V \subset \mathbb{P}_K^{\boldsymbol{n}}$ be an irreducible subvariety of dimension r and $\boldsymbol{d} \in (\mathbb{N}^{m+1} \setminus \{\boldsymbol{0}\})^{r+1}$. The *eliminant* of V of index $\boldsymbol{d}$, denoted by $\mathrm{Elim}_{\boldsymbol{d}}(V)$, is defined as any irreducible polynomial in $A[\boldsymbol{u}]$ giving an equation for the image $\rho(\Omega_{V,\boldsymbol{d}})$ if it is a hypersurface, and as 1 otherwise.

Definition 2.4. Let $V \subset \mathbb{P}_K^{\boldsymbol{n}}$ be an irreducible subvariety of dimension r and $\boldsymbol{d} \in (\mathbb{N}^{m+1} \setminus \{\boldsymbol{0}\})^{r+1}$. The *resultant* of V of index $\boldsymbol{d}$, denoted by $\mathrm{Res}_{\boldsymbol{d}}(X)$, is defined as any primitive polynomial in $A[\boldsymbol{u}]$ giving an equation for the direct image $\rho_* \Omega_{V,\boldsymbol{d}}$.

More generally, let $X \in Z_r(\mathbb{P}_K^{\boldsymbol{n}})$ and write $X = \sum_V m_V V$ as in (2.3). Then, the *resultant* of X of index $\boldsymbol{d}$ is defined as

$$\mathrm{Res}_{\boldsymbol{d}}(X) = \prod_{V} \mathrm{Res}_{\boldsymbol{d}}(V)^{m_V}.$$

Both eliminants and resultants are well-defined up to an scalar factor in $A^{\times}$, the group of units of A.

The eliminant $\mathrm{Elim}_{\boldsymbol{d}}(V)$ can be alternatively defined as an irreducible equation for the support of the direct image $\rho_*\Omega_{V,\boldsymbol{d}}$. Hence

$$\mathrm{Res}_{\boldsymbol{d}}(V) = \lambda\, \mathrm{Elim}_{\boldsymbol{d}}(V)^{\deg(\rho|_{\Omega_{V,\boldsymbol{d}}})}$$

with $\lambda \in A^\times$. The exponent $\deg(\rho|_{\Omega_{V,\boldsymbol{d}}})$ is not necessarily equal to 1 and so eliminants and resultants do not necessarily coincide, see for instance [DKS13, Example 1.31].

The definitions of these objects in [Rém01, DKS13] are given in more algebraic terms. We now show that our present definitions coincide with theirs.

Proposition 2.5. *The notions of eliminants and resultants in Definitions 2.3 and 2.4 respectively coincide, up to a scalar factor in $A^\times$, with those in* [DKS13, Definitions 1.25 and 1.26].

Proof. Let notation be as Definitions 2.3 and 2.4, and denote temporarily by $\widetilde{\mathrm{Elim}}_{\boldsymbol{d}}(V)$ and $\widetilde{\mathrm{Res}}_{\boldsymbol{d}}(V)$ the eliminant and the resultant from [DKS13]. By Proposition 1.37(2) and Lemma 1.34 in *loc. cit.*, all four zero sets of $\mathrm{Elim}_{\boldsymbol{d}}(V)$, $\mathrm{Res}_{\boldsymbol{d}}(V)$, $\widetilde{\mathrm{Elim}}_{\boldsymbol{d}}(V)$ and $\widetilde{\mathrm{Res}}_{\boldsymbol{d}}(V)$ coincide. By construction, both $\mathrm{Elim}_{\boldsymbol{d}}(V)$ and $\widetilde{\mathrm{Elim}}_{\boldsymbol{d}}(V)$ are irreducible and so they coincide up to a scalar factor in $A^\times$, proving the statement for the eliminants.

Both resultants are powers of the same irreducible polynomial. Hence, to prove the rest of the statement it is enough to show that their mixed degrees coincide.

Let $0 \le i \le r$. By [DKS13, Propositions 1.10(4)] and Lemma 2.2,

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{d}}(V)) = \deg_{\boldsymbol{N}-\boldsymbol{e}_i}(\rho_*\Omega_{V,\boldsymbol{d}}) = \deg_{\boldsymbol{0},\boldsymbol{N}-\boldsymbol{e}_i}(\Omega_{V,\boldsymbol{d}}), \tag{2.11}$$

where $\boldsymbol{e}_i$ denotes the $(i+1)$-th vector in the standard basis of $\mathbb{Z}^{r+1}$.

Let θ_i, $i = 0, \dots, m$, and ζ_j, $j = 0, \dots, r$, respectively denote the variables in the Chow rings $A^*(\mathbb{P}_K^{\boldsymbol{n}})$ and $A^*(\mathbb{P}_K^{\boldsymbol{N}})$ as in (2.4). Let $[\Omega_{V,\boldsymbol{d}}]$ denote the class of the incidence variety in the Chow ring $A^*(\mathbb{P}_K^{\boldsymbol{n}} \times \mathbb{P}_K^{\boldsymbol{N}}) \simeq A^*(\mathbb{P}_K^{\boldsymbol{n}}) \otimes A^*(\mathbb{P}_K^{\boldsymbol{N}})$. By (2.5),

$$\deg_{\boldsymbol{0},\boldsymbol{N}-\boldsymbol{e}_i}(\Omega_{V,\boldsymbol{d}}) = \mathrm{coeff}_{\boldsymbol{\theta}^{\boldsymbol{n}}\zeta_i}([\Omega_{V,\boldsymbol{d}}]). \tag{2.12}$$

By the multiprojective Bézout theorem in (2.6), $[\Omega_{V,\boldsymbol{d}}] = [V \times \mathbb{P}^{\boldsymbol{N}}] \cdot \prod_{i=0}^n [Z(F_i)]$, where F_i is the general polynomial as in (2.8). By [DKS13, Propositions 1.19(2) and 1.10(2,4)], the classes in $A^*(\mathbb{P}_K^{\boldsymbol{n}}) \otimes A^*(\mathbb{P}_K^{\boldsymbol{N}})$ of $V \times \mathbb{P}_K^{\boldsymbol{N}}$ and $Z(F_i)$ are given by

$$[V \times \mathbb{P}^{\boldsymbol{N}}] = [V] \otimes 1 \quad \text{and} \quad [Z(F_i)] = \zeta_i + \sum_{j=0}^m d_{i,j}\theta_j,$$

where $[V]$ denotes the class of V in $A^*(\mathbb{P}_K^{\boldsymbol{n}})$. Hence,

$$\begin{aligned} \mathrm{coeff}_{\boldsymbol{\theta}^{\boldsymbol{n}}\zeta_i}([\Omega_{V,d}]) &= \mathrm{coeff}_{\boldsymbol{\theta}^{\boldsymbol{n}}\zeta_i}\Big(([V]\otimes 1) \cdot \prod_{i=0}^n \Big(\zeta_i + \sum_{j=0}^m d_{i,j}\theta_j\Big)\Big) \\ &= \mathrm{coeff}_{\boldsymbol{\theta}^{\boldsymbol{n}}}\Big([V] \cdot \prod_{\ell\neq i}^n \sum_{j=0}^m d_{\ell,j}\theta_j\Big). \end{aligned} \tag{2.13}$$

Then, (2.11), (2.12) and (2.13) together with Proposition 1.32 in *loc. cit.*, imply that

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{d}}(V)) = \deg_{\boldsymbol{u}_i}(\widetilde{\mathrm{Res}}_{\boldsymbol{d}}(V)).$$

Hence, both resultants coincide up to a scalar factor in $A^\times$. The general case when X is a cycle of pure dimension r follows by linearity. □

Let $V \subset \mathbb{P}^{\boldsymbol{n}}_K$ be an irreducible subvariety of dimension r and $\boldsymbol{d} \in (\mathbb{N}^{m+1} \setminus \{\boldsymbol{0}\})^{r+1}$. Each set of variables $\boldsymbol{u}_i$ corresponds to the coefficients of a multihomogeneous polynomial of degree $\boldsymbol{d}_i$. Hence, given $f_i \in \mathbb{K}[\boldsymbol{x}]_{\boldsymbol{d}_i}$, $i = 0, \dots, r$, we can write

$$\mathrm{Elim}_{\boldsymbol{d}}(V)(f_0, \dots, f_r) \quad \text{and} \quad \mathrm{Res}_{\boldsymbol{d}}(X)(f_0, \dots, f_r)$$

for the evaluation of the eliminant and of the resultant at the coefficients of the f_i's, respectively.

Eliminants and resultants are polynomials whose vanishing at a given family of multihomogeneous polynomials corresponds to the condition that this family has a common root on V: if $\rho(\Omega_{V,\boldsymbol{d}})$ is a hypersurface, then

$$\mathrm{Res}_{\boldsymbol{d}}(V)(f_0, \dots, f_r) = 0 \Longleftrightarrow V \cap V(f_0, \dots, f_r) \neq \emptyset, \tag{2.14}$$

and a similar statement holds for the eliminant.

A central property of resultants is that they translate intersection of cycles and Cartier divisors into evaluation. In precise terms, let $X \in Z_r(\mathbb{P}^{\boldsymbol{n}}_K)$ be a cycle of pure dimension r, $\boldsymbol{d} = (\boldsymbol{d}_0, \dots, \boldsymbol{d}_r) \in (\mathbb{N}^{m+1} \setminus \{\boldsymbol{0}\})^{r+1}$, and $f \in K[\boldsymbol{x}]_{\boldsymbol{d}_r}$ such that $\mathrm{div}(f)$ intersects X properly. Then

$$\mathrm{Res}_{\boldsymbol{d}_0,\boldsymbol{d}_1\dots,\boldsymbol{d}_r}(X)(\boldsymbol{u}_0, \dots, \boldsymbol{u}_{r-1}, f) = \lambda \ \mathrm{Res}_{\boldsymbol{d}_0,\dots,\boldsymbol{d}_{r-1}}(X \cdot \mathrm{div}(f))(\boldsymbol{u}_0, \dots, \boldsymbol{u}_{r-1}),$$

with $\lambda \in K^\times$, see [Rém01, Proposition 3.6] or [DKS13, Proposition 1.40].

Resultants also behave well with respect to other geometric constructions including linear projections, products and ruled joins. Both eliminants and resultants are invariant under index permutations and field extensions. The partial degrees of a resultant are given by the mixed degrees of the underlying cycle, a fact already exploited in the proof of Proposition 2.5. The statements of these properties and their proofs can be found in [Rém01, DKS13].

2.3. **Multiprojective toric varieties and cycles.** In this subsection, we set the standard notation for multiprojective toric varieties and cycles, and prove some preliminary results, most notably a formula for the intersection of a multiprojective toric cycle and a toric Cartier divisor (Proposition 2.8). We assume a basic knowledge of the theory of normal toric varieties as explained in [Ful93, CLS11].

Let $n \geq 0$ and $M \simeq \mathbb{Z}^n$ a lattice of rank n, and set $N = M^\vee = \mathrm{Hom}(M, \mathbb{Z})$ for its dual lattice. Set also $M_\mathbb{R} = M \otimes \mathbb{R}$ and $N_\mathbb{R} = N \otimes \mathbb{R}$. The pairing between $x \in M_\mathbb{R}$ and $u \in N_\mathbb{R}$ is denoted by $\langle x, u \rangle$.

For a field K, we set

$$\mathbb{T}_{M,K} = \mathrm{Spec}(K[M]) \tag{2.15}$$

for the *(algebraic) torus* over K corresponding to M. For simplicity, we will focus on the case $K = \mathbb{K}$ is algebraically closed, although all notions and results in this subsection are valid, with suitable modifications, over an arbitrary field. In our situation, we write $\mathbb{T}_M = \mathbb{T}_{M,\mathbb{K}}$ for short. Since $\mathbb{K}$ is algebraically closed, we can identify this torus with its set of points. With this identification,

$$\mathbb{T}_M = \mathrm{Hom}(M, \mathbb{K}^\times) = N \otimes \mathbb{K}^\times \simeq (\mathbb{K}^\times)^n$$

For $a \in M$, we denote by $\chi^a \colon \mathbb{T}_M \to \mathbb{K}^\times$ the corresponding group homomorphism or *character* of $\mathbb{T}_M$.

For $m \geq 0$, consider a family of nonempty finite subsets $\mathcal{A}_i = \{a_{i,0}, \dots, a_{i,c_i}\} \subset M$, $i = 0, \dots, m$, and set $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \dots, \mathcal{A}_m)$. Set $\boldsymbol{c} = (c_0, \dots, c_m)$ and consider the associated multiprojective space over $\mathbb{K}$

$$\mathbb{P}^{\boldsymbol{c}} = \mathbb{P}^{\boldsymbol{c}}_{\mathbb{K}} = \prod_{i=0}^{m} \mathbb{P}^{c_i}_{\mathbb{K}}.$$

For each i, we denote by $\boldsymbol{x}_i = \{x_{i,0}, \dots, x_{i,c_i}\}$ a set of $c_i + 1$ variables and we put $\boldsymbol{x} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\}$, so that $\mathbb{K}[\boldsymbol{x}] = \mathbb{K}[\boldsymbol{x}_1, \dots, \boldsymbol{x}_m]$ is the multihomogeneous coordinate ring of $\mathbb{P}^{\boldsymbol{c}}$.

Let $\varphi_{\boldsymbol{\mathcal{A}}} \colon \mathbb{T}_M \to \mathbb{P}^{\boldsymbol{c}}$ be the monomial map given, for $\xi \in \mathbb{T}_M$, by

$$\varphi_{\boldsymbol{\mathcal{A}}}(\xi) = ((\chi^{a_{0,0}}(\xi) : \dots : \chi^{a_{0,c_0}}(\xi)), \dots, (\chi^{a_{m,0}}(\xi) : \dots : \chi^{a_{m,c_m}}(\xi))). \tag{2.16}$$

We then set

$$X_{\boldsymbol{\mathcal{A}}} = \overline{\varphi_{\boldsymbol{\mathcal{A}}}(\mathbb{T}_M)}, \quad Z_{\boldsymbol{\mathcal{A}}} = (\varphi_{\boldsymbol{\mathcal{A}}})_* \mathbb{T}_M \tag{2.17}$$

for the associated multiprojective toric subvariety and toric cycle, respectively.

For $i = 0, \dots, m$, consider the sublattice of M given by

$$L_{\mathcal{A}_i} = \sum_{j=1}^{c_i} (a_{i,j} - a_{i,0}) \mathbb{Z}, \tag{2.18}$$

and put $L_{\boldsymbol{\mathcal{A}}} = \sum_{i=0}^{m} L_{\mathcal{A}_i}$. By [CLS11, Proposition 1.1.8], it follows that

$$\dim(X_{\boldsymbol{\mathcal{A}}}) = \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}}). \tag{2.19}$$

In particular, $X_{\boldsymbol{\mathcal{A}}}$ coincides with the support of $Z_{\boldsymbol{\mathcal{A}}}$ if and only if $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}}) = n$. Otherwise, $\dim(X_{\boldsymbol{\mathcal{A}}}) \leq n - 1$ and $Z_{\boldsymbol{\mathcal{A}}} = 0$.

For $i = 0, \dots, m$, consider the convex hull

$$\Delta_i = \operatorname{conv}(\mathcal{A}_i) \subset M_{\mathbb{R}}.$$

It is a lattice polytope lying in a translate of the linear space $L_{\mathcal{A}_i,\mathbb{R}} = L_{\mathcal{A}_i} \otimes \mathbb{R}$. We also set $\Delta = \sum_{i=0}^{m} \Delta_i$ for its Minkowski sum, which is a lattice polytope lying in a translate of $L_{\boldsymbol{\mathcal{A}},\mathbb{R}} = L_{\boldsymbol{\mathcal{A}}} \otimes \mathbb{R}$. We denote by Σ_Δ the conic polyhedral complex on $N_{\mathbb{R}}$ given by the inner directions of Δ as in [Ful93, page 26] or [CLS11, Proposition 6.2.3]. If $\dim(\Delta) = n$, then Σ_Δ is a fan.

