RATIONAL PARAMETRIZATIONS, INTERSECTION THEORY, AND NEWTON POLYTOPES*

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Abstract. The study of the Newton polytope of a parametric hypersurface is currently receiving a lot of attention both because of its computational interest and its connections with Tropical Geometry, Singularity Theory, Intersection Theory and Combinatorics. We introduce the problem and survey the recent progress on it, with emphasis in the case of curves.

Key words. Parametric curve, implicit equation, Newton polytope, tropical geometry, Intersection Theory.

AMS(MOS) 2000 subject classifications. Primary 14Q05; Secondary 12Y05, 52B20, 14C17.

1. Introduction. Parametric curves and surfaces play a central role in Computer Aided Geometric Design (CAGD) because they provide shapes which are easy to plot. Indeed, a rational parametrization allows to produce many points in the variety using only the elementary operations ($\pm$, $\times$, $\div$) of the base field.

For instance, consider the *folium of Descartes* (Figure 1). This plane curve can be defined either by the equation $x^3 + y^2 - 3xy = 0$ or as the image of the rational map

$$\mathbb{C} \dashrightarrow \mathbb{C} \quad , \quad t \mapsto \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right). \tag{1}$$

The parametric representation is certainly more suitable for plotting the curve. If instead we plot it using only its implicit equation, the result is bound to be poor, specially around the singular point $(0,0)$ (Figure 2).

This is because in order to produce many points in the folium in this way, we have to solve as many cubic equations. This is certainly more expensive than evaluating the parametrization and moreover, the resulting points are typically not rational but live in different cubic extensions of $\mathbb{Q}$.

On the other hand, if we are to decide whether a given point lies in the folium or not, it is better to use the implicit equation. For instance, it is straightforward to conclude that $(-2,1)$ does not belong to the folium

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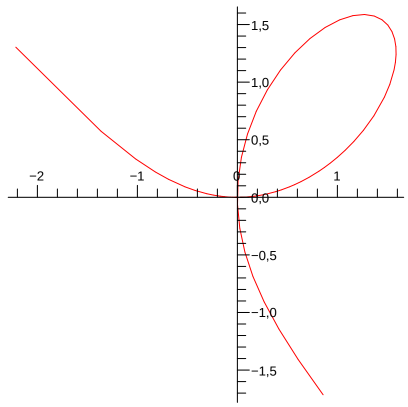

FIG. 1. *The folium of Descartes.*

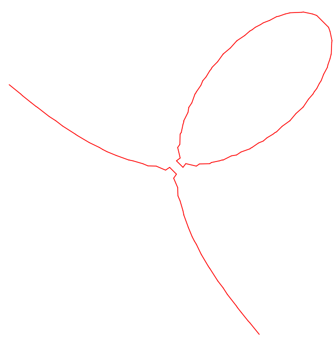

FIG. 2. *The folium of Descartes according to the Maple command* `implicitplot`.

by evaluating the equation: $(-2)^3 + 1^3 - 3(-2) = -1 \neq 0$. If we were to find that out from the parametrization (1), we would have to determine if the system of equations

$$-2 = \frac{3t}{1+t^3}, \qquad 1 = \frac{3t^2}{1+t^3}$$

admits a solution for $t \in \mathbb{C}$ or not, which is a harder task.

Depending on which kind of operation one needs to perform on a certain parametric variety, it may be convenient to dispose of the parametric representation or of the implicit one. Efficiently performing the passage from one representation to the other is one of the central problems of Computational Algebraic Geometry. In the present text we will mostly concentrate in one these directions: the *implicitization problem*, consisting in computing equations for an algebraic variety given in parametric form.

In precise terms, the implicitization problem is: let $\rho_1, \dots, \rho_n \in \mathbb{C}(t_1, \dots, t_{n-1})$ be a family of rational functions and consider the map

$$\boldsymbol{\rho} : \mathbb{C}^{n-1} \dashrightarrow \mathbb{C}^n \quad , \quad \boldsymbol{t} = (t_1, \dots, t_{n-1}) \mapsto (\rho_1(\boldsymbol{t}), \dots, \rho_n(\boldsymbol{t})). \tag{2}$$

Suppose that the Zariski closure $\overline{\mathrm{Im}(\rho)}$ of the image of this map is a hypersurface or equivalently, that the Jacobian matrix $(\frac{\partial \rho_i}{\partial t_j}(\boldsymbol{t}))_{i,j}$ has maximal rank $n-1$ for generic $\boldsymbol{t} \in \mathbb{C}^{n-1}$. The ideal of this parametric (or *unirational*) hypersurface is generated by a single irreducible polynomial and the problem consists in computing this "implicit equation".