The multiprojective toric variety $X_{\boldsymbol{\mathcal{A}}}$ is not necessarily normal. The next lemma shows that we can construct a proper normal toric variety dominating it by considering any fan refining Σ_Δ. As it is customary, we denote by X_Σ the normal toric variety over $\mathbb{K}$ corresponding to a fan Σ on $N_{\mathbb{R}}$.

Lemma 2.6. *Let Σ be a fan in $N_{\mathbb{R}}$ refining Σ_Δ. The map $\varphi_{\boldsymbol{\mathcal{A}}}$ in* (2.16) *extends to a morphism of proper toric varieties*

$$\Phi_{\boldsymbol{\mathcal{A}}} \colon X_\Sigma \longrightarrow \mathbb{P}^{\boldsymbol{c}}. \tag{2.20}$$

In particular, $X_{\boldsymbol{\mathcal{A}}} = \Phi_{\boldsymbol{\mathcal{A}}}(X_\Sigma)$ and $Z_{\boldsymbol{\mathcal{A}}} = (\Phi_{\boldsymbol{\mathcal{A}}})_* X_\Sigma$.

Proof. Let Σ^{c_i} be the normal fan of the standard simplex of $\mathbb{R}^{c_i}$, $i = 0, \dots, m$, and set $\Sigma^{\boldsymbol{c}} = \prod_{i=0}^{r} \Sigma^{c_i}$, which is a fan on $\mathbb{R}^{\boldsymbol{c}}$. For each i, the toric variety associated to Σ_i is $\mathbb{P}^{c_i}$ and so, by [CLS11, Proposition 3.1.14], the toric variety associated to $\Sigma^{\boldsymbol{c}}$ is the multiprojective space $\mathbb{P}^{\boldsymbol{c}}$.

The map

$$(\mathbb{K}^\times)^{|\boldsymbol{c}|} \longrightarrow \mathbb{P}^{\boldsymbol{c}}, \quad (\boldsymbol{z}_0, \dots, \boldsymbol{z}_m) \longmapsto ((1 : \boldsymbol{z}_0), \dots, (1 : \boldsymbol{z}_m))$$

gives an isomorphism between the torus $(\mathbb{K}^\times)^{|\boldsymbol{c}|}$ and the open orbit $\mathbb{P}_0^{\boldsymbol{c}}$ of $\mathbb{P}^{\boldsymbol{n}}$. The image of $\varphi_{\boldsymbol{\mathcal{A}}}$ is contained in this orbit and the map $\varphi_{\boldsymbol{\mathcal{A}}}\colon \mathbb{T}_M \to \mathbb{P}_0^{\boldsymbol{c}}$ is a homomorphism of tori. Under the correspondence in [CLS11, Theorem 3.3.4], this homomorphism corresponds to the linear map $A\colon N \to \mathbb{Z}^{\boldsymbol{c}}$ given, for $u \in N$, by

$$A(u) = (\langle a_{i,j} - a_{i,0}, u\rangle)_{0\le i\le m, 1\le j\le c_i}. \tag{2.21}$$

We have that $A^{-1}(\Sigma^{\boldsymbol{c}}) = \Sigma_\Delta$. Since Σ refines Σ_Δ, it follows that this linear map is compatible with the fans Σ and $\Sigma^{\boldsymbol{c}}$ in the sense of [CLS11, Definition 3.3.1]. By Theorem 3.3.4(a) in *loc. cit.*, $\varphi_{\boldsymbol{\mathcal{A}}}$ extends to a proper toric map $\Phi_{\boldsymbol{\mathcal{A}}}\colon X_\Sigma \to \mathbb{P}^{\boldsymbol{c}}$.

Since $\Phi_{\boldsymbol{\mathcal{A}}}$ is a map of proper toric varieties and $\mathbb{T}_M$ is a dense open subset of X_Σ,

$$\Phi_{\boldsymbol{\mathcal{A}}}(X_\Sigma) = \overline{\varphi_{\boldsymbol{\mathcal{A}}}(\mathbb{T}_M)} = X_{\boldsymbol{\mathcal{A}}}, \quad (\Phi_{\boldsymbol{\mathcal{A}}})_* X_\Sigma = (\varphi_{\boldsymbol{\mathcal{A}}})_*\mathbb{T}_M = Z_{\boldsymbol{\mathcal{A}}},$$

which completes the proof. □

Lemma 2.7. *Let notation be as in* (2.16) *and* (2.17)*. Then*

$$X_{\boldsymbol{\mathcal{A}}} \setminus \bigcup_{i=0}^{n}\bigcup_{j=0}^{c_i} V(x_{i,j}) = \varphi_{\boldsymbol{\mathcal{A}}}(\mathbb{T}_M).$$

Proof. By translating the subsets $\mathcal{A}_i$ and restricting them to the sublattice $L_{\boldsymbol{\mathcal{A}}}$, we can reduce without loss of generality to the case when $M = L_{\boldsymbol{\mathcal{A}}}$. Assume that we are in this situation, and consider the morphism of proper toric varieties $\Phi_{\boldsymbol{\mathcal{A}}}\colon X_\Sigma \longrightarrow \mathbb{P}^{\boldsymbol{c}}$ in (2.20) and its associated linear map $A\colon N \to \mathbb{Z}^{\boldsymbol{c}}$ as in (2.21). For each cone $\sigma \in \Sigma$ we denote by $O(\sigma)$ the associated orbit under the orbit-cone correspondence explained in [Ful93, §3.1] and [CLS11, §3.2].

The correspondence $\sigma \mapsto O(\sigma)$ is a bijection and so there is a decomposition

$$X_\Sigma = \bigsqcup_{\sigma\in\Sigma} O(\sigma).$$

We have that $O(0) = \mathbb{T}_M$ and $\Phi_{\boldsymbol{\mathcal{A}}}(O(0)) = \varphi_{\boldsymbol{\mathcal{A}}}(\mathbb{T}_M)$ is contained in $\mathbb{P}_0^{\boldsymbol{c}}$, the open orbit of $\mathbb{P}^{\boldsymbol{c}}$. On the other hand, the hypothesis that $M = L_{\boldsymbol{\mathcal{A}}}$ implies that the linear map A is injective and so, given $\sigma \in \Sigma \setminus \{0\}$, we have that $A(\sigma) \neq 0$. By [CLS11, Lemma 3.3.21(b)], $\Phi_{\boldsymbol{\mathcal{A}}}(O(\sigma))$ is contained in $\mathbb{P}^{\boldsymbol{c}} \setminus \mathbb{P}_0^{\boldsymbol{c}} = \bigcup_{i,j} V(x_{i,j})$. It follows that

$$X_{\boldsymbol{\mathcal{A}}} \setminus \bigcup_{i,j} V(x_{i,j}) = \Big(\bigcup_{\sigma\in\Sigma} \Phi_{\boldsymbol{\mathcal{A}}}(O(\sigma))\Big) \setminus \bigcup_{i,j} V(x_{i,j}) = \Phi_{\boldsymbol{\mathcal{A}}}(O(0)) = \varphi_{\boldsymbol{\mathcal{A}}}(\mathbb{T}_M),$$

as stated. □

Now suppose that the lattice polytope Δ has dimension n and let Γ be a facet, that is, a face of Δ of codimension 1. Let $L_{\Gamma\cap M} \simeq \mathbb{Z}^{n-1}$ be the sublattice of M generated by the differences of the lattice points of Γ and $\mathbb{T}_{L_{\Gamma\cap M}} \simeq (\mathbb{K}^\times)^{n-1}$ its associated torus. Let $v(\Gamma) \in N$ denote the primitive inner normal vector of Γ and, for each i, set Γ_i for the face of Δ_i which minimizes the functional $v(\Gamma)\colon M_\mathbb{R} \to \mathbb{R}$ on Δ_i.

We consider the morphism $\varphi_{\boldsymbol{\mathcal{A}},\Gamma}\colon \mathbb{T}_{L_{\Gamma\cap M}} \to \mathbb{P}^{\boldsymbol{c}}$ given, for $\xi \in \mathbb{T}_{L_{\Gamma\cap M}}$, by

$$\varphi_{\boldsymbol{\mathcal{A}},\Gamma}(\xi)_{i,j} = \begin{cases} \chi^{a_{i,j}}(\xi) & \text{if } a_{i,j} \in \Gamma_i, \\ 0 & \text{otherwise.}\end{cases} \tag{2.22}$$

Set $Z_{\boldsymbol{\mathcal{A}},\Gamma} = (\varphi_{\boldsymbol{\mathcal{A}},\Gamma})_*(\mathbb{T}_{L_{\Gamma\cap M}}) \in Z_{n-1}(\mathbb{P}^{\boldsymbol{c}})$ for the associated multiprojective toric cycle.

For a bounded subset $P \subset M_{\mathbb{R}}$, we define its *support function* as the function $h_P\colon N_{\mathbb{R}} \to \mathbb{R}$ given, for $v \in N_{\mathbb{R}}$, by

$$h_P(v) = \inf_{x \in P} \langle v, x \rangle. \tag{2.23}$$

The usual convention in convex analysis is to define support functions as convex functions by putting a "sup" instead of the "inf" in the formula above as it is done, for instance, in [Sch93, page 37]. Our notion of support function gives a *concave* function, and is better suited to toric geometry.

Proposition 2.8. *Let notation be as above and let* $0 \le i \le m$ *and* $0 \le j \le c_i$. *If* $\dim(\Delta) = n$, *then*

$$Z_{\boldsymbol{\mathcal{A}}} \cdot \operatorname{div}(x_{i,j}) = \sum_{\Gamma} -h_{\Delta_i - a_{i,j}}(v(\Gamma))\, Z_{\boldsymbol{\mathcal{A}},\Gamma}, \tag{2.24}$$

where the sum is over the facets Γ *of* Δ. *Otherwise,* $Z_{\boldsymbol{\mathcal{A}}} \cdot \operatorname{div}(x_{i,a_{i,j}}) = 0$.

Proof. By symmetry, we can suppose without loss of generality that $i = j = 0$. Consider first the case when $\dim(\Delta) = n$. Then Σ_Δ is a fan and so, by Lemma 2.6, the map $\varphi_{\boldsymbol{\mathcal{A}}}$ extends to a morphism of proper toric varieties

$$\Phi_{\boldsymbol{\mathcal{A}}}\colon X_{\Sigma_\Delta} \longrightarrow X_{\boldsymbol{\mathcal{A}}}$$

and $Z_{\boldsymbol{\mathcal{A}}} = (\Phi_{\boldsymbol{\mathcal{A}}})_* X_{\Sigma_\Delta}$.

Set $D = (\Phi_{\boldsymbol{\mathcal{A}}})^* \operatorname{div}(x_{0,0}) \in \operatorname{Div}(X_{\Sigma_\Delta})$. By the projection formula (2.7),

$$Z_{\boldsymbol{\mathcal{A}}} \cdot \operatorname{div}(x_{0,0}) = (\Phi_{\mathcal{A}})_*(X_{\Sigma_\Delta} \cdot D). \tag{2.25}$$

Let $\Psi_D\colon N_{\mathbb{R}} \to \mathbb{R}$ be the virtual support function of D under the correspondence in [CLS11, Theorem 4.2.12]. Using either Theorem 4.2.12(b) in *loc. cit.* or [Ful93, Lemma, page 61], it follows that

$$X_{\Sigma_\Delta} \cdot D = \sum_{\tau} -\Psi_D(v_\tau) V(\tau), \tag{2.26}$$

the sum being over the rays τ of Σ_Δ, where v_τ denotes the first nonzero vector in $\tau \cap N$ and $V(\tau)$ denotes the $\mathbb{T}_M$-invariant prime Weil divisor of X_{Σ_Δ} determined by τ.

Let $\Delta^{c_0} = \operatorname{conv}(\mathbf{0}, \boldsymbol{e}_{0,1}, \dots, \boldsymbol{e}_{0,c_0})$ be the standard simplex of $\mathbb{R}^{c_0}$ and $\Delta^{c_0} \times \{\mathbf{0}\}$ its immersion into $\mathbb{R}^{\boldsymbol{c}}$. We can verify that the virtual support function associated to the Cartier divisor $\operatorname{div}(x_{0,0}) \in \operatorname{Div}(\mathbb{P}^{\boldsymbol{c}})$ under the correspondence in [CLS11, Theorem 4.2.12] coincides with $h_{\Delta^{c_0} \times \{\mathbf{0}\}}$, the support function of this polytope. By *loc. cit*, Proposition 6.2.7, $\Psi_D = h_{\Delta^{c_0} \times \{\mathbf{0}\}} \circ A$ where $A\colon N \to \mathbb{Z}^{\boldsymbol{c}}$ denotes the linear map in (2.21). This implies that

$$\Psi_D = h_{\Delta_0 - a_{0,0}}, \tag{2.27}$$

the support function of the translated polytope $\Delta_0 - a_{0,0} \subset M_{\mathbb{R}}$.

By construction, the rays of Σ_Δ are the inner normal directions of the facets of Δ. For each ray τ, the prime Weil divisor $V(\tau)$ is the closure of the orbit $O(\tau)$ associated to τ under the orbit-cone correspondence. We denote by $\tau^\perp$ the subspace of $M_{\mathbb{R}}$ orthogonal to τ and by $\iota_\tau\colon \mathbb{T}_{\tau^\perp \cap M} \to O(\tau)$ the isomorphism in [CLS11, Lemma 3.2.5].

Let Γ be the facet of Δ corresponding to τ. Hence, $v_\tau = v(\Gamma)$, the primitive inner normal vector of Γ. We can verify that $\tau^\perp \cap M = L_{\Gamma \cap M}$ and so $\mathbb{T}_{\tau^\perp \cap M} = \mathbb{T}_{L_{\Gamma \cap M}}$, and that the composition $\Phi_{\boldsymbol{\mathcal{A}}} \circ \iota_\tau$ coincides with the map $\varphi_{\boldsymbol{\mathcal{A}},\Gamma}$ in (2.22). Hence

$$(\Phi_{\boldsymbol{\mathcal{A}}})_* V(\tau) = Z_{\boldsymbol{\mathcal{A}},\Gamma}. \tag{2.28}$$

The formula (2.24) then follows from (2.25), (2.26), (2.27) and (2.28).

In the case when $\dim(\Delta) < n$, we have that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}}) < n$. It follows that $Z_{\boldsymbol{\mathcal{A}}} = 0$ by (2.19) and, *a fortiori*, that $Z_{\boldsymbol{\mathcal{A}}} \cdot \operatorname{div}(x_{0,0}) = 0$. □

2.4. **Root counting on algebraic tori.** It is well-known that the number of roots of a family of Laurent polynomial is related to the combinatorics of the exponents appearing in its monomial expansion. For the convenience of the reader, we recall the results in this direction that we will use in the sequel.

We denote by vol_M the Haar measure on $M_{\mathbb{R}}$ normalized so that M has covolume 1. The *mixed volume* of a family of compact bodies $Q_1, \dots, Q_n \subset M_{\mathbb{R}}$ is defined as

$$\operatorname{MV}_M(Q_1, \dots, Q_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \le i_1 < \cdots < i_j \le n} \operatorname{vol}_M(Q_{i_1} + \cdots + Q_{i_j}). \tag{2.29}$$

For $n = 0$ we agree that $\operatorname{MV}_M = 1$.

We have that $\operatorname{MV}_M(Q, \dots, Q) = n! \operatorname{vol}_M(Q)$ and so the mixed volume can be seen as a generalization of the volume of a convex body. The mixed volume is symmetric and linear in each variable Q_i with respect to the Minkowski sum, invariant with respect to isomorphisms of lattices, and monotone with respect to the inclusion of compact bodies of $M_{\mathbb{R}}$, see for instance [CLO05, §7.4] or [Sch93, Chapter 5].

Let K be a field and $\overline{K}$ its algebraic closure. Given a square family of Laurent polynomials $f_i \in K[M]$, $i = 1, \dots, n$, we denote by $Z(f_1, \dots, f_n)$ the cycle on $\mathbb{T}_{M,\overline{K}}$, given by its isolated roots together with their corresponding multiplicities. In precise terms,

$$Z(f_1, \dots, f_n) = \sum_{\xi} m_\xi \, \xi, \tag{2.30}$$

the sum being over the isolated points ξ of $V(f_1, \dots, f_n) \subset \mathbb{T}_{M,\overline{K}}$ and where, if $I(\xi) \subset \overline{K}[M]$ denotes the ideal of ξ, the multiplicity m_ξ is given by

$$m_\xi = \dim_{\overline{K}}(\overline{K}[M]/(f_1, \dots, f_n))_{I(\xi)}. \tag{2.31}$$

Write

$$f_i = \sum_{j=0}^{c_i} \alpha_{i,j} \chi^{a_{i,j}}, \quad i = 1, \dots, n, \tag{2.32}$$

with $\alpha_{i,j} \in K^\times$ and $a_{i,j} \in M$. The *Newton polytope* of f_i is given by

$$\Delta_i = \operatorname{N}(f_i) = \operatorname{conv}(a_{i,0}, \dots, a_{i,c_i}) \subset M_{\mathbb{R}}.$$

For $v \in N_{\mathbb{R}}$, we denote by $\Delta_{i,v} \subset M_{\mathbb{R}}$ the subset of points of Δ_i whose weight in the direction of v is minimal. It is a face of Δ_i. We also set

$$f_{i,v} = \sum_j \alpha_{i,j} \chi^{a_{i,j}} \in K[M], \quad i = 1, \dots, n,$$

the sum being over $0 \le j \le c_i$ such that $a_{i,j} \in \Delta_{i,v}$.

Bernstein's theorem [Ber75, Theorem B] states that, if $\operatorname{char}(K) = 0$ and, for all $v \in N \setminus \{0\}$, the family $f_{i,v}$, $i = 1, \dots, n$, has no root in $\mathbb{T}_{M,\overline{K}}$, then $V(f_1, \dots, f_n)$ is finite and

$$\deg(Z(f_1, \dots, f_n)) = \operatorname{MV}_M(\Delta_1, \dots, \Delta_n). \tag{2.33}$$

This statement also holds for an arbitrary field K [PS08, Proposition 1.4].

When K is endowed with a discrete valuation $\mathrm{val}\colon K^\times \to \mathbb{R}$, there is a refinement of Bernstein's theorem due to Smirnov [Smi96], that gives a combinatorial expression for the number of roots with a given valuation.

To state it properly, let K° and $K^{\circ\circ}$ denote the valuation ring and its maximal ideal associated to the pair (K, val). Let κ be a uniformizer of K°, that is, a generator of $K^{\circ\circ}$, and $k = K^\circ/K^{\circ\circ}$ the residue field. For $\alpha \in K$, the *initial part* of α with respect to κ, denoted by $\mathrm{init}_\kappa(\alpha)$, is defined as the class in k of the element $\kappa^{-\mathrm{val}(\alpha)}\alpha \in K^\circ$.

Consider also an arbitrary extension of the valuation to $\overline{K}$. Since $\mathbb{T}_{M,\overline{K}} = N_\mathbb{R} \otimes \overline{K}^\times$, this valuation induces a map $\mathbb{T}_{M,\overline{K}} \to N_\mathbb{R}$, that we also denote by val. For a square family of Laurent polynomials as before and $w \in N_\mathbb{R}$, we consider the cycle on $\mathbb{T}_{M,\overline{K}}$ given by

$$Z(f_1, \dots, f_n)_w = \sum_\xi m_\xi \, \xi,$$

the sum being over the isolated points ξ of $V(f_1, \dots, f_n) \subset \mathbb{T}_{M,\overline{K}}$ such that $\mathrm{val}(\xi) = w$, and with multiplicities m_ξ as in (2.30).

For $i = 1, \dots, n$, we consider the lifted polytope of f_i defined as

$$\widetilde{\Delta}_i = \mathrm{conv}((a_{i,0}, -\mathrm{val}(\alpha_{i,0})), \dots, (a_{i,c_i}, -\mathrm{val}(\alpha_{i,c_i}))) \subset M_\mathbb{R} \times \mathbb{R}. \tag{2.34}$$

Given $w \in N_\mathbb{R}$, we denote by $\widetilde{\Delta}_{i,(w,1)} \subset M_\mathbb{R} \times \mathbb{R}$ the subset of points of $\widetilde{\Delta}_i$ whose weight in the direction of $(w, 1)$ is minimal. It is a face of this lifted polytope contained in its upper envelope. Then we set $\Delta_{i,(w,1)} \subset M_\mathbb{R}$ for the image of this face under the projection $M_\mathbb{R} \times \mathbb{R} \to M_\mathbb{R}$. We also set

$$f_{i,(w,1)} = \sum_j \mathrm{init}_\kappa(\alpha_{i,j})\chi^{a_{i,j}} \in k[M], \quad i = 1, \dots, n,$$

the sum being over $0 \le j \le c_i$ such that $(a_{i,j}, -\mathrm{val}(\alpha_{i,j})) \in \widetilde{\Delta}_{i,(w,1)}$.

In this situation, Smirnov's theorem [Smi96, Theorem 3.2.2(b)] states that if, for all $w \in N$ such that $\dim(\sum_{i=1}^n \widetilde{\Delta}_{i,(w,1)}) < n$ the family of Laurent polynomials $f_{i,(w,1)} \in k[M]$, $i = 1, \dots, n$, has no root in $\mathbb{T}_{M,\overline{k}}$, then, for any $w_0 \in N_\mathbb{R}$, the set of points of $V(f_1, \dots, f_n)$ with valuation w_0 is finite and

$$\deg(Z(f_1, \dots, f_n)_{w_0}) = \mathrm{MV}_M(\Delta_{1,(w_0,1)}, \dots, \Delta_{n,(w_0,1)}). \tag{2.35}$$

We are interested in the following generic situation. For $i = 1, \dots, n$, let $\boldsymbol{u}_i = \{u_{i,0}, \dots, u_{i,c_i}\}$ be a set of $c_i + 1$ variables and set $\overline{\boldsymbol{u}} = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$. For a polynomial $R = \sum_{\boldsymbol{b}} \beta_{\boldsymbol{b}} \overline{\boldsymbol{u}}^{\boldsymbol{b}} \in K[\overline{\boldsymbol{u}}]$, we set

$$\mathrm{val}(R) = \min_{\boldsymbol{b}} \mathrm{val}(\beta_{\boldsymbol{b}}). \tag{2.36}$$

By Gauss' lemma, this gives a discrete valuation on the field $\mathbb{F} := K(\overline{\boldsymbol{u}})$ that extends val. We then consider an arbitrary extension of this valuation to the algebraic closure $\overline{\mathbb{F}}$ and the associated map $\mathrm{val}\colon \mathbb{T}_{M,\overline{\mathbb{F}}} \to N_\mathbb{R}$ as before. We denote by $\mathfrak{f}$ the residue field of $\mathbb{F}$.

Proposition 2.9. *With notation as above, set*

$$F_i = \sum_{j=0}^{c_i} u_{i,j}\chi^{a_{i,j}} \in K[\boldsymbol{u}_i][M], \quad i = 1, \dots, n.$$

Then $V(F_1,\dots,F_n)\subset \mathbb{T}_{M,\overline{\mathbb{F}}}$ *is finite,* $\deg(Z(F_1,\dots,F_n))=\mathrm{MV}_M(\Delta_1,\dots,\Delta_n)$*, and* $\mathrm{val}(\xi)=0$ *for all* $\xi\in V(F_1,\dots,F_n)$.

Proof. Let $v\in N\setminus\{0\}$. Since $\Delta_{i,v}$ lies in a translate of the orthogonal space $v^\perp$, the roots of in the torus of the system $F_{i,v}$, $i=1,\dots,n$, are the roots of an equivalent system of n general Laurent polynomials in $n-1$ variables. Hence, this set of roots is empty, and Bernstein's theorem (2.33) implies the first and the second claims.

For the last claim, denote by $\widetilde{\Delta}_i\subset M_\mathbb{R}\times\mathbb{R}$ the lifted polytope in associated to F_i as in (2.34), $i=1,\dots,n$. Since $\mathrm{val}(u_{i,j})=0$ for all j, we have that $\widetilde{\Delta}_{i,(w,1)}=\Delta_{i,w}\times\{0\}$. We deduce that, for $w\in N$, $\widetilde{\Delta}_{i,(w,1)}=\Delta_{i,w}\times\{0\}$ and $F_{i,(w,1)}$ coincides with the class of $F_{i,w}$ in the polynomial ring $\mathfrak{f}[\overline{\boldsymbol{u}}]$.

Suppose now that $\dim(\sum_{i=1}^n\widetilde{\Delta}_{i,(w,1)})<n$. Then $\dim(\sum_{i=1}^n\Delta_{i,w})<n$ and, similarly as before, the system $F_{i,(w,1)}$, $i=1,\dots,n$, has no roots in $\mathbb{T}_{M,\bar{\mathfrak{f}}}$. Smirnov's theorem applied to the case when $f_i=F_i$, $i=1,\dots,n$, and $w_0=0$, implies that

$$\deg(Z(F_1,\dots,F_n)_0)=\mathrm{MV}_M(\Delta_1,\dots,\Delta_n)=\deg(Z(F_1,\dots,F_n)).$$

Hence all points of $V(F_1,\dots,F_n)$ have valuation 0, which concludes the proof. □

3. Basic properties of sparse eliminants and resultants

In this section, we show that the sparse eliminant and the sparse resultant respectively coincide with the eliminant and the resultant of a multiprojective toric variety/cycle. Using this interpretation, we derive some of their basic properties from the corresponding ones for general eliminants and resultants.

We will freely use the notation in §2.3 with $K=\mathbb{Q}$ and $\mathbb{K}=\mathbb{C}$. We also set $m=n$ so that, in particular, we have that $\mathcal{A}_i=\{a_{i,0},\dots,a_{i,c_i}\}$, $i=0,\dots,n$, is a family of $n+1$ nonempty finite subsets of M or *supports*. We denote by $\Delta_i=\mathrm{conv}(\mathcal{A}_i)$ the convex hull of $\mathcal{A}_i$.

For $i=0,\dots,n$, let $\boldsymbol{u}_i=\{u_{i,0},\dots,u_{i,c_i}\}$ be a set of c_i+1 variables. Set $\boldsymbol{u}=\{\boldsymbol{u}_0,\dots,\boldsymbol{u}_n\}$, so that $\mathbb{C}[\boldsymbol{u}]=\mathbb{C}[\boldsymbol{u}_0,\dots,\boldsymbol{u}_n]$ is the multihomogeneous coordinate ring of the multiprojective space

$$\mathbb{P}^{\boldsymbol{c}}=\prod_{i=0}^n\mathbb{P}^{c_i}_{\mathbb{C}}.$$

For each i, we consider the general Laurent polynomial with support $\mathcal{A}_i$ given by

$$F_i=\sum_{j=0}^{c_i}u_{i,j}\chi^{a_{i,j}}\in\mathbb{Q}[\boldsymbol{u}_i][M]. \tag{3.1}$$

We set for short

$$\boldsymbol{\mathcal{A}}=(\mathcal{A}_0,\dots,\mathcal{A}_n),\quad \Delta=\sum_{i=0}^n\Delta_i\quad\text{and}\quad \boldsymbol{F}=(F_0,\dots,F_n) \tag{3.2}$$

The incidence variety of the family $\boldsymbol{F}$ is

$$\Omega_{\boldsymbol{\mathcal{A}}}=\{(\xi,\boldsymbol{u})\mid F_0(\boldsymbol{u}_0,\xi)=\dots=F_n(\boldsymbol{u}_n,\xi)=0\}\subset\mathbb{T}_M\times\mathbb{P}^{\boldsymbol{c}},$$

which is an irreducible subvariety of codimension $n+1$ defined over $\mathbb{Q}$. We denote by $\pi\colon\mathbb{T}_M\times\mathbb{P}^{\boldsymbol{c}}\to\mathbb{P}^{\boldsymbol{c}}$ the projection onto the second factor.

Definition 3.1. The $\boldsymbol{\mathcal{A}}$-*eliminant* or *sparse eliminant*, denoted by $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$, is defined as any irreducible polynomial in $\mathbb{Z}[\boldsymbol{u}]$ giving an equation for the closure of the image $\overline{\pi(\Omega_{\boldsymbol{\mathcal{A}}})}$, if this is a hypersurface, and as 1 otherwise.

The $\boldsymbol{\mathcal{A}}$-*resultant* or *sparse resultant*, denoted by $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$, is defined as any primitive polynomial in $\mathbb{Z}[\boldsymbol{u}]$ giving an equation for the direct image $\pi_*\Omega_{\boldsymbol{\mathcal{A}}}$.

Both the sparse eliminant and the sparse resultant are well-defined up to a sign. It follows from these definitions that there exists $d_{\boldsymbol{\mathcal{A}}} \in \mathbb{N}$ such that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{d_{\boldsymbol{\mathcal{A}}}}, \tag{3.3}$$

with $d_{\boldsymbol{\mathcal{A}}}$ equal to the degree of the restriction of π to the incidence variety $\Omega_{\boldsymbol{\mathcal{A}}}$.

Let $Z_{\boldsymbol{\mathcal{A}}}$ be the multiprojective toric cycle on $\mathbb{P}^{\boldsymbol{c}}$ as in (2.17) and $|Z_{\boldsymbol{\mathcal{A}}}|$ its support. Both are defined over $\mathbb{Q}$, and we will consider their eliminants and resultants, in the sense of Definitions 2.3 and 2.4, with respect to the ring $A = \mathbb{Z}$.

Proposition 3.2. *Let notation be as before and set $\boldsymbol{e}_i$, $i = 0, \dots, n$, for the standard basis of $\mathbb{Z}^{n+1}$. Then*

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(|Z_{\boldsymbol{\mathcal{A}}}|) \quad \textit{and} \quad \mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}}).$$

Proof. Let $\boldsymbol{x} = \{x_{i,j}\}_{i,j}$ and $\boldsymbol{u} = \{u_{i,j}\}_{i,j}$ respectively denote the homogeneous coordinates of the first and the second factor in the product $\mathbb{P}^{\boldsymbol{c}} \times \mathbb{P}^{\boldsymbol{c}}$, respectively. For $i = 0, \dots, n$, consider the general linear form on $\mathbb{P}^{c_i}$ given by

$$L_i = \sum_{j=0}^{c_i} u_{i,j} x_{i,j} \in \mathbb{Q}[\boldsymbol{u}][\boldsymbol{x}_i]. \tag{3.4}$$

Let Σ be a fan refining Σ_Δ and $\Phi_{\boldsymbol{\mathcal{A}}}\colon X_\Sigma \to \mathbb{P}^{\boldsymbol{c}}$ the corresponding morphism of proper toric varieties as in Lemma 2.6. For each i, set

$$D_i = (\Phi_{\boldsymbol{\mathcal{A}}} \times \mathrm{id}_{\mathbb{P}^{\boldsymbol{c}}})^*(\mathrm{div}(L_i)) \in \mathrm{Div}(X_\Sigma \times \mathbb{P}^{\boldsymbol{c}}).$$

This is a Cartier divisor whose restriction to $\mathbb{T}_M \times \mathbb{P}^{\boldsymbol{c}}$ coincides with $\mathrm{div}(F_i)$ for the general Laurent polynomial F_i as in (3.1).