This problem is equivalent to the elimination of the parameter variables from some system of equations. For instance, to compute the implicit equation of the folium from the parametrization (1), one should eliminate the variable t from the system of equations

$$(1+t^3)\, x - 3t = 0 \quad , \quad (1+t^3)\, y - 3t^2 = 0, \tag{3}$$

that is, we have to find the irreducible polynomial in $\mathbb{C}[x,y]$ vanishing at the points (x,y,t) satisfying (3) for some $t \in \mathbb{C}$.

The same procedure works in general. For a parametrization like in (2), write $\rho_i(\boldsymbol{t}) = \frac{p_i(\boldsymbol{t})}{q_i(\boldsymbol{t})}$ for some coprime polynomials p_i, q_i for $1 \leq i \leq n$. The implicit equation of the hypersurface $\overline{\mathrm{Im}(\boldsymbol{\rho})}$ can then be obtained by eliminating the variables $t_1, \dots, t_{n-1}$ from the system of equations

$$q_1(\boldsymbol{t})x_1 - p_1(\boldsymbol{t}) = 0, \dots, q_n(\boldsymbol{t})x_n - p_n(\boldsymbol{t}) = 0.$$

This elimination task can be effectively done either with Gröbner bases or with resultants [3] but in practice, this can be too expensive. For instance, for $a \in \mathbb{N}$ consider the parametrization

$$\boldsymbol{\rho} : \mathbb{C} \to \mathbb{C}^2 \quad , \quad t \mapsto \left(\frac{t(t-1)^a}{(t+1)^{a+1}}, \frac{(t+1)^a}{t(t-1)^{a-1}} \right). \tag{4}$$

It is not hard to check by hand that the implicit equation of the image of this map is

$$2 - x^{a-1}y^a - x^a y^{a+1} = 0.$$

However, all current implementations of Gröbner bases and resultant algorithms fail to solve the problem for moderately large values of a, because of the increasing number of intermediate computations involved.

2. The Newton polytope of the implicit equation. Instead of trying to compute the implicit equation of a parametric hypersurface, we will focus in the problem of determining its Newton polytope. We will work with *Laurent polynomials*, that is expressions of the form $x_2^{-1} + x_1^{-2}x_2$ where the exponents can be any integer numbers.

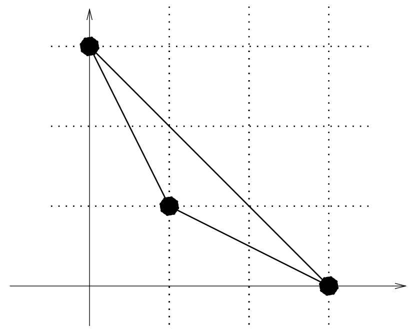

FIG. 3. *The Newton polygon of the folium of Descartes.*

DEFINITION 2.1. *The Newton polytope* $\mathrm{N}(F) \subset \mathbb{R}^n$ *of a Laurent polynomial* $F \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ *is the convex hull of the exponents in its monomial expansion.*

This notion readily extends to hypersurfaces: we define the *Newton polytope* $\mathrm{N}(Z)$ of a hypersurface $Z \subset \mathbb{C}^n$ as the Newton polytope of its implicit equation; this polytope is well defined because the equation is unique up to a scalar factor. For the case $n = 2$ we will apply the more usual terminology of "polygon" instead of polytope. For instance, the Newton polygon of the folium $x_1^3 + x_2^3 - 3x_1x_2 = 0$ is the convex hull $\mathrm{Conv}\big((1,1),(3,0),(0,3)\big)$ (Figure 3).

The Newton polytope tells us which are the possible exponents occuring in a given Laurent polynomial: if the polytope is small, then the polynomial is *sparse*, in the sense that it has few monomials. It is an important refinement of the notion of degree: if we denote by $S := \mathrm{Conv}(\mathbf{0}, e_1, \dots, e_n)$ the standard simplex of $\mathbb{R}^n$, the degree of a polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ is the least integer d such that $\mathrm{N}(F) \subset dS$. Note that the Newton polytope of a polynomial (and *a fortiori* that of a hypersurface) is always contained in the octant $(\mathbb{R}_{\geq 0})^n$.

From now on, we will focus in the following problem: determine the Newton polytope of a hypersurface given by a rational map $\boldsymbol{\rho} : \mathbb{C}^{n-1} \dashrightarrow \mathbb{C}^n$. This problem is currently receiving a lot of attention because of its connections with Tropical Geometry, Singularity Theory, Intersection Theory and Combinatorics. The Newton polytope does not characterize the hypersurface but retains a lot of relevant information and as a consequence of the research done during the last years, we now know that in plenty of cases its computation is much simpler than that of the full implicit equation.