By Lemma 2.6, $Z_{\boldsymbol{\mathcal{A}}} \times \mathbb{P}^{\boldsymbol{c}} = (\Phi_{\boldsymbol{\mathcal{A}}} \times \mathrm{id}_{\mathbb{P}^{\boldsymbol{c}}})_*(X_\Sigma \times \mathbb{P}^{\boldsymbol{c}})$ and the family $\mathrm{div}(L_i)$, $i = 0, \dots, n$, intersects this cycle properly. By the projection formula (2.7), it follows that

$$(\Phi_{\boldsymbol{\mathcal{A}}} \times \mathrm{id}_{\mathbb{P}^{\boldsymbol{c}}})_*\Big((X_\Sigma \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} D_i\Big) = (Z_{\boldsymbol{\mathcal{A}}} \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} \mathrm{div}(L_i).$$

Let $\rho\colon \mathbb{P}^{\boldsymbol{c}} \times \mathbb{P}^{\boldsymbol{c}} \to \mathbb{P}^{\boldsymbol{c}}$ be the projection onto the second factor as in (2.10). By the functoriality of the direct image, $\pi_* = \rho_* \circ (\Phi_{\boldsymbol{\mathcal{A}}} \times \mathrm{id}_{\mathbb{P}^{\boldsymbol{c}}})_*$. Hence

$$\pi_*\Big((X_\Sigma \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} D_i\Big) = \rho_*\Big((Z_{\boldsymbol{\mathcal{A}}} \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} \mathrm{div}(L_i)\Big) = \rho_*\Omega_{Z_{\boldsymbol{\mathcal{A}}},(\boldsymbol{e}_0,\dots,\boldsymbol{e}_n)} \tag{3.5}$$

for the incidence cycle $\Omega_{Z_{\boldsymbol{\mathcal{A}}},(\boldsymbol{e}_0,\dots,\boldsymbol{e}_n)}$ as in (2.9).

On the other hand, the general linear form L_i does not vanish identically on $\xi \times \mathbb{P}^{\boldsymbol{c}}$ for any $\xi \in X_{\boldsymbol{\mathcal{A}}}$. Hence, the support of D_i does not contain $\zeta \times \mathbb{P}^{\boldsymbol{c}}$ for any $\zeta \in X_\Sigma$. This implies that no component of the intersection cycle $(X_\Sigma \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} D_i$ is supported in $(X_\Sigma \setminus \mathbb{T}_M) \times \mathbb{P}^{\boldsymbol{c}}$. It follows that

$$\pi_*\Big((X_\Sigma \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} D_i\Big) = \pi_*\Big((\mathbb{T}_M \times \mathbb{P}^{\boldsymbol{c}}) \cdot \prod_{i=0}^{n} \mathrm{div}(F_i)\Big) = \pi_*\Omega_{\boldsymbol{\mathcal{A}}}. \tag{3.6}$$

From (3.5) and (3.6) we deduce the equality of cycles $\rho_*\Omega_{Z_{\boldsymbol{\mathcal{A}}},(\boldsymbol{e}_0,\dots,\boldsymbol{e}_n)} = \pi_*\Omega_{\boldsymbol{\mathcal{A}}}$, which implies the statement for the resultants and, *a fortiori*, for the eliminants. □

We devote the rest of this section to the study of the basic properties of sparse eliminants and resultants.

Proposition 3.3. *Both the sparse eliminant and the sparse resultant are invariant, up to a sign, under permutations and translations of the supports.*

Proof. The first statement follows directly from Proposition 3.2 and [DKS13, Proposition 1.27]. The second claim is a consequence of the fact that the monomial map $\varphi_{\boldsymbol{\mathcal{A}}}$ in (2.16) is invariant under translations of the supports. □

The following proposition gives the partial degrees of the sparse resultant. It is the analogue of the well-known formula for the partial degrees of the sparse eliminant given in [GKZ94, Chapter 8, Proposition 1.6] under some hypothesis, and by [PS93, Corollary 2.4] in the general case.

Proposition 3.4. *For $i = 0, \dots, n$,*

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n),$$

where $\Delta_i \subset M_{\mathbb{R}}$ denotes the convex hull of $\mathcal{A}_i$ and MV_M is the mixed volume of convex bodies as in (2.29).

Proof. By Proposition 3.2 and [DKS13, Proposition 1.32],

$$\deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \deg_{\boldsymbol{u}_i}(\mathrm{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}})) = \deg\Big(Z_{\boldsymbol{\mathcal{A}}} \cdot \prod_{j\neq i} H_j\Big), \tag{3.7}$$

where $H_j \subset \mathbb{P}^{\boldsymbol{c}}$ is the inverse image under the projection $\mathbb{P}^{\boldsymbol{c}} \to \mathbb{P}^{c_j}$ of a generic hyperplane of $\mathbb{P}^{c_j}$.

Let Σ be a fan refining Σ_Δ and $\Phi_{\boldsymbol{\mathcal{A}}}\colon X_\Sigma \to \mathbb{P}^{\boldsymbol{c}}$ the morphism of proper toric varieties as in Lemma 2.6. For $j = 0, \dots, n$, set

$$D_j = (\Phi_{\boldsymbol{\mathcal{A}}})^* H_j \in \mathrm{Div}(X_\Sigma).$$

Observe that the restriction of D_j to $\mathbb{T}_M$ coincides with the Cartier divisor of a generic Laurent polynomial $f_j \in \mathbb{C}[M]$ with support $\mathcal{A}_j$. By the projection formula (2.7),

$$Z_{\boldsymbol{\mathcal{A}}} \cdot \prod_{j\neq i} H_j = (\Phi_{\boldsymbol{\mathcal{A}}})_*\Big(X_\Sigma \cdot \prod_{j\neq i} \mathrm{div}(D_j)\Big). \tag{3.8}$$

Since the hyperplanes H_j are generic, the cycle $X_\Sigma \cdot \prod_{j\neq i} \mathrm{div}(D_j)$ is supported on $\mathbb{T}_M$ and so

$$X_\Sigma \cdot \prod_{j\neq i} \mathrm{div}(D_j) = \mathbb{T}_M \cdot \prod_{j\neq i} \mathrm{div}(f_j). \tag{3.9}$$

By Bernstein's theorem (2.33), the degree of the cycle in the right-hand side of (3.9) coincides with the mixed volume $\mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n)$. The statement then follows from (3.7), (3.8) and (3.9). □

We recall here the notion of essential subfamily of supports introduced by Sturmfels in [Stu94]. For $J \subset \{0, \dots, n\}$, we set

$$L_{\boldsymbol{\mathcal{A}}_J} = \sum_{j\in J} L_{\mathcal{A}_j}$$

with $L_{\mathcal{A}_j}$ as in (2.18).

Definition 3.5. Let $J \subset \{0, \dots, n\}$. The subfamily $\boldsymbol{\mathcal{A}}_J = (\mathcal{A}_j)_{j \in J}$ is *essential* if the following conditions hold:

(1) $\#J = \operatorname{rank}\left(L_{\boldsymbol{\mathcal{A}}_J}\right) + 1$;
(2) $\#J' \leq \operatorname{rank}\left(L_{\boldsymbol{\mathcal{A}}'_J}\right)$ for all $J' \subsetneq J$.

Remark 3.6. When $J = \emptyset$, we have that $L_{\boldsymbol{\mathcal{A}}_J} = 0$ and so $\#J = \operatorname{rank}\left(L_{\boldsymbol{\mathcal{A}}'_J}\right) = 0$. In particular, if $\boldsymbol{\mathcal{A}}_J$ is an essential subfamily, then $J \neq \emptyset$. On the other extreme, when the family $\boldsymbol{\mathcal{A}}$ is essential, $\boldsymbol{\mathcal{A}}_J$ is essential if and only if $J = \{0, \dots, n\}$.

Lemma 3.7. *Let $I \subset \{0, \dots, n\}$ such that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_I}) < \#I$. Then there exists $J \subset I$ such that $\boldsymbol{\mathcal{A}}_J$ is essential.*

Proof. Choose a subset $J \subset I$ which is minimal with respect to the inclusion, under the condition that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_J}) < \#J$. Such a minimal subset exists because of the hypothesis that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_I}) < \#I$. We have that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{J'}}) \geq \#I$ for all $J' \subsetneq J$, and the minimality of J implies that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_J}) = \#J - 1$. Hence, J is essential. □

The notion of essential subfamily gives a combinatorial criterion to decide when $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$ and, in that case, to determine which are the sets of variables that actually appear in the sparse eliminant and the sparse resultant.

Proposition 3.8. *Let notation be as above.*

(1) *The following conditions are equivalent:*
   (a) $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$*;*
   (b) $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$*;*
   (c) $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_I}) \geq \#I - 1$ *for all* $I \subset \{0, \dots, n\}$*;*
   (d) *there exists a unique essential subfamily of* $\boldsymbol{\mathcal{A}}$.
(2) *Suppose that* $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ *or equivalently, that* $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$*, and let* $\boldsymbol{\mathcal{A}}_J$ *be the unique essential subfamily of* $\boldsymbol{\mathcal{A}}$. *Then the following conditions are equivalent:*
   (a) $\deg_{\boldsymbol{u}_i}(\operatorname{Elim}_{\boldsymbol{\mathcal{A}}}) > 0$*;*
   (b) $\deg_{\boldsymbol{u}_i}(\operatorname{Res}_{\boldsymbol{\mathcal{A}}}) > 0$*;*
   (c) $i \in J$.

Proof. We first prove (1). The equivalence between (1a) and (1b) follows directly from (3.3).

By Proposition 3.2, we have that $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ if and only if $\operatorname{Elim}_{\boldsymbol{e}_0, \dots, \boldsymbol{e}_n}(|Z_{\boldsymbol{\mathcal{A}}}|) \neq 1$. By [DKS13, Lemmas 1.34 and 1.37(2)], this is equivalent to

$$\dim(\operatorname{pr}_I(|Z_{\boldsymbol{\mathcal{A}}}|)) \geq \#I - 1 \quad \text{for all } I \subset \{0, \dots, n\}, \tag{3.10}$$

where pr_I denotes the projection $\prod_{i=0}^{n} \mathbb{P}^{c_i} \to \prod_{i \in I} \mathbb{P}^{c_i}$. We claim that this condition is equivalent to (1c).

To prove this, suppose that (3.10) holds. In particular, $\dim(|Z_{\boldsymbol{\mathcal{A}}}|) = n$ and so $|Z_{\boldsymbol{\mathcal{A}}}| = X_{\boldsymbol{\mathcal{A}}}$. Hence, $\operatorname{pr}_I(|Z_{\boldsymbol{\mathcal{A}}}|) = \operatorname{pr}_I(X_{\boldsymbol{\mathcal{A}}}) = X_{\boldsymbol{\mathcal{A}}_I}$. Applying (2.19), we deduce that $\dim(\operatorname{pr}_I(|Z_{\boldsymbol{\mathcal{A}}}|)) = \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_I})$ and so (1c) follows. Conversely, suppose that (1c) holds. In particular, $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}}) = n$. By (2.19), this implies that $\dim(X_{\boldsymbol{\mathcal{A}}}) = n$ and so $|Z_{\boldsymbol{\mathcal{A}}}| = X_{\boldsymbol{\mathcal{A}}}$. Hence $\dim(\operatorname{pr}_I(|Z_{\boldsymbol{\mathcal{A}}}|)) = \dim(\operatorname{pr}_I(X_{\boldsymbol{\mathcal{A}}})) = \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_I}) \geq \#I - 1$, and (3.10) follows.

We now show the equivalence of (1c) and the existence of a unique essential subfamily of supports. First, assume that (1c) holds. Lemma 3.7 applied to the subset

$I = \{0, \dots, n\}$ shows that there exists at least one essential subfamily $\boldsymbol{\mathcal{A}}_J$. Suppose that there exist a further essential subfamily $\boldsymbol{\mathcal{A}}_{J'}$. Then

$$L_{\boldsymbol{\mathcal{A}}_{J\cup J'}} = L_{\boldsymbol{\mathcal{A}}_J} + L_{\boldsymbol{\mathcal{A}}_{J'}} \quad \text{and} \quad L_{\boldsymbol{\mathcal{A}}_{J\cap J'}} \subset L_{\boldsymbol{\mathcal{A}}_J} \cap L_{\boldsymbol{\mathcal{A}}_{J'}}.$$

We deduce that

$$\begin{aligned}\text{(3.11)} \quad \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{J\cup J'}}) &\leq \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_J}) + \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{J'}}) - \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{J\cap J'}}) \\ &\leq \#J - 1 + \#J' - 1 - \#(J \cap J') = \#(J \cup J') - 2,\end{aligned}$$

since both $\boldsymbol{\mathcal{A}}_J$ and $\boldsymbol{\mathcal{A}}_{J'}$ are essential and $\boldsymbol{\mathcal{A}}_{J\cap J'}$ is a proper subfamily of them. The inequality (3.11) contradicts (1c), showing that there is a unique essential subfamily.

Conversely, suppose that (1c) does not hold. Then, there exists a subset $I_0 \subset \{0, \dots, n\}$ such that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{I_0}}) \leq \#I - 2$. By Lemma 3.7, there exists $J \subset I_0$ such that $\boldsymbol{\mathcal{A}}_J$ is essential. Choose $i_0 \in J$. Then $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_{I_0 \setminus \{i_0\}}}) \leq \#(I_0 \setminus \{i_0\}) - 1$. Again, Lemma 3.7 implies that there exists an essential subfamily of support $\boldsymbol{\mathcal{A}}_{J'}$ with $J' \subset I_0 \setminus \{i_0\}$. By construction, the essential subfamilies $\boldsymbol{\mathcal{A}}_J$ and $\boldsymbol{\mathcal{A}}_{J'}$ are different, concluding the proof of (1).

We now turn to the proof of (2). Suppose that $\operatorname{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ or $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$ and let $\boldsymbol{\mathcal{A}}_J$ denote the unique essential subfamily. The equivalence between (2a) and (2b) follows again from (3.3).

Choose $i \notin J$. Then $J \subset \{0, \dots, n\} \setminus \{i\}$ and $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_J}) = \#J - 1$. By [Sch93, Theorem 5.1.7], we have that $\operatorname{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) = 0$.

Now let $i \in J$. There is no essential subfamily of supports $\boldsymbol{\mathcal{A}}_{J'}$ with $J' \not\ni i$. Lemma 3.7 then implies that $\operatorname{rank}(L_{\boldsymbol{\mathcal{A}}_I}) \geq \#I$ for all $I \subset \{0, \dots, n\} \setminus \{i\}$. Applying again [Sch93, Theorem 5.1.7], we deduce that $\operatorname{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) > 0$, as stated. □

Given a family of Laurent polynomials $f_i \in \mathbb{C}[M]$ with support contained in $\mathcal{A}_i$, $i = 0, \dots, n$, we denote by

$$\operatorname{Elim}_{\boldsymbol{\mathcal{A}}}(f_0, \dots, f_r), \quad \operatorname{Res}_{\boldsymbol{\mathcal{A}}}(f_0, \dots, f_r) \quad \in \mathbb{C}$$

the evaluation of the sparse eliminant and the sparse resultant, respectively, at the coefficients of the f_i's.

Typically, the fact that the family of Laurent polynomials has a common root in the torus implies the vanishing of the sparse eliminant and of the sparse resultant. In precise terms, if $\overline{\pi(\Omega_{\boldsymbol{\mathcal{A}}})}$ is a hypersurface,

$$\text{(3.12)} \qquad V(f_0, \dots, f_n) \neq \emptyset \Longrightarrow \operatorname{Res}_{\boldsymbol{\mathcal{A}}}(f_0, \dots, f_r) = 0,$$

and a similar statement holds for the sparse eliminant. In Lemma 3.9 below, we give sufficient conditions such that the vanishing of the sparse eliminant at a given family of Laurent polynomials implies the existence of a common root in the torus.

Lemma 3.9. *Let*

$$\text{(3.13)} \qquad \boldsymbol{f} = (f_0, \dots, f_n) \in V(\operatorname{Elim}_{\boldsymbol{\mathcal{A}}}) \setminus \bigcup_{i=0}^{n} \bigcup_{j=0}^{c_i} V\left(\frac{\partial \operatorname{Elim}_{\boldsymbol{\mathcal{A}}}}{\partial u_{i,j}}\right) \subset \mathbb{P}^{\boldsymbol{c}}.$$

Then, $V(\boldsymbol{f}) \neq \emptyset$ and, for all $\xi \in V(\boldsymbol{f})$,

$$\text{(3.14)} \qquad (\chi^{a_{i,j}}(\xi))_{0\leq i\leq n, 0\leq j\leq c_i} = \left(\frac{\partial \operatorname{Elim}_{\boldsymbol{\mathcal{A}}}}{\partial u_{i,j}}(\boldsymbol{f})\right)_{0\leq i\leq n, 0\leq j\leq c_i} \in \mathbb{P}^{\boldsymbol{c}}.$$

Proof. If $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}=1$, then $V(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}})=\emptyset$ and the statement is trivially verified. Hence, we suppose that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}\neq 1$.

By Proposition 3.8(1) and (2.19), $\dim(X_{\boldsymbol{\mathcal{A}}})=n$. Hence, by the definition of the toric cycle $Z_{\boldsymbol{\mathcal{A}}}$ in (2.17), it follows that $|Z_{\boldsymbol{\mathcal{A}}}|=X_{\boldsymbol{\mathcal{A}}}$. By Proposition 3.2, $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}=\mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(X_{\boldsymbol{\mathcal{A}}})$ and so $\overline{\pi(\Omega_{\boldsymbol{\mathcal{A}}})}=\rho(\Omega_{X_{\boldsymbol{\mathcal{A}}},(\boldsymbol{e}_0,\dots,\boldsymbol{e}_n)})$. In particular, the latter is a hypersurface that contains the point $\boldsymbol{f}$. By (2.14),

$$X_{\boldsymbol{\mathcal{A}}}\cap V(\ell_0,\dots,\ell_n)\neq\emptyset, \tag{3.15}$$

where ℓ_i denotes the linear form on $\mathbb{P}^{c_i}$ associated to f_i *via* the monomial map $\varphi_{\boldsymbol{\mathcal{A}}}$ given in (2.16).

Take a point $\boldsymbol{\zeta}\in X_{\boldsymbol{\mathcal{A}}}\cap V(\ell_0,\dots,\ell_n)$ and, for $j=0,\dots,n$, choose $0\leq l_j\leq c_j$ such that $\zeta_{j,l_j}\neq 0$. We assume without loss of generality that $\zeta_{j,l_j}=1$ for all j. By [DKS13, Proposition 1.37], there exists $\kappa\gg 0$ such that

$$\Big(\prod_{j=0}^n u_{j,l_j}^{\kappa}\Big)\,\mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(X_{\boldsymbol{\mathcal{A}}})\in(L_0,\dots,L_n)\quad\subset\mathbb{C}[\boldsymbol{u}][\boldsymbol{x}]/I(X_{\boldsymbol{\mathcal{A}}}),$$

where L_i denotes the general linear form as in (3.4). Choose $G_j\in\mathbb{C}[\boldsymbol{u}][\boldsymbol{x}]$ such that

$$\Big(\prod_{j=0}^n u_{j,l_j}^{\kappa}\Big)\,\mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(X_{\boldsymbol{\mathcal{A}}})=\sum_{j=0}^n G_jL_j\qquad (\mathrm{mod}\ I(X_{\boldsymbol{\mathcal{A}}})\otimes\mathbb{C}[\boldsymbol{u}]). \tag{3.16}$$

Computing partial derivatives, evaluating at the point $(\boldsymbol{\zeta},\boldsymbol{f})$ and using the fact that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}=\mathrm{Elim}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(X_{\boldsymbol{\mathcal{A}}})$, we deduce from (3.16) that

$$\frac{\partial\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}}{\partial u_{i,j}}(\boldsymbol{f})=G_i(\boldsymbol{f},\boldsymbol{\zeta})\zeta_{i,j}\qquad\text{for } i=0,\dots,n \text{ and } j=0,\dots,c_i.$$

By the choice of $\boldsymbol{f}$ in (3.13),

$$(\zeta_{i,j})_{i,j}=\bigg(\frac{\partial\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}}{\partial u_{i,j}}(\boldsymbol{f})\bigg)_{i,j}\in\mathbb{P}^{\boldsymbol{c}}. \tag{3.17}$$

It follows that $\boldsymbol{\zeta}\in X_{\boldsymbol{\mathcal{A}}}\setminus\bigcup_{i,j}V(x_{i,j})$. By Lemma 2.7, this latter subset coincides with the image of the map $\varphi_{\boldsymbol{\mathcal{A}}}$. It follows that $\varphi_{\boldsymbol{\mathcal{A}}}^{-1}(\boldsymbol{\zeta})$ is a nonempty subset of $V(\boldsymbol{f})$, proving the first statement.

Now let $\xi\in V(\boldsymbol{f})$. The point $\boldsymbol{\zeta}=\varphi_{\boldsymbol{\mathcal{A}}}(\xi)$ satisfies (3.15) and so it also satisfies (3.17), which implies the formula (3.14) and completes the proof. □

Proposition 3.10. *Suppose that $L_{\boldsymbol{\mathcal{A}}}=M$ and that $\boldsymbol{\mathcal{A}}$ is essential. Then*

$$\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}})=1\qquad\textit{and}\qquad \mathrm{Res}_{\boldsymbol{\mathcal{A}}}=\pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}.$$

Proof. As $\boldsymbol{\mathcal{A}}$ is the unique essential subfamily, by Proposition 3.8(1) we have that $\overline{\pi(\Omega_{\boldsymbol{\mathcal{A}}})}$ is a hypersurface of $\mathbb{P}^{\boldsymbol{c}}$ with defining equation $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$. Consider the open subset of this hypersurface given by

$$U=V(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}})\setminus\bigcup_{i=0}^n\bigcup_{j=0}^{c_i}V\Big(\frac{\partial\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}}{\partial u_{i,j}}\Big).$$

By Proposition 3.8(2), $\deg_{\boldsymbol{u}_i}(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}})>0$ for all i and so $U\neq\emptyset$.

Take $\boldsymbol{f} \in U$. By Lemma 3.9, $V(\boldsymbol{f}) \neq \emptyset$ and, given $\xi \in V(\boldsymbol{f}) \subset \mathbb{T}_M$, one can compute $\chi^{a-b}(\xi)$ for all $a, b \in \mathcal{A}_i$, $i = 0, \dots, n$, in terms of $\boldsymbol{f}$. Hence, one can compute $\chi^a(\xi)$ for all $a \in L_{\boldsymbol{\mathcal{A}}}$. Since $L_{\boldsymbol{\mathcal{A}}} = M$, it follows that ξ is univocally determined and so

$$\#\pi_{\boldsymbol{\mathcal{A}}}(\boldsymbol{f}) = 1 \quad \text{for all } \boldsymbol{f} \in U.$$

By [Sha94, §II.6, Theorem 4], $\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}}) = 1$, which proves the first statement.

The second statement follows directly from the first one and (3.3). □

Suppose that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ and let $\boldsymbol{\mathcal{A}}_J$ be the unique essential subfamily of supports. For each $i \in J$, choose $b_i \in M$ such that $\mathcal{A}_i - b_i \subset L_{\boldsymbol{\mathcal{A}}_J}$. Then $L_{\boldsymbol{\mathcal{A}}_J}$ has rank $\#J - 1$ and $\mathcal{A}_i - b_i$, $i \in J$, is a family of nonempty finite subsets of $L_{\boldsymbol{\mathcal{A}}_J}$. We define $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}_J} \in \mathbb{Z}[\{\boldsymbol{u}_i\}_{i\in J}]$ as the sparse eliminant associated to the lattice $L_{\boldsymbol{\mathcal{A}}_J}$ and this family of supports. This polynomial does not depend on the choice of the vectors b_i.

Proposition 3.11. *Suppose that* $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ *and let* $\boldsymbol{\mathcal{A}}_J$ *be the unique essential subfamily of* $\boldsymbol{\mathcal{A}}$. *Then*

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm \, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}_J}$$

and, for $i \in J$,

$$\deg_{\boldsymbol{u}_i}(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}_{L_{\boldsymbol{\mathcal{A}}_J}}(\{\Delta_j - b_j\}_{j\in J\setminus\{i\}}).$$

Proof. The inclusion of lattices $L_{\boldsymbol{\mathcal{A}}_J} \hookrightarrow M$ induces a surjective homomorphism of tori $\psi\colon \mathbb{T}_M \to \mathbb{T}_{L_{\boldsymbol{\mathcal{A}}_J}}$. Consider the incidence variety $\Omega_{\boldsymbol{\mathcal{A}}_J} \subset \mathbb{T}_{L_{\boldsymbol{\mathcal{A}}_J}} \times \prod_{i\in J} \mathbb{P}^{c_i}$. Then there is a commutative diagram

$$\begin{array}{ccc}
\Omega_{\boldsymbol{\mathcal{A}}} & \xrightarrow{\psi\times \mathrm{pr}_J} & \Omega_{\boldsymbol{\mathcal{A}}_J} \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_J} \\
\pi(\Omega_{\boldsymbol{\mathcal{A}}}) & \xrightarrow{\mathrm{pr}_J} & \pi_J(\Omega_{\boldsymbol{\mathcal{A}}_J})
\end{array}$$

where π_J and pr_J are induced by the projections $\mathbb{T}_{L_{\boldsymbol{\mathcal{A}}_J}} \times \prod_{i\in J}\mathbb{P}^{c_i} \to \prod_{i\in J}\mathbb{P}^{c_i}$ and $\mathbb{P}^{\boldsymbol{c}} \to \prod_{i\in J}\mathbb{P}^{c_i}$, respectively. Let $\mathbb{Q}[\{\boldsymbol{u}_i\}_{i\in J}] \hookrightarrow \mathbb{Q}[\boldsymbol{u}]$ be the inclusion of algebras corresponding to the arrow in the bottom row. Then there is an inclusion of ideals

$$(\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}) \cap \mathbb{Q}[\{\boldsymbol{u}_i\}_{i\in J}] \supset (\mathrm{Elim}_{\boldsymbol{\mathcal{A}}_J}).$$

The hypothesis that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} \neq 1$ implies that both ideals are principal and irreducible. We conclude that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}_J}$, which gives the first statement.

The second statement follows from the first one together with Propositions 3.10 and 3.4. □

Lemma 3.12. *Let* $L \subset M$ *be a saturated sublattice of rank* m *and* P_i, $i = 1, \dots, n$, *convex bodies of* $M_{\mathbb{R}}$ *such that* $P_i \subset L_{\mathbb{R}}$ *for* $i = 1, \dots, m$. *Then*

$$\mathrm{MV}_M(P_1, \dots, P_n) = \mathrm{MV}_L(P_1, \dots, P_m)\, \mathrm{MV}_{M/L}(\varpi(P_{m+1}), \dots, \varpi(P_n)), \tag{3.18}$$

where ϖ *denotes the projection* $M_{\mathbb{R}} \to M_{\mathbb{R}}/L_{\mathbb{R}}$.

Proof. The fact that L is saturated implies that there is an isomorphism $M \simeq \mathbb{Z}^n$ identifying L with $\mathbb{Z}^m \times \{\boldsymbol{0}\}$. The mixed volumes in (3.18) are invariant under isomorphism of lattices, and so it suffices to prove this formula in the case when $M = \mathbb{Z}^n$ and $L = \mathbb{Z}^m \times \{\boldsymbol{0}\}$.

Let $P, Q \subset \mathbb{R}^n$ be compact bodies such that $P \subset \mathbb{R}^m \times \{\mathbf{0}\}$. The function on $\mathbb{R}_{\geq 0}$ given by $\lambda \mapsto \mathrm{vol}_{\mathbb{Z}^n}(\lambda P + Q)$ is polynomial in λ, and

$$\mathrm{MV}_{\mathbb{Z}^n}(\overbrace{P, \dots, P}^{m}, \overbrace{Q, \dots, Q}^{n-m}) = \mathrm{coeff}_{\lambda^m}(\mathrm{vol}_{\mathbb{Z}^n}(\lambda P + Q)). \tag{3.19}$$

Let $\lambda \in \mathbb{R}_{\geq 0}$. By Fubini's theorem,

$$\begin{aligned} \mathrm{vol}_{\mathbb{Z}^n}(\lambda P + Q) &= \int_{\mathbb{R}^{n-m}} \mathrm{vol}_{\mathbb{Z}^m}((\lambda P + Q) \cap (\mathbb{R}^m + \boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^{n-m}} \mathrm{vol}_{\mathbb{Z}^m}(\lambda P + (Q \cap (\mathbb{R}^m + \boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.20}$$

with $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1 \dots \mathrm{d}x_{n-m}$. The m-dimensional volume of $\lambda P + (Q \cap (\mathbb{R}^m + \boldsymbol{x}))$ is different from 0 if and only if $Q \cap (\mathbb{R}^m + \boldsymbol{x}) \neq \emptyset$ or, equivalently, if and only if $\boldsymbol{x} \in \varpi(Q)$. In that case, $\mathrm{coeff}_{\lambda^m}(\mathrm{vol}_{\mathbb{Z}^m}(\lambda P + (Q \cap (\mathbb{R}^m + \boldsymbol{x}))) = \mathrm{vol}_{\mathbb{Z}^m}(P)$. Hence,

$$\begin{aligned} &\mathrm{coeff}_{\lambda^m}\Big(\int_{\mathbb{R}^{n-m}} \mathrm{vol}_{\mathbb{Z}^m}(\lambda P + (Q \cap (\mathbb{R}^m + \boldsymbol{x}))\, \mathrm{d}\boldsymbol{t}\Big) \\ &\qquad = \mathrm{vol}_{\mathbb{Z}^m}(P) \int_{\varpi(Q)} \mathrm{d}\boldsymbol{x} = \mathrm{vol}_{\mathbb{Z}^m}(P)\, \mathrm{vol}_{\mathbb{Z}^{n-m}}(\varpi(Q)). \end{aligned} \tag{3.21}$$

By (3.19), (3.20) and (3.21), it follows that

$$\mathrm{MV}_{\mathbb{Z}^n}(\overbrace{P, \dots, P}^{m}, \overbrace{Q, \dots, Q}^{n-m}) = \mathrm{vol}_{\mathbb{Z}^m}(P)\, \mathrm{vol}_{\mathbb{Z}^{n-m}}(\varpi(Q)),$$

which gives the formula (3.18) for the case when $P_1 = \dots = P_m = P$ and $P_{m+1} = \dots = P_n = Q$. The general case follows by a standard polarization argument. □

The following result shows that the degree of the restriction of π to the incidence variety $\Omega_{\boldsymbol{\mathcal{A}}}$ and, *a fortiori*, the relation between the sparse resultant and the sparse eliminant, can be expressed in combinatorial terms. This formula already appears in [Est07, Theorem 2.23].