A preliminary version of this question was first posed by B. Sturmfels and J.-T. Yu. In the context of the sparse elimination theory, their question can be resumed in: "can I predict the Newton polytope of the implicit equation from the Newton polytopes of the input parametrization?" In precise terms:

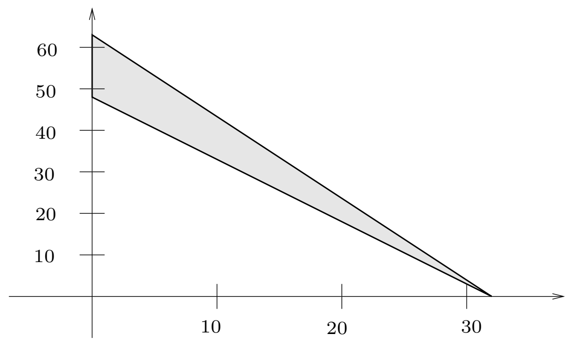

FIG. 4. *The Newton polygon of the implicit equation of (5).*

PROBLEM 2.2. *Let $P_1, \dots, P_n \subset \mathbb{R}^{n-1}$ be lattice polytopes with non-empty interior and consider the family of n Laurent polynomials in $n-1$ variables*

$$\rho_i = \sum_{a \in P_i \cap \mathbb{Z}^{n-1}} \lambda_{i,a}\, t_1^{a_1} \cdots t_{n-1}^{a_{n-1}} \in \mathbb{C}\Big[t_1^{\pm 1}, \dots, t_{n-1}^{\pm 1}\Big]$$

for $1 \le i \le n$ and $\lambda_{i,a} \in \mathbb{C}$ generic. Determine the Newton polytope of the image of the parametrization $\boldsymbol{t} \mapsto (\rho_1(\boldsymbol{t}), \dots, \rho_n(\boldsymbol{t}))$.

A *lattice polytope* in $\mathbb{R}^{n-1}$ is a polytope whose vertices lie in $\mathbb{Z}^{n-1}$. The hypothesis that the P_i's have non empty interior ensures that the image of the parametrization is a hypersurface for a generic choice of the coefficients $\lambda_{i,a}$ (that is, in some non empty open set of the space of parameters). The Newton polytope of this hypersurface does not depend on this generic choice although the equation itself does.

As an example, let us consider the parametrization proposed by A. Dickenstein and R. Fröberg [8]:

$$\boldsymbol{\rho} : \mathbb{C} \to \mathbb{C}^2 \quad , \quad t \mapsto \big(t^{48} - t^{56} - t^{60} - t^{62} - t^{63}, t^{32}\big). \tag{5}$$

The Newton polytopes of the defining polynomials are relatively small: the real interval $[48, 63]$ and the singleton $\{32\}$. The exponents are rather large, but in any case the implicit equation can be computed *via* the Sylvester resultant. It's Newton polygon is the triangle with vertices $(32, 0)$, $(0, 48)$, $(0, 63)$.

This example was studied by I. Emiris and I. Kotsireas, who succeeded in determining the polygon by analysing the behavior of the resultant under specialization, thus showing that it is sometimes possible to access to the Newton polytope without computing the implicit equation [8]; see also [7] for further applications of this method.

The recent irruption of Tropical Geometry in the mathematical panorama has boosted the interest in the problem. The tropical variety associated to an affine hypersurface is a polyhedral object, equivalent to its Newton polytope in the sense that one can be recovered from the other and viceversa. In this direction, Sturmfels, J. Tevelev and J. Yu succeeded in determining the tropical variety (and thus the Newton polytope) of a hypersurface parametrized by generic Laurent polynomials [15, 16] and implemented the resulting algorithm [17, 18].

From another point of view, A. Esterov and A. Khovanskiĭ have shown that the Newton polytope of the implicit equation of a generic parametrization can be identified with the *mixed fiber polytope* in the sense of P. McMullen, hence providing a different characterization of this object [9].

2.1. The Newton polygon of a parametric curve. If one wants to determine the Newton polytope in *all* cases and not just in the generic ones, it is clear that finer invariants of the parametrization must be taken into account.

In this section we will focus in the case of parametric plane curves, which has been recently solved in the papers [4, 5, 15, 20]. In this case, the Newton polygon is determined by the multiplicities of the parametrization. Let $\rho : \mathbb{C} \dashrightarrow \mathbb{C}^2$ be a map given by rational function $f, g \in \mathbb{C}(t) \setminus \mathbb{C}$. For a point v in the projective line $\mathbb{P}^1$, the *multiplicity of ρ in v* is

$$\mathrm{ord}_v(\rho(t)) := \big(\,\mathrm{ord}_v(f(t)), \mathrm{ord}_v(g(t))\big) \in \mathbb{Z}^2,$$

where $\mathrm{ord}_v(f)$ denotes the order of vanishing of f at v. Recall that the order of vanishing at $v = \infty$ of a rational function $\frac{p}{q} \in \mathbb{C}(t)$ $(p, q \in \mathbb{C}[t])$ equals $\deg(q) - \deg(p)$.