Proposition 3.13. *Suppose that* $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$ *and let* $\boldsymbol{\mathcal{A}}_J$ *be the unique essential subfamily of* $\boldsymbol{\mathcal{A}}$*. Then*

$$\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}}) = [L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J} : L_{\boldsymbol{\mathcal{A}}_J}]\, \mathrm{MV}_{M/L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J}}(\{\varpi(\Delta_i)\}_{i\notin J}),$$

where $L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J} = (L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J} \otimes \mathbb{Q}) \cap M$ *denotes the saturation of the sublattice* $L_{\boldsymbol{\mathcal{A}}_J}$*, and* ϖ *the projection* $M_{\mathbb{R}} \to M/L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J} \otimes \mathbb{R}$*. In particular,*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{[L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J} : L_{\boldsymbol{\mathcal{A}}_J}]\, \mathrm{MV}_{M/L^{\mathrm{sat}}_{\boldsymbol{\mathcal{A}}_J}}(\{\varpi(\Delta_i)\}_{i\notin J})}.$$

Proof. Suppose for simplicity that $J = \{0, \dots, m\}$ and set $L = L_{\boldsymbol{\mathcal{A}}_J}$ for short. By comparing the degree with respect to $\boldsymbol{u}_0$ of $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ and of $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$ using Propositions 3.4 and 3.11, we deduce that

$$\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}}) = \frac{\mathrm{MV}_M(\Delta_1, \dots, \Delta_n)}{\mathrm{MV}_L(\Delta_1 - a_{1,0}, \dots, \Delta_m - a_{m,0})}.$$

We have that $[L^{\mathrm{sat}} : L]\, \mathrm{vol}_L = \mathrm{vol}_{L^{\mathrm{sat}}}$ and so $[L^{\mathrm{sat}} : L]\, \mathrm{MV}_L = \mathrm{MV}_{L^{\mathrm{sat}}}$. Lemma 3.12 then implies that

$$\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}}) = [L^{\mathrm{sat}} : L]\, \mathrm{MV}^{\mathrm{sat}}_{M/L}(\varpi(\Delta_{m+1}), \dots, \varpi(\Delta_n)),$$

which proves the first statement. The second claim follows then from (3.3). □

Example 3.14. Let $\mathcal{A}_0, \dots, \mathcal{A}_n$ be a family of $n+1$ nonempty finite subsets of M with $\mathcal{A}_0 = \{a\}$ for $a \in M$. Suppose that $\mathcal{A}_0$ is the unique essential subfamily, and set $\Delta_i = \operatorname{conv}(\mathcal{A}_i)$, $i = 1, \dots, n$. By Propositions 3.11 and 3.13, it follows that

$$\operatorname{Elim}_{\boldsymbol{\mathcal{A}}} = \pm u_{0,a}, \quad \operatorname{Res}_{\mathcal{A}} = \pm u_{0,a}^{\mathrm{MV}_M(\Delta_1,\dots,\Delta_n)}$$

and $\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}}) = \mathrm{MV}_M(\Delta_1, \dots, \Delta_n)$.

In [Som04], the second author gave a bound for the height of the $\boldsymbol{\mathcal{A}}$-eliminant in the case when the family $\boldsymbol{\mathcal{A}}$ is essential. The following result extends this bound to an arbitrary family of supports. Recall that, given a polynomial $R = \sum_{\boldsymbol{a}} \alpha_{\boldsymbol{a}} \boldsymbol{u}^{\boldsymbol{a}} \in \mathbb{Z}[\boldsymbol{u}]$, its *height* and its *sup-norm* are respectively defined as

$$\mathrm{h}(R) = \log(\max_{\boldsymbol{a}} |\alpha_{\boldsymbol{a}}|) \quad \text{and} \quad \|R\|_{\sup} = \sup_{|u_{i,j}|=1} |R(\boldsymbol{u})|.$$

Proposition 3.15. *Let notation be as above. Then*

$$\mathrm{h}(\operatorname{Res}_{\boldsymbol{\mathcal{A}}}) \leq \sum_{i=0}^{n} \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) \log(\#\mathcal{A}_i).$$

Proof. We suppose that $\operatorname{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$ because otherwise, the inequality is trivially satisfied. Let $\boldsymbol{\mathcal{A}}_J$ be the unique essential subfamily of supports. By [Som04, Lemma 1.3],

$$\log \| \operatorname{Elim}_{\boldsymbol{\mathcal{A}}_J} \|_{\sup} \leq \frac{1}{[L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}} : L_{\boldsymbol{\mathcal{A}}_J}]} \sum_{i \in J} \mathrm{MV}_{L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}}(\{\Delta_j - a_{j,0}\}_{j \neq i}) \log(\#\mathcal{A}_i).$$

Multiplying both sides of this inequality by $\deg(\pi_{\boldsymbol{\mathcal{A}}}|_{\Omega_{\boldsymbol{\mathcal{A}}}})$ and applying Proposition 3.13, it follows that

$$\log \| \operatorname{Res}_{\boldsymbol{\mathcal{A}}} \|_{\sup} \leq \sum_{i \in J} \mathrm{MV}_{M/L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}}(\{\varpi(\Delta_k)\}_{k \notin J}) \, \mathrm{MV}_{L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}}(\{\Delta_j - a_{j,0}\}_{j \in J \setminus \{i\}}) \log(\#\mathcal{A}_i).$$

For short, write μ_i for the product of the two mixed volumes in the right-hand side of this formula. By Lemma 3.12 and Propositions 3.4 and 3.8(2),

$$\mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) = \begin{cases} \mu_i & \text{if } i \in J, \\ 0 & \text{if } i \notin J. \end{cases}$$

It follows that

$$\log \| \operatorname{Res}_{\boldsymbol{\mathcal{A}}} \|_{\sup} \leq \sum_{i=0}^{n} \mathrm{MV}_M(\Delta_0, \dots, \Delta_{i-1}, \Delta_{i+1}, \dots, \Delta_n) \log(\#\mathcal{A}_i).$$

The statement follows from the fact that $\mathrm{h}(\operatorname{Res}_{\boldsymbol{\mathcal{A}}}) \leq \log \| \operatorname{Res}_{\boldsymbol{\mathcal{A}}} \|_{\sup}$, the latter being a consequence of Cauchy's integral formula, see page 1255 in *loc. cit* for details. □

4. The Poisson formula

In this section, we prove the Poisson formula in Theorem 1.1. We also derive some of its consequences, including the formula for the product of the roots in Corollary 1.3, the product formula for the addition of supports, and the extension of the "hidden variable" technique to the sparse setting.

We keep the notation at (3.2). Furthermore, we set

$$\overline{\boldsymbol{\mathcal{A}}}=(\mathcal{A}_1,\dots,\mathcal{A}_n),\quad \overline{\Delta}=\sum_{i=1}^{n}\Delta_i \quad\text{and}\quad \overline{\boldsymbol{F}}=(F_1,\dots,F_n) \tag{4.1}$$

Let $\mathcal{B}\subset M$ be a nonempty finite subset and $f=\sum_{b\in\mathcal{B}}\beta_b\chi^b\in K[M]$ a Laurent polynomial over a field K with support contained in $\mathcal{B}$. Given $v\in N_{\mathbb{R}}$, we set

$$\mathcal{B}_v=\{b\in\mathcal{B}\mid\langle b,v\rangle=h_{\mathcal{B}}(v)\}\quad\text{and}\quad f_v=\sum_{b\in\mathcal{B}_v}\beta_b\chi^b,$$

with $h_{\mathcal{B}}$ the support function of $\mathcal{B}$ as in (2.23). We also set $\mathbb{F}=\mathbb{Q}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_n)$.

Definition 4.1. Let $v\in N\setminus\{0\}$ and $v^{\perp}\subset M_{\mathbb{R}}$ the orthogonal subspace. Then $M\cap v^{\perp}$ is a lattice of rank $n-1$ and, for $i=1,\dots,n$, there exists $b_{i,v}\in M$ such that $\mathcal{A}_{i,v}-b_{i,v}\subset M\cap v^{\perp}$. The *sparse resultant of* $\mathcal{A}_1,\dots,\mathcal{A}_n$ *in the direction of* v, denoted by $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}$, is defined as the sparse resultant of the family $\mathcal{A}_{i,v}-b_{i,v}$, $i=1,\dots,n$, considered as a family of nonempty finite subsets of $M\cap v^{\perp}$.

Let $F_i\in\mathbb{F}[M]$ be the general polynomial with support $\mathcal{A}_i$ as in (3.1), $i=1,\dots,n$. For each i, write $F_{i,v}=\chi^{b_{i,v}}G_{i,v}$ for the general Laurent polynomial $G_{i,v}\in\mathbb{F}[M\cap v^{\perp}]$ with support $\mathcal{A}_{i,v}-b_{i,v}$. The expression

$$\mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)=\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}(F_{1,v},\dots,F_{n,v})\in\mathbb{Z}[\boldsymbol{u}_1,\dots,\boldsymbol{u}_n]$$

is defined as the evaluation of this directional sparse resultant at the coefficients of the $G_{i,v}$'s. These constructions are independent of the choice of the $b_{i,v}$'s.

By Proposition 3.8(1), we have that $\mathrm{Res}_{\mathcal{A}_{1,v},\dots,\mathcal{A}_{n,v}}\neq 1$ only if v is an inner normal to a face of $\overline{\Delta}$ of dimension $n-1$. In particular, the number of non-trivial directional sparse resultants of the family $\overline{\boldsymbol{\mathcal{A}}}$ is finite.

We first prove the following Poisson formula for the general Laurent polynomials.

Theorem 4.2. *Let notation be as in* (4.1). *Then*

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{F})=\pm\prod_{v}\mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{-h_{\mathcal{A}_0}(v)}\cdot\prod_{\xi}F_0(\xi)^{m_\xi}, \tag{4.2}$$

the first product being over the primitive vectors $v\in N$ *and the second over the roots* $\xi\in\mathbb{T}_{M,\mathbb{F}}$ *of* $F_1,\dots,F_n$, *and where* m_ξ *denotes the multiplicity of* ξ *as in* (2.31).

Proof. First suppose that $\dim(\Delta)\le n-1$. By Proposition 3.8(1), the sparse resultant in the left-hand side of (4.2) is 1. Since $\dim(\overline{\Delta})\le\dim(\Delta)\le n-1$, the family $\overline{\boldsymbol{F}}$ has no roots and so the second product in the right-hand side is also 1. When $\dim(\overline{\Delta})=n-2$, Proposition 3.8(1) also implies that all directional sparse resultants of $\overline{\boldsymbol{\mathcal{A}}}$ in the first product of (4.2) are equal to 1. When $\dim(\overline{\Delta})=n-1$, there are two directional sparse resultants which might be nontrivial, corresponding to a primitive normal vector of $\overline{\Delta}$ and its opposite. Both directional sparse resultants coincide, but they appear with opposite exponents in the first product of (4.2). In all these cases, the formula reduces to the equality $1=\pm1$.

From now on, we assume that $\dim(\Delta)=n$. Let $Z_{\boldsymbol{\mathcal{A}}}$ be the multiprojective toric cycle in (2.17). This cycle is defined over $\mathbb{Q}$ and so it can be considered as a cycle of $\mathbb{P}^{\boldsymbol{n}}_{\mathbb{Q}}$. Let $Z_{\boldsymbol{\mathcal{A}},\mathbb{F}}=Z_{\boldsymbol{\mathcal{A}}}\times\mathrm{Spec}(\mathbb{F})$ be the cycle on $\mathbb{P}^{\boldsymbol{n}}_{\mathbb{F}}$ induced by the base change $\mathbb{Q}\hookrightarrow\mathbb{F}$. Consider the linear forms $L_i=\sum_{j=0}^{c_i}u_{i,j}x_{i,j}\in\mathbb{F}[\boldsymbol{x}]$, $i=1,\dots,n$, and set $\mathrm{div}(L_i)$ for the corresponding Cartier divisor on $\mathbb{P}^{\boldsymbol{c}}_{\mathbb{F}}$. These Cartier divisors intersect

$Z_{\boldsymbol{\mathcal{A}},\mathbb{F}}$ properly, and applying [DKS13, Propositions 1.28 and 1.40 and Corollary 1.38] we deduce that

$$\operatorname{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}}) = \lambda_1 \operatorname{Res}_{\boldsymbol{e}_0}\Big(Z_{\boldsymbol{\mathcal{A}},\mathbb{F}} \cdot \prod_{i=1}^{n} \operatorname{div}(L_i)\Big) = \lambda_2 \prod_{\xi} F_0(\xi)^{m_\xi} \tag{4.3}$$

with $\lambda_i \in \mathbb{F}^\times$, the product in the right-hand side being as in (4.2).

Suppose for the moment that $a_{0,0} = 0$. Then, by evaluating (4.3) at $F_0 = 1$, we obtain that $\lambda_2 = \operatorname{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}})(1, \overline{\boldsymbol{F}})$. By [DKS13, Propositions 1.40] and Proposition 2.8, there exist $\nu_i \in \mathbb{Q}^\times$ such that

$$\begin{aligned}\operatorname{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}})(1, \overline{\boldsymbol{F}}) &= \nu_1 \operatorname{Res}_{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}} \cdot \operatorname{div}(x_{0,0})) \\ &= \nu_2 \prod_{\Gamma} \operatorname{Res}_{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}},\Gamma})(\overline{\boldsymbol{F}}_{v(\Gamma)})^{-h_{\mathcal{A}_0}(v(\Gamma))},\end{aligned} \tag{4.4}$$

the product being over the facets Γ of Δ, and where $v(\Gamma)$ denotes the primitive inner normal vector of Γ. By Proposition 3.2, $\operatorname{Res}_{\boldsymbol{e}_0,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}}}) = \operatorname{Res}_{\boldsymbol{\mathcal{A}}}$ and, for each facet Γ,

$$\operatorname{Res}_{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n}(Z_{\boldsymbol{\mathcal{A}},\Gamma}) = \operatorname{Res}_{\overline{\boldsymbol{\mathcal{A}}}_{v(\Gamma)}}. \tag{4.5}$$

By Proposition 3.8(1c), $\operatorname{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v} = 1$ for every primitive vector $v \in N$ which is not an inner normal to a facet of Δ.

From (4.3), (4.4) and (4.5), it follows that

$$\operatorname{Res}_{\boldsymbol{\mathcal{A}}} = \nu_2 \prod_{v} \operatorname{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\boldsymbol{F}_v)^{-h_{\mathcal{A}_0}(v)} \cdot \prod_{\xi} F_0(\xi)^{m_\xi} \tag{4.6}$$

with $\nu_2 \in \mathbb{Q}^\times$, the product being over the primitive vectors $v \in N$.

To prove that ν_2 is actually equal to ± 1, we will show its p-adic valuation is zero for every prime p of $\mathbb{Z}$. To do this, let p be such a prime and consider the p-adic valuation ord_p on $\mathbb{Q}$. We extend this valuation to the field $\mathbb{F}(\boldsymbol{u}_0) = \mathbb{Q}(\boldsymbol{u}_0, \boldsymbol{u}_1, \dots, \boldsymbol{u}_n)$ as in (2.36), and we also denote it by ord_p. By Proposition 2.9, $\operatorname{ord}_p(\xi) = 0$ for every root $\xi \in \mathbb{T}_{M,\overline{\mathbb{F}(\boldsymbol{u}_0)}}$ of $F_1, \dots, F_n$. Hence

$$\operatorname{ord}_p\Big(\prod_{\xi} F_0(\xi)^{m_\xi}\Big) = 0.$$

It follows that $\operatorname{ord}_p(\nu_2) = 0$. Since this holds for every p, we deduce that $\nu_2 = \pm 1$, which proves the theorem for the case when $a_{0,0} = 0$.