The basic properties of these multiplicities are:

- $\mathrm{ord}_v(\rho) = (0, 0)$ except for a finite number of $v \in \mathbb{P}^1$ and
- $\sum_{v\in\mathbb{P}^1} \mathrm{ord}_v(\rho) = (0, 0)$.

We next define an auxiliary operation which produces a convex lattice polygon from a balanced family of vectors of the plane. Let $B \subset \mathbb{Z}^2$ be a family of vectors which are zero except for a finite number of them and such that $\sum_{b\in B} b = (0, 0)$. We denote by $\mathcal{P}(B) \subset (\mathbb{R}_{\geq 0})^2$ the (unique) convex polygon obtained by: 1) rotating $-90°$ the non-zero vectors of B, 2) concatenating them following their directions counterclockwise and 3) translating the resulting polygon to the first quadrant $(\mathbb{R}_{\geq 0})^2$ in such a way that it "touches" the coordinate axes (Figure 5). The zero-sum condition warrants that the polygonal line closes at the end of the concatenation step.

The *tracing index* (or *degree*) $\mathrm{ind}(\rho) \geq 1$ is the number of times the parametrization ρ runs over the curve when t runs over $\mathbb{C}$. When $\mathrm{ind}(\rho) = 1$, we say that the parametrization is *birational*.

The solution to the problem of the computation of the Newton polygon of a parametric plane curve can be found in the papers of Dickenstein, E.-M. Feichtner, Sturmfels and Tevelev [5, 15, 20] and also in ours [4].

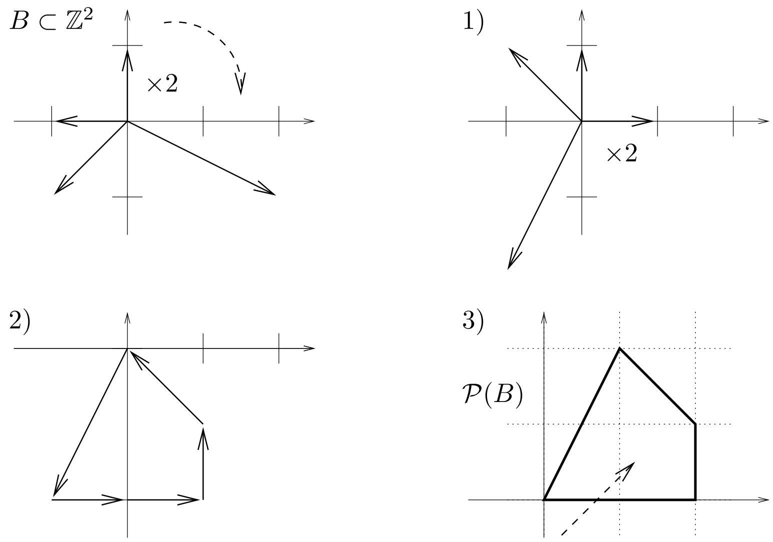

FIG. 5. *The operation $\mathcal{P}(B)$.*

THEOREM 2.1. *Let $\rho : \mathbb{C} \dashrightarrow \mathbb{C}^2$ be a rational map and set $C := \overline{\mathrm{Im}(\rho)}$, then*

$$\mathrm{N}(C) = \frac{1}{\mathrm{ind}(\rho)} \mathcal{P}\big((\mathrm{ord}_v(\rho))_{v \in \mathbb{P}^1}\big). \tag{6}$$

EXAMPLE 1. *Consider the parametrization*

$$\rho : t \mapsto \Big(\frac{1}{t(t-1)}, \frac{t^2 - 5t + 2}{t}\Big).$$

Its multiplicities are

$$\mathrm{ord}_0(\rho) = (-1,-1) \quad , \quad \mathrm{ord}_1(\rho) = (-1,0) \quad , \quad \mathrm{ord}_\infty(\rho) = (2,-1)$$

and $\mathrm{ord}_{v_i}(\rho) = (0,1)$ *for each of the two zeros v_1, v_2 of the equation $t^2 - 5t + 2 = 0$, while* $\mathrm{ord}_v(\rho) = (0,0)$ *for $v \neq 0, \pm 1, \infty, v_1, v_2$. Figure 5 illustrates the family B and the associated polygon $\mathcal{P}(B)$.*

Theorem 2.1 tells us that this polygon is $\mathrm{ind}(\rho)$ *times the actual Newton polygon of the curve. It is easy to check that the constructed polygon is non contractible, in the sense that it is not a non trivial integer multiple of another lattice polygon. We conclude that the parametrization is birational (that is,* $\mathrm{ind}(\rho) = 1$*) and that* $\mathrm{N}(C) = \mathcal{P}(B)$*. These results can be* *contrasted with the implicit equation of the curve:* $1 - 16x - 4x^2 - 9xy - 2x^2y - xy^2 = 0$.

Similarly, Theorem 2.1 allows to determine the Newton polygon of the folium of Descartes and of the Dickenstein-Fröberg example. Again, the additional data $\mathrm{ind}(\rho) = 1$ is a consequence of the non contractibility of the resulting polygon.

2.2. Tropical geometry and Intersection Theory. We will sketch here the two different methods for proving Theorem 2.1: Tropical Geometry and Intersection Theory.

Tropical Geometry can be regarded as the geometry of the min-plus algebra $(\mathbb{R}, \oplus, \odot)$, where the operations are defined as

$$x \oplus y = \min(x, y) \quad , \quad x \odot y = x + y.$$

To simplify the exposition, we will only deal with polynomials in $\mathbb{C}[x, y]$, although the theory extends naturally to multivariate polynomials with coefficients in a valuated field.

The *tropicalization* of a polynomial $F = \sum_{j=0}^{N} \lambda_j x^{a_j} y^{b_j} \in \mathbb{C}[x, y]$ is the concave piecewise linear function

$$t_F : \mathbb{R}^2 \to \mathbb{R} \quad , \quad x \mapsto \bigoplus_{j=0}^{N} x^{\odot a_j} y^{\odot b_j} = \min_j \langle (a_j, b_j), (x, y) \rangle. \tag{7}$$

Here, $\bigoplus$ stands for the tropical sum and $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{R}^2$. The *tropical variety* $\mathcal{T}_F \subset \mathbb{R}^2$ is defined as the set of points in $\mathbb{R}^2$ where this function is not smooth. It can be deduced from (7) that $\mathcal{T}_F$ consists exactly in the union of the outer normal directions to the edges of $\mathrm{N}(F)$. To each of these directions δ we can assign a *multiplicity* $m_\delta \geq 1$, which coincides with the lattice length of the edge of $\mathrm{N}(F)$ normal to the given direction. We recall that the *lattice length* $\ell(S)$ of a lattice segment S is the number of points in $\mathbb{Z}^2 \cap S$ minus 1.

This setting allows to interpret the Newton polygon as a certain degeneration of the curve and to study it with tools of Tropical Geometry. The proof given in [5,12] of Theorem 2.1 is based in the so-called "Kapranov Theorem" [6] and the Bieri-Groves Theorem. Moreover, their method allows them to treat higher-dimensional hypersurfaces parametrized by products of linear forms [5, 15].

As an illustration, Figure 6 shows the tropical variety associated to the curve in the example 1:

As we can see, the tropical variety plus the corresponding multiplicities are in correspondence with the vectors in Figure 5 and Theorem 2.1 can be easily reformulated in tropical terms.

On the other hand, in our paper [4] we propose a method which reduces the determination of the Newton polygon to the computation of the number of solutions of some polynomial systems of equations.

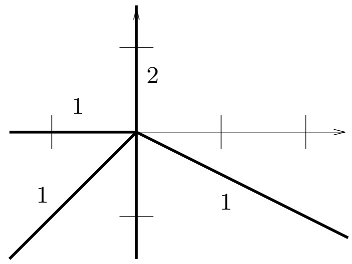

FIG. 6. *The tropical curve associated with* $C : 1 - 16x - 4x^2 - 9xy - 2x^2y - xy^2 = 0$.

The *support function* of a polygon $Q \subset \mathbb{R}^2$ is defined as

$$h_Q : \mathbb{R}^2 \to \mathbb{R} \quad , \quad x \mapsto \max\{\langle u, x\rangle : u \in Q\}.$$

It is a convex piecewise affine function which completely characterizes Q. Let $\rho : \mathbb{C} \dashrightarrow \mathbb{C}^2$ be a rational parametrization and set $C := \overline{\mathrm{Im}(\rho)}$. Then, for $\sigma \in (\mathbb{N} \setminus \{0\})^2$ it can be shown that

$$h_{\mathrm{N}(C)}(\sigma) = \frac{1}{\mathrm{ind}(\rho)} \#\big\{(t,x,y) \in \mathbb{C}^3 : x^{\sigma_1} = f(t), y^{\sigma_2} = g(t), \\ \ell_0 + \ell_1 x + \ell_2 y = 0\big\} \tag{8}$$

for generic $\ell_0, \ell_1, \ell_2 \in \mathbb{C}$. The proof of Theorem 2.1 reduces then to the determination of this number of solutions, which can be obtained *via* the refinement of the Bernštein-Kušnirenko-Khovanskiĭ (BKK) Theorem recently obtained by P. Philippon and the second author [13].