In particular, let $a \in M$ and set $\mathcal{A}_0 = \{0, a\}$, and $-\mathcal{A}_0 = \{0, -a\}$. Note that $-\mathcal{A}_0$ is the translate of $\mathcal{A}_0$ by the point $-a$. By Proposition 3.3,

$$\operatorname{Res}_{\mathcal{A}_0,\overline{\boldsymbol{\mathcal{A}}}}(u_{0,0} + u_{0,1}\chi^a, \overline{\boldsymbol{F}}) = \pm \operatorname{Res}_{-\mathcal{A}_0,\overline{\boldsymbol{\mathcal{A}}}}(u_{0,0}\chi^{-a} + u_{0,1}, \overline{\boldsymbol{F}}).$$

Since both $\mathcal{A}_0$ and $-\mathcal{A}_0$ contain 0, we can apply the previous case to both presentations of this sparse resultant to deduce that

$$\begin{aligned}&\prod_{v} \operatorname{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{-\min(0,\langle a,v\rangle)} \cdot \prod_{\xi}(u_{0,0} + u_{0,1}\chi^a(\xi))^{m_\xi} \\ &\qquad = \pm \prod_{v} \operatorname{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{-\min(0,\langle -a,v\rangle)} \cdot \prod_{\xi}(u_{0,0}\chi^{-a}(\xi) + u_{0,1})^{m_\xi}.\end{aligned}$$

Using that $\min(0,\langle a,v\rangle)-\min(0,-\langle a,v\rangle)=\langle a,v\rangle$, we deduce from here that

$$\prod_{\xi}\chi^{a}(\xi)^{m_\xi}=\pm\prod_{v}\mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{\langle a,v\rangle}. \tag{4.7}$$

Now we consider the general case when $a_{0,0}$ is an arbitrary element of M. Applying Proposition 3.3, the formula for the case when $a_{0,0}=0$, and (4.7), we get

$$\begin{aligned}\mathrm{Res}_{\mathcal{A}_0,\overline{\boldsymbol{\mathcal{A}}}}(F_0,\overline{\boldsymbol{F}})&=\mathrm{Res}_{\mathcal{A}_0-a_{0,0},\overline{\boldsymbol{\mathcal{A}}}}(\chi^{-a_{0,0}}F_0,\overline{\boldsymbol{F}})\\&=\pm\prod_{v}\mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{\langle a_{0,0},v\rangle-h_{\mathcal{A}_0}(v)}\cdot\prod_{\xi}(\chi^{a_{0,0}}(\xi)F_0(\xi))^{m_\xi}\\&=\pm\prod_{v}\mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{-h_{\mathcal{A}_0}(v)}\cdot\prod_{\xi}F_0(\xi)^{m_\xi},\end{aligned}$$

completing the proof. □

Remark 4.3. Let notation be as in Theorem 4.2. By the structure theorem for Artin rings, there is a decomposition into local Artin rings

$$\mathbb{F}[M]/(F_1,\dots,F_n)=\bigoplus_{\xi}A_\xi,$$

where the direct sum is over the roots ξ of the family F_i, $i=1,\dots,n$. Each local Artin ring A_ξ is a $\mathbb{F}$-algebra of dimension m_ξ. Hence

$$\deg(Z(F_1,\dots,F_n))=\sum_{\xi}m_\xi,\quad\text{and}\quad\prod_{\xi}F_0(\xi)^{m_\xi}=\mathrm{norm}_{S/\mathbb{F}}(F_0), \tag{4.8}$$

with $S=\mathbb{F}[M]/(F_1,\dots,F_n)$, and where $\mathrm{norm}_{S/\mathbb{F}}(F_0)$ denotes the norm of F_0 as an element of this $\mathbb{F}$-algebra that is, the determinant of the $\mathbb{F}$-linear endomorphism of S defined by the multiplication by F_0.

We now study the genericity conditions allowing to specialize the Poisson formula (4.2).

Lemma 4.4. *Let $f_i,g_i\in\mathbb{C}[M]$, $i=1,\dots,n$, such that $V(f_1,\dots,f_n)\subset\mathbb{T}_M$ has dimension 0. Let t be a variable and consider the ideal*

$$I=(f_1+tg_1,\dots,f_n+tg_n)\subset\mathbb{C}[t][M].$$

Then t is not a zero divisor modulo I.

Proof. Let $V(I)$ be the subvariety of $\mathbb{T}_M\times\mathbb{A}^1$ defined by I. This ideal is generated by n elements and so, as a consequence of Krull's Hauptidealsatz, all irreducible components of $V(I)$ have dimension ≥ 1.

We have that $I+(t)=(f_1,\dots,f_n,t)$ and so $V(I)\cap V(t)$ is 0-dimensional. This implies that, if W is an irreducible component of $V(I)$ such that $W\cap V(t)\neq\emptyset$, then $\dim(W)=1$. Hence, there is an open subset $U\subset\mathbb{T}_M\times\mathbb{A}^1$ containing the hyperplane $V(t)$ where the family f_i+tg_i, $i=1,\dots,n$, forms a complete intersection. In particular, I has no embedded components supported on U. We conclude that t does not belong to any of the associated prime ideals of I and so it is not a zero divisor modulo I. □

Lemma 4.5. *Let $f_i\in\mathbb{C}[M]$ with support contained in $\mathcal{A}_i$ and F_i the general Laurent polynomial with support $\mathcal{A}_i$ as in* (3.1)*, $i=0,\dots,n$. Set $D=\mathrm{MV}_M(\Delta_1,\dots,\Delta_n)$ and consider the quotient algebras*

$$R=\mathbb{C}[M]/(f_1,\dots,f_n),\quad S=\mathbb{C}\otimes\mathbb{F}(t)[M]/(f_1+tF_1,\dots,f_n+tF_n).$$

Suppose that $\dim_{\mathbb{C}}(R) = D$ *and let* $g_k \in \mathbb{C}[M]$, $k = 1, \dots, D$, *giving a basis of* R *over* $\mathbb{C}$. *Then,*

(1) $\dim_{\mathbb{C}\otimes\mathbb{F}(t)}(S) = D$ *and* g_k, $k = 1, \dots, D$, *is a basis of* S *over* $\mathbb{F}(t)$;

(2) $\mathrm{norm}_{S/\mathbb{C}\otimes\mathbb{F}(t)}(f_0 + tF_0)\big|_{t=0} = \mathrm{norm}_{R/\mathbb{C}}(f_0)$.

Proof. We first prove (1). Set $\mathbb{L} = \mathbb{C} \otimes \mathbb{F}$ for short. The family $f_i + tF_i$, $i = 1, \dots, n$, verifies the hypothesis of Bernstein's theorem in (2.33). Then $V(f_1 + tF_1, \dots, f_n + tF_n)$ is of dimension 0 and, by (4.8),

$$\dim_{\mathbb{L}(t)}(S) = \deg(Z(f_1 + tF_1, \dots, f_n + tF_n)) = D = \dim_{\mathbb{C}}(R).$$

Hence, to prove that the g_k's form a basis of S over $\mathbb{L}(t)$, it is enough to show that they are linearly independent. Suppose that this is not the case and take a nontrivial linear combination

$$\sum_{l=1}^{D} \gamma_l g_l = 0 \quad \text{on } S \tag{4.9}$$

with $\gamma_l \in \mathbb{L}(t)$, not all of them simultaneously zero. Set $I \subset \mathbb{L}[t][M]$ for the ideal generated in this ring by the family $f_i + tF_i$, $i = 1, \dots, n$. Multiplying (4.9) by a suitable denominator in $\mathbb{L}[t] \setminus \{0\}$, we can assume without loss of generality that $\gamma_l \in \mathbb{C}[\boldsymbol{u}_1, \dots, \boldsymbol{u}_n][t]$ and that $\sum_{l=1}^{D} \gamma_l g_l \in I$. Moreover, by Lemma 4.4, the variable t is not a zero divisor modulo I and so we can also assume that $t \nmid \gcd_{\mathbb{L}[t]}(\gamma_1, \dots, \gamma_D)$.

We then obtain a nontrivial linear combination over $\mathbb{C}$ for the g_k's by specializing (4.9) at $t = 0$ and taking any nonzero coefficient in the expansion with respect to the variables $\boldsymbol{u}_i$. This contradicts our assumption and hence it follows that the g_k's form a basis of S over $\mathbb{L}(t)$, which proves (1).

Now we turn to (2). For $j = 0, \dots, c_0$ and $k = 1, \dots, D$ write

$$\chi^{a_{0,j}} g_k = \sum_{l=1}^{D} p_{j,k,l}\, g_l \in R \quad \text{and} \quad \chi^{a_{0,j}} g_k = \sum_{l=1}^{D} P_{j,k,l}\, g_l \in S \tag{4.10}$$

with $p_{j,k,l} \in \mathbb{C}$ and $P_{j,k,l} \in \mathbb{L}(t)$. Using the fact that t is not a zero divisor modulo I, we can deduce that none of the $P_{j,k,l}$'s has a pole at $t = 0$ and that $\chi^{a_{0,j}} g_k - \sum_{l=1}^{D} P_{j,k,l}\, g_l \in I$. Evaluating the right equation in (4.10) at $t = 0$ we obtain

$$\chi^{a_{0,j}} g_k = \sum_{l=1}^{D} P_{j,k,l}\Big|_{t=0} g_l \in (f_1, \dots, f_n) \subset \mathbb{L}[M]. \tag{4.11}$$

Since the g_k's are a basis of R over $\mathbb{C}$, they are also a basis of $R \otimes \mathbb{L}$ over $\mathbb{L}$. It then follows from (4.10) and (4.11) that $P_{j,k,l}\big|_{t=0} = p_{j,k,l} \in \mathbb{C}$ for all j, k, l.

Let m_{f_0} and m_{f_0} respectively denote the matrix of the multiplication by F_0 on S and by f_0 on R, with respect to the basis g_k, $k = 1, \dots, D$. Then,

$$m_{f_0 + tF_0}\big|_{t=0} = (m_{f_0} + t m_{F_0})\big|_{t=0} = m_{f_0},$$

and hence

$$\mathrm{norm}_{S/\mathbb{L}(t)}(F_0)\big|_{t=0} = \det(m_{f_0 + tF_0}\big|_{t=0}) = \det(m_{f_0}) = \mathrm{norm}_{R/\mathbb{C}}(f_0),$$

as stated. □

We finally prove the results stated in the introduction.

Proof of Theorem 1.1 and Corollary 1.3. By (3.12), the hypothesis $\mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{f}_v) \neq 0$ implies that the family $f_{i,v}$, $i = 1, \dots, n$, has no roots in $\mathbb{T}_M$. Then, by Bernstein's theorem in (2.33), the variety $V(f_1, \dots, f_n)$ is of dimension 0 and

$$\dim_{\mathbb{C}}(\mathbb{C}[M]/(f_1, \dots, f_n)) = \deg(Z(f_1, \dots, f_n)) = D.$$

Then we can apply Lemma 4.5(2) and Remark 4.3 to deduce that

$$\begin{aligned}(4.12) \quad \prod_{\xi} f_0(\xi)^{m_\xi} &= \mathrm{norm}_{R/\mathbb{C}}(f_0) \\ &= \mathrm{norm}_{S/\mathbb{C}\otimes\mathbb{F}(t)}(f_0 + tF_0)\big|_{t=0} = \Big(\prod_{\xi}(f_0(\xi) + tF_0(\xi))^{m_\xi}\Big)\Big|_{t=0}.\end{aligned}$$

Applying the Poisson formula (4.2) to the general Laurent polynomials $f_i + tF_i$, $i = 0, \dots, n$, we deduce that the second product in (4.12) is equal to

$$\pm \mathrm{Res}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{f} + t\boldsymbol{F}) \cdot \prod_{v} \mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{f}}_v + t\overline{\boldsymbol{F}}_v)^{h_{\mathcal{A}_0}(v)}, \tag{4.13}$$

the product being over the primitive vectors $v \in N$. Theorem 1.1 then follows from (4.12) by evaluation (4.13) at $t = 0$.

Corollary 1.3 follows from Theorem 1.1 applied to the supports $\{a\}, \mathcal{A}_1, \dots, \mathcal{A}_n$. □

From the Poisson formula, we can deduce a number of other properties for the sparse resultant. The following is the product formula for the addition of supports.

Corollary 4.6. *Let $\mathcal{A}_0, \mathcal{A}_0', \mathcal{A}_1, \dots, \mathcal{A}_n \subset M$ be nonempty finite subsets and F_0, F_0', $F_1, \dots, F_n$ the general Laurent polynomials with support $\mathcal{A}_0, \mathcal{A}_0', \mathcal{A}_1, \dots, \mathcal{A}_n$, respectively. Then*

$$\begin{aligned}&\mathrm{Res}_{\mathcal{A}_0+\mathcal{A}_0', \mathcal{A}_1, \dots, \mathcal{A}_n}(F_0F_0', F_1, \dots, F_n) \\ &\qquad = \pm \mathrm{Res}_{\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n}(F_0, F_1, \dots, F_n) \cdot \mathrm{Res}_{\mathcal{A}_0', \mathcal{A}_1, \dots, \mathcal{A}_n}(F_0', F_1, \dots, F_n).\end{aligned}$$

Proof. This follows from Theorem 4.2 and the additivity of support functions with respect to the addition of sets. □

We devote the rest of this section to the proof of Theorem 1.4 in the introduction. Let $n \geq 1$ and set $M = \mathbb{Z}^n$ and let be the general Laurent polynomials $F_i \in \mathbb{Q}[\boldsymbol{u}_i][t_1^{\pm1}, \dots, t_n^{\pm1}]$ with support $\mathcal{A}_i$, $i = 1, \dots, n$. Let $\mathrm{Res}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{t_n}$ as defined in (1.3).

Proposition 4.7. *Let notation be as above. Then, there exists $d \in \mathbb{Z}$ such that*

$$\mathrm{Res}_{\mathcal{A}_1, \dots, \mathcal{A}_n}^{t_n} = \pm t_n^d \, \mathrm{Res}_{\{\boldsymbol{0}, \boldsymbol{e}_1\}, \mathcal{A}_1, \dots, \mathcal{A}_n}(z - t_n, F_1, \dots, F_n)\big|_{z=t_n}, \tag{4.14}$$

with $\boldsymbol{e}_n = (0, \dots, 0, 1) \in \mathbb{Z}^n$.

Proof. Let $\overline{\boldsymbol{\mathcal{A}}} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$, $\overline{\boldsymbol{F}} = (F_1, \dots, F_n)$ and $\overline{\boldsymbol{u}} = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ as before, and set for short

$$R = \mathrm{Res}_{\overline{\boldsymbol{\mathcal{A}}}}^{t_n} \in \mathbb{Q}[\overline{\boldsymbol{u}}][t_n^{\pm1}] \quad \text{and} \quad E = \mathrm{Res}_{\{\boldsymbol{0}, \boldsymbol{e}_1\}, \overline{\boldsymbol{\mathcal{A}}}}(z - t_n, \overline{\boldsymbol{F}}) \in \mathbb{Q}[\overline{\boldsymbol{u}}][z].$$

Set also $\varpi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ for the projection onto the first $n-1$ coordinates of $\mathbb{R}^n$.