Identity (8) holds also in higher dimensions. However, there is no analogue for $n \geq 3$ of the estimation in [13] and so for the moment, this method cannot be extended to higher dimension.

3. Some applications and consequences. Besides of its theoretical interest, the Newton polytope is useful for computational purposes. Its knowledge allows to speed-up computations and gives interesting information about the solutions of polynomial systems of equations. Here we point out two applications.

3.1. Computing the implicit equation with numerical interpolation. The Newton polytope tells us which exponents might occur in the implicit equation and thus allows us to compute it *via* a suitable interpolation algorithm. Suppose we are given a parametrization $\boldsymbol{\rho} = (f, g) : \mathbb{C} \dashrightarrow \mathbb{C}^2$ and that we want to compute the implicit equation $E(x, y) \in \mathbb{C}[x, y]$ of its image curve. A possible strategy is to apply Theorem 2.1 to obtain its Newton polygon Q and use this information to recover E. We have

$$E(x,y) = \sum_{j=0}^{N} \lambda_j x^{a_j} y^{b_j}$$

where the (a_j, b_j)'s are the integer points in Q and the $\lambda_j \in \mathbb{C}$ are unknown. To determine these coefficients, we can evaluate $\boldsymbol{\rho}$ in $N+1$ points $\tau_0, \ldots, \tau_N \in \mathbb{C}$ where $\rho(\tau_i)$ is defined. We then obtain a homogeneous system of linear equations in the λ_j's, of size $(N+1) \times (N+1)$:

$$E(\boldsymbol{\rho}(\tau_k)) = \sum_{j=0}^{N} \lambda_j f(\tau_k)^{a_j} g(\tau_k)^{b_j} = 0 \quad \text{for } 0 \le k \le N.$$

If the interpolation points τ_k are generic enough, the solution space of this system is of dimension 1 and the polynomial $E(x,y)$ can be computed as some (any) generator of this space. This approach is most useful when Q has few points integer points, which for instance is the case of the parametrization (4), where the number of integer points is 3 for any $a \in \mathbb{N}$.

3.2. Intersecting parametric curves. In the practice of CAGD, it is important to be able to determine where two modelled shapes cut each other. Typically, this amounts to compute the intersection of two curves or surfaces given in parametric form. This task can be done by computing the implicit equation of one of the two varieties but as explained, this can be too expensive. If we only have access to the Newton polytope, we will not be able to compute this intersection but we still can say something about the number of intersection points of two parametric curves or about the degree of the intersection curve of two parametric surfaces.

For two plane curves $C, D \subset \mathbb{C}^2$, the BKK Theorem says that the number of their intersection points in $(\mathbb{C}^\times)^2$ is bounded above by the *mixed volume*

$$\mathrm{Area}(\mathrm{N}(C) + \mathrm{N}(D)) - \mathrm{Area}(\mathrm{N}(C)) - \mathrm{Area}(\mathrm{N}(D))$$

with equality in the generic case. Here, the "+" denotes the Minkowski (that is, pointwise) sum of polygons in the plane. For instance, let $C, D \subset (\mathbb{C}^\times)^2$ be the curves respectively parametrized by

$$t \mapsto \Big(\frac{(t+1)^2}{2\,t\,(1-t)}, \frac{4\,t\,(t-1)^3}{(t+1)^5} \Big) \quad , \quad t \mapsto \Big(t, \frac{10}{t^3} \Big).$$

In Figure 7 we see the corresponding polygons and their Minkowski sum. The mixed volume is the area of the shaded zone, which is equal to 2.

Hence C and D have at most two points in common, which turn out to be $(1.33, 4.22)$ and $(-4.17, -0.14)$ (Figure 8).

3.3. Generic parametrizations. With Theorem 2.1 at our disposal, we can easily answer Problem 2.2 for the case $n = 2$:

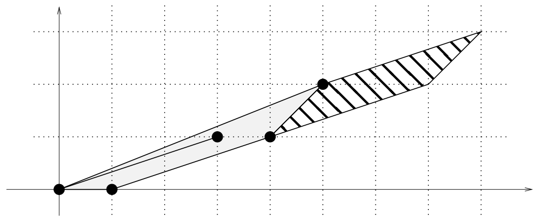

FIG. 7. *The mixed volume of two polygons.*

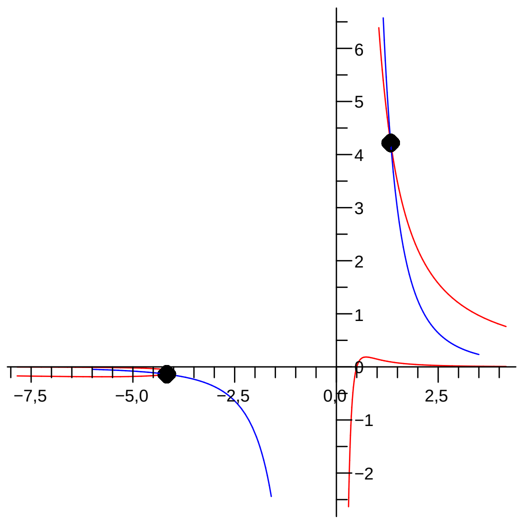

FIG. 8. *The intersection of two parametric curves.*

COROLLARY 3.1. *For $D \geq d$, $E \geq e$ let*

$$p(t) = \alpha_d t^d + \cdots + \alpha_D t^D \quad , \quad q(t) = \beta_e t^e + \cdots + \beta_E t^E \quad \in \mathbb{C}[t^{\pm 1}] \tag{1}$$

such that $\alpha_d, \alpha_D, \beta_e, \beta_E \neq 0$ and $\gcd(t^{-d}p(t), t^{-e}q(t)) = 1$. Set $\rho = (p, q)$ and $C := \overline{\operatorname{Im}(\rho)}$, then

$$\mathrm{N}(C) = \frac{1}{\operatorname{ind}(\rho)} \mathcal{P}\big((D-d, 0), (0, E-e), (-D, -E), (d, e)\big).$$

In particular, parametrizations by generic Laurent polynomials produce equations whose Newton polygon is typically a quadrilateral (Figure 9).

The proof of this corollary is simple: we have $\operatorname{ord}_0(\rho) = (d, e)$ and $\operatorname{ord}_\infty(\rho) = (-D, -E)$. Let $v_1, \ldots, v_r \neq 0$ be the different roots of $t^{-d}p(t)$ and $m_i \geq 1$ the multiplicity of v_i in p, then

$$\operatorname{ord}_{v_i}(\rho) = (m_i, 0) \quad \text{for } 1 \leq i \leq r$$

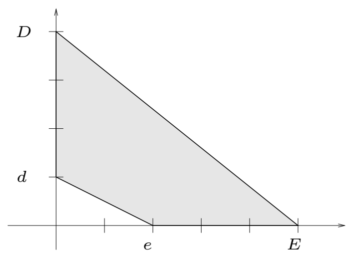

FIG. 9. *The Newton polygon of a generic Laurent polynomial parametrization.*

as we assume that p and q do not share roots in the torus. Similarly, let $w_1, \dots, w_s \neq 0$ be the roots of $t^{-e}q(t)$ and $n_j \geq 1$ be their respective multiplicities. For the same reasons as above

$$\mathrm{ord}_{w_j}(\rho) = (0, n_j).$$

Theorem 2.1 then shows that $\mathrm{ind}(\rho)\,\mathrm{N}(C)$ is obtained by rotating $-90°$ and concatenating the vectors (d, e), $(-D, -E)$, $(m_i, 0)$ and $(0, n_j)$, for $1 \leq i \leq r$ and $1 \leq j \leq s$. But the $(m_i, 0)$'s are all pointing in the same direction and so they concatenate together into the vector $\sum_i (m_i, 0) = (D - d, 0)$. Similarly, the $(0, n_j)$'s concatenate together into $\sum_j (0, n_j) = (0, E - e)$, which concludes the proof.

Moreover, it can be shown that for a parametrization like (1), the Newton polygon of the implicit equation equals

$$\frac{1}{\mathrm{ind}(\rho)} \mathcal{P}\big((D - d, 0), (0, E - e), (-D, -E), (d, e)\big)$$

if and only if $\alpha_d, \alpha_D, \beta_e, \beta_E \neq 0$ and $\gcd(t^{-d}p(t), t^{-e}q(t)) = 1$. If besides the vectors $(D-d, 0), (0, E-e), (d, e)$ are not collinear, then ρ is birational.

Note that the polygon does not depend on the actual values of the roots of p and q, it only depends on the hypothesis that they are disjoint and that we know the sum of their multiplicities. This is a general principle: for computing the Newton polygon of a parametrization $\rho = (f, g)$, we do not need full access to the zeros and poles of f and g. It suffices with partial factorizations of the form

$$f(t) = \alpha \prod_{p \in P} p(t)^{d_p} \quad , \quad g(t) = \beta \prod_{p \in P} p(t)^{e_p}$$

where $\mathcal{P} \subset \mathbb{C}[t]$ is a finite set of relatively prime polynomials, $d_p, e_p \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}^\times$. Such factorizations can be obtained with gcd's operations only, which is certainly easier than extracting roots and poles.