We will prove the statement by induction on the number of variables. When $n = 1$,

$$R = \pm F_1 \quad \text{and} \quad E = z^{-\mathrm{ord}_{t_1}(F_1)} F_1(z). \tag{4.15}$$

These identities can be respectively proven using Example 3.14 and the formula (4.2). This implies (4.14) in this case, with $d = \mathrm{ord}_{t_1}(F_1)$.

Suppose now that $n \geq 2$. Applying the formula (4.2) both to R and to E, we get

$$R = \pm \prod_v \mathrm{Res}_{\varpi(\mathcal{A}_1)_v,\dots,\varpi(\mathcal{A}_{n-1})_v}(F_{1,v},\dots,F_{n-1,v})^{-h_{\varpi(\mathcal{A}_n)}(v)} \prod_\xi F_n(\xi)^{m_\xi},$$

$$E(t_n) = \pm \prod_w \mathrm{Res}_{\{\mathbf{0},\boldsymbol{e}_n\}_w,\mathcal{A}_{1,w},\dots,\mathcal{A}_{n-1,w}}((z-t_n)_w, F_{1,w},\dots,F_{n-1,w})^{-h_{\mathcal{A}_n}(w)} \prod_\eta F_n(\eta)^{m_\eta}.$$

In these formulae, the first product is over all primitive vectors v in $\mathbb{Z}^{n-1}$, the second is over the roots ξ of $F_1,\dots,F_{n-1}$ in $(\overline{\mathbb{C}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n-1})(t_n)}^\times)^{n-1}$, the third is over all primitive vectors w in $\mathbb{Z}^n$, and the fourth is over the roots η of $z-t_n, F_1,\dots,F_{n-1}$ in $(\overline{\mathbb{C}(\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n-1})(z)}^\times)^n$.

Using Remark 4.3, we can verify that

$$\prod_\xi F_n(\xi)^{m_\xi} = \prod_\eta F_n(\eta)^{m_\eta}\Big|_{z=t_n}$$

Let $w = (w_1,\dots,w_n) \in \mathbb{Z}^n$. If w is of the form $(v,0)$ with $v \in \mathbb{Z}^{n-1}$, then $h_{\mathcal{A}_n}(w) = h_{\varpi(\mathcal{A}_n)}(v)$. Applying the inductive hypothesis, we get that, in this case,

$$\begin{aligned} &\mathrm{Res}_{\varpi(\mathcal{A}_1)_v,\dots,\varpi(\mathcal{A}_{n-1})_v}(F_{1,v},\dots,F_{n-1,v})^{-h_{\varpi(\mathcal{A}_n)}(v)} \\ &\quad = t_n^{-h_{\mathcal{A}_n}(w)d_w} \mathrm{Res}_{\{\mathbf{0},\boldsymbol{e}_n\},\mathcal{A}_{1,w},\dots,\mathcal{A}_{n-1,w}}(z-t_n, F_{1,w},\dots,F_{n-1,w})^{-h_{\mathcal{A}_n}(w)}\Big|_{z=t_n} \end{aligned} \tag{4.16}$$

with $d_w \in \mathbb{Z}$. On the other hand, if $w_n \neq 0$, then

$$(z-t_n)_w = \begin{cases} z & \text{if } w_n > 0, \\ -t_n & \text{if } w_n < 0. \end{cases}$$

Example 3.14 implies that

$$\mathrm{Res}_{\{\mathbf{0},\boldsymbol{e}_n\}_w,\mathcal{A}_{1,w},\dots,\mathcal{A}_{n-1,w}}((z-t_n)_w, F_{1,w},\dots,F_{n-1,w})^{-h_{\mathcal{A}_n}(w)} = \pm z^{c_w} \tag{4.17}$$

with

$$c_w = \begin{cases} -h_{\mathcal{A}_n}(w)\,\mathrm{MV}_{\mathbb{Z}^n\cap w^\perp}(\Delta_{1,w},\dots,\Delta_{n,w}) & \text{if } w_n > 0, \\ 0 & \text{if } w_n < 0, \end{cases}$$

where $\Delta_{i,w}$ is the face in the direction w of the convex hull of $\mathcal{A}_i$. The statement then follows from (4.16) and (4.17) with

$$d = -\sum_w h_{\mathcal{A}_n}(w)d_w - \sum_w h_{\mathcal{A}_n}(w)\,\mathrm{MV}_{\mathbb{Z}^n\cap w^\perp}(\Delta_{1,w},\dots,\Delta_{n,w}) \in \mathbb{Z}, \tag{4.18}$$

for d_w as in (4.16). □

Remark 4.8. The exponent d in (4.14) can be made explicit in terms of mixed integrals in the sense of [PS08, Definition 1.1] or, equivalently, shadow mixed volumes as in [Est08, Definition 1.7]. Indeed, let $\iota\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $(x_1,\dots,x_{n-1},x_n) \mapsto (x_1,\dots,x_{n-1},-x_n)$. Then d coincides with the mixed integral of the family of concave functions on $\varpi(\Delta_i) \to \mathbb{R}$, $i = 1,\dots,n$, parametrizing the upper envelope of $\iota(\Delta_i)$. This can be shown by induction on the number of variables n by using (4.15), plus the recursive formulae (4.18) and [PS08, (8.6)].

Proof of Theorem 1.4. This follows directly from Proposition 4.7 and Theorem 1.1. □

5. Comparison with previous results and further examples

Using the relation between sparse resultants and sparse eliminants given in Proposition 3.13, we can easily translate any results for sparse resultants in terms of sparse eliminants and viceversa: with notation as in Proposition 3.13, we have that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm \,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{d_{\boldsymbol{\mathcal{A}}}}$$

with

$$d_{\boldsymbol{\mathcal{A}}} = \begin{cases} [L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}} : L_{\boldsymbol{\mathcal{A}}_J}]\, \mathrm{MV}_{M/L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}}(\{\varpi(\Delta_i)\}_{i\notin J}) & \text{if } \exists! \text{ essential subfamily } \boldsymbol{\mathcal{A}}_J, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the Poisson formula in Theorem 4.2 can be translated in terms of sparse eliminants as follows. Let notation be as in that result. For each primitive vector $v \in N$ we choose $b_{i,v} \in M$ such that $\mathcal{A}_{i,v} - b_{i,v} \subset M \cap v^{\perp} \simeq \mathbb{Z}^{n-1}$, $i = 1, \dots, n$, and we set

$$d_{\overline{\boldsymbol{\mathcal{A}}}_v} := d_{\mathcal{A}_{1,v}-b_{1,v},\dots,\mathcal{A}_{n,v}-b_{n,v}}.$$

Then, the formula (4.2) can be rewritten as

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{d_{\boldsymbol{\mathcal{A}}}}(\boldsymbol{F}) = \pm \prod_v \mathrm{Elim}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{-d_{\overline{\boldsymbol{\mathcal{A}}}_v} h_{\mathcal{A}_0}(v)} \cdot \prod_{\xi} F_0(\xi)^{m_\xi}. \tag{5.1}$$

On the other hand, the product formula in [PS93, Theorem 1.1] can be reformulated with our notation as

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \lambda \cdot \prod_v \mathrm{Elim}_{\overline{\boldsymbol{\mathcal{A}}}_v}(\overline{\boldsymbol{F}}_v)^{-\delta_v} \cdot \prod_{\xi} F_0(\xi)^{m_\xi}, \tag{5.2}$$

with $\lambda \in \mathbb{Q}^{\times}$ and where, for each primitive vector $v \in N$, the exponent δ_v is given by

$$\delta_v = \begin{cases} [L_{\overline{\boldsymbol{\mathcal{A}}}_v}^{\mathrm{sat}} : L_{\overline{\boldsymbol{\mathcal{A}}}_v}] & \text{if } v \text{ is normal to a facet of } \overline{\Delta}, \\ 0 & \text{otherwise.} \end{cases}$$

In [PS93, Theorem 1.1], it is implicitly assumed that $L_{\boldsymbol{\mathcal{A}}} = \mathbb{Z}^n$ and that the family $\boldsymbol{\mathcal{A}}$ is essential. These assumptions imply that $d_{\boldsymbol{\mathcal{A}}} = 1$. Hence, (5.2) actually holds if and only if, for every primitive vector $v \in N$ such that $\mathrm{Elim}_{\overline{\boldsymbol{\mathcal{A}}}_v} \neq 1$,

$$\delta_v = d_{\overline{\boldsymbol{\mathcal{A}}}_v} h_{\mathcal{A}_0}(v).$$

This set of equalities does hold when, for each v such that $\overline{\boldsymbol{\mathcal{A}}}_v$ has a unique essential subfamily, this subfamily actually coincides with $\overline{\boldsymbol{\mathcal{A}}}_v$. The Pedersen-Sturmfels product formula is correct in that case, which includes the unmixed case when $\mathcal{A}_0 = \cdots = \mathcal{A}_n = \mathcal{A}$ for a nonempty finite subset $\mathcal{A} \subset \mathbb{Z}^n$ such that $L_{\mathcal{A}} = \mathbb{Z}^n$.

Example 1.2 in the introduction illustrates how (5.2) can fail in degenerate cases. In the setting of this example, $L_{\boldsymbol{\mathcal{A}}} = \mathbb{Z}^2$ and $\boldsymbol{\mathcal{A}}$ is essential. However, for the vector $(1, 0)$, the unique essential subfamily $\overline{\boldsymbol{\mathcal{A}}}_{(1,0)}$ is the point $\{(-1, 0)\}$. The exponent of the directional eliminant $\mathrm{Elim}_{\mathcal{A}_{1,(1,0)},\mathcal{A}_{2,(1,0)}} = u_{1,1}$ in the formula (5.1) is the 1-dimensional volume of the segment $\mathrm{conv}((-1, 0), (-1, 2))$, which is equal to 2. On the other hand, $\delta_{(1,0)} = 1$ because $L_{\overline{\boldsymbol{\mathcal{A}}}_{(1,0)}}$ is saturated, and so (5.2) fails in this case.

In [Min03], Minimair reformulated (5.2) in the course of his study of sparse resultants under vanishing coefficients, but this reformulation has also flaws. In particular, the definition of the exponent $\mathrm{e}_{\mathcal{A}_1,\dots,\mathcal{A}_n}$ in [Min03, Remark 3] depends on the construction of a supplement of the sublattice $L_{\boldsymbol{\mathcal{A}}_J}$ associated to an essential subfamily of supports: if this sublattice is not saturated, the supplement does not exists and the exponent cannot be defined. Moreover, [Min03, Theorem 8] is meaningless in many situations as it leads to expressions of the form $\frac{0}{0}$ like the one shown in Example 1.2.

Next we give two further examples. The first one shows that the condition that $\boldsymbol{\mathcal{A}}$ is essential, which is implicitly assumed in [PS93], is necessary for (5.2) to hold.

Example 5.1. Let $M = \mathbb{Z}$, and set $\mathcal{A}_0 = \{0\}$ and $\mathcal{A}_1 = \{0,1,2\}$. Then $\mathcal{A}_0$ is the unique essential subfamily and $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm u_{0,0}^2$. We also have that $h_{\mathcal{A}_0}(v) = 0$ for all $v \in N$. Hence, the Poisson formula (4.2) reads in this case as

$$\pm u_{0,0}^2 = \pm F_0(\xi_1)F_0(\xi_2),$$

where ξ_i are the roots of F_1. On the other hand, $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \pm u_{0,0}$ and so (5.2) does not hold.

The next example exhibits a phenomenon similar to the one in Example 1.2.

Example 5.2. Let $M = \mathbb{Z}^2$ and set

$$\mathcal{A}_0 = \{(0,1),(1,0)\}, \quad \mathcal{A}_1 = \{(0,0),(1,0)\}, \quad \mathcal{A}_2 = \{(0,0),(0,1),(0,2)\}.$$

Then $L_{\boldsymbol{\mathcal{A}}} = \mathbb{Z}^2$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2)$ is essential. It can be verified that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = u_{0,0}^2 u_{1,0}^2 u_{2,0} + u_{0,0}u_{0,1}u_{1,0}u_{1,1}u_{2,1} + u_{0,1}^2 u_{1,1}^2 u_{2,2},$$

We can verify that the formula (4.2) reads in this case as

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm u_{1,1}^2 u_{2,2} F_0(\xi_1)F_0(\xi_2)$$

where ξ_i are the roots of the family F_1, F_2. We have that $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ but the formula (5.2) gives the exponent 1 to the directional sparse eliminant $u_{1,1}$. Hence, this formula also fails in this case.

The product formula for the addition of supports in Corollary 4.6 can also be rewritten in terms of sparse eliminants. Indeed, with notation as in that statement, set $\boldsymbol{\mathcal{A}} = (\mathcal{A}_0, \mathcal{A}_1, \cdots, \mathcal{A}_n)$, $\boldsymbol{\mathcal{A}}' = (\mathcal{A}_0', \mathcal{A}_1, \cdots, \mathcal{A}_n)$, $\boldsymbol{F} = (F_0, F_1, \ldots, F_n)$ and $\boldsymbol{F}' = (F_0', F_1, \ldots, F_n)$ for short. Then

$$\mathrm{Elim}_{\mathcal{A}_0+\mathcal{A}_0', \mathcal{A}_1, \ldots, \mathcal{A}_n}(F_0F_0', F_1, \ldots, F_n)^{d_{\mathcal{A}_0+\mathcal{A}_0', \mathcal{A}_1, \ldots, \mathcal{A}_n}} = \pm\, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{F})^{d_{\boldsymbol{\mathcal{A}}}} \cdot \mathrm{Elim}_{\boldsymbol{\mathcal{A}}'}(\boldsymbol{F}')^{d_{\boldsymbol{\mathcal{A}}'}}.$$

On the other hand, the analogous formula in [PS93, Proposition 7.1] can be reformulated with our notation as

$$\begin{aligned}(5.3)\quad \mathrm{Elim}_{\mathcal{A}_0+\mathcal{A}_0', \mathcal{A}_1, \ldots, \mathcal{A}_n}&(F_0F_0', F_1, \ldots, F_n)\\ &= \lambda\, \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{F})^{[L_{\boldsymbol{\mathcal{A}}}:L_{\boldsymbol{\mathcal{A}}'}]} \cdot \mathrm{Elim}_{\boldsymbol{\mathcal{A}}'}(\boldsymbol{F}')^{[L_{\boldsymbol{\mathcal{A}}}:L_{\boldsymbol{\mathcal{A}}''}]}\end{aligned}$$

with $\lambda \in \mathbb{Q}^\times$. These two formulae are equivalent, up to the scalar factor λ, in the case when both $\boldsymbol{\mathcal{A}}'$ and $\boldsymbol{\mathcal{A}}''$ are essential. Otherwise, (5.3) might fail, as shown by the following example.

Example 5.3. Let $M = \mathbb{Z}$ and set $\mathcal{A}_0' = \{0\}$, $\mathcal{A}_0'' = \{0,1\}$ and $\mathcal{A}_1 = \{0,1,2\}$. Then the formula in Corollary 4.6 reads in this case as

$$\mathrm{Elim}_{\{0,1\},\{0,1,2\}}(u_{0,0}'(u_{0,0}'' + u_{0,1}''x), f_1) = \pm {u_{0,0}'}^2\, \mathrm{Elim}_{\{0,1\},\{0,1,2\}}(u_{0,0}'' + u_{0,1}''x, f_1),$$

since $\mathrm{Res}_{\{0\},\{0,1,2\}} = {u_{0,0}'}^2$. However, the formula (5.3) gives the exponent 1 to the sparse eliminant $\mathrm{Elim}_{\{0\},\{0,1,2\}} = u_{0,0}'$, instead of 2.

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