Back to the world of generic parametrizations, the second case to tackle is when we have two rational functions with the same denominator. It turns out that the resulting Newton polygon has at most five edges (Figure 10).

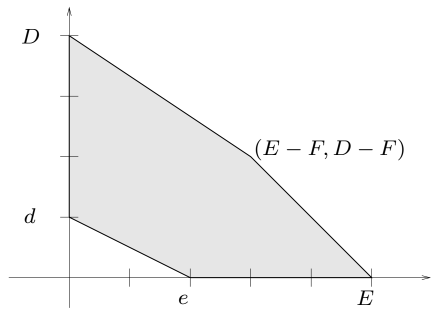

Fig. 10. *The Newton polygon of a parametrization by generic rational functions with the same denominator.*

Corollary 3.2. *Given $D \geq d$, $E \geq e$ and $F \geq 0$, let*

$$p(t) = \alpha_d t^d + \cdots + \alpha_D t^D, \quad q(t) = \beta_e t^e + \cdots + \beta_E t^E, \quad r(t) = \gamma_0 + \cdots + \gamma_F t^F.$$

Set $\rho = \left(\frac{p}{r}, \frac{q}{r}\right) \in \mathbb{C}(t)^2$ and $C := \overline{\mathrm{Im}(\rho)}$, then

$$\mathrm{N}(C) = \frac{1}{\mathrm{ind}(\rho)} \mathcal{P}\big((D-d,0),(0,E-e),(F-D,F-E),(d,e),(-F,-F)\big)$$

if and only if $\alpha_d, \alpha_D, \beta_e, \beta_E, \gamma_0, \gamma_F \neq 0$ and $t^{-d}p(t), t^{-e}q(t), r(t)$ are pairwise coprime.

Finally, we consider the case when the parametrization is given by two generic rational functions with different denominators. The resulting polygon has at most six edges (Figure 11).

Corollary 3.3. *Given $D \geq d$, $E \geq e$, $F, G \geq 0$, let*

$$p(t) = \alpha_d t^d + \cdots + \alpha_D t^D \quad , \quad q(t) = \beta_e t^e + \cdots + \beta_E t^E \quad \in \mathbb{C}[t^{\pm 1}]$$

and

$$r(t) = \gamma_0 + \cdots + \gamma_F t^F \quad , \quad s(t) = \delta_0 + \cdots + \delta_G t^G \quad \in \mathbb{C}[t].$$

Set $\rho = \left(\frac{p}{r}, \frac{q}{s}\right)$ and $C := \overline{\mathrm{Im}(\rho)}$, then

$$\mathrm{N}(C) = \frac{1}{\mathrm{ind}(\rho)} \mathcal{P}\big((D-d,0),(0,E-e),(F-D,G-E),(d,e),(-F,0),(0,-G)\big)$$

if and only if $\alpha_d, \alpha_D, \beta_e, \beta_E, \gamma_0, \gamma_F, \delta_0, \delta_G \neq 0$ and $t^{-d}p(t), t^{-e}q(t), r(t), s(t)$ are pairwise coprime.

4. The general case *vs* the generic case. Now suppose we start from the other endpoint, that is suppose that we are given the equation $E(x, y)$ of a parametric curve. What does its Newton polytope tell us about the (unknown) parametrization?

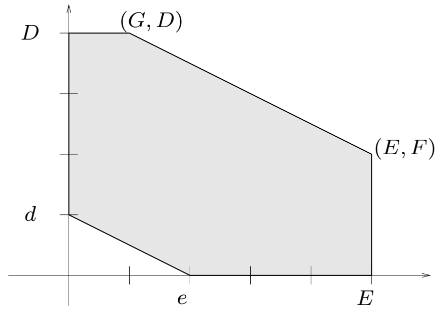

FIG. 11. *The Newton polygon of a generic parametrization by rational functions with different denominators.*

A first natural question is whether $\mathrm{N}(E(x,y))$ can be *any* lattice polygon. As we have seen, the polygons produced by generic parametrizations are very special: they have at most six edges and some of them are in prefixed directions.

Before answering this question, let us fix a lattice polygon $Q \subset (\mathbb{R}_{\geq 0})^2$ with non empty interior and touching the coordinate axes. We will identify $\mathbb{C}^{\#(Q\cap\mathbb{Z}^2)}$ with the $\mathbb{C}$-vector space of polynomials whose Newton polygon is contained in Q. Consider the set

$$M_Q^\circ := \left\{F \in \mathbb{C}[x,y] : \ \mathrm{N}(F) = Q, \ F \text{ defines a parametric curve in } \mathbb{C}^2\right\} \subset \mathbb{C}^{\#(Q\cap\mathbb{Z}^2)}$$

and let M_Q denote its Zariski closure. Recall that ∂Q denotes the *border* of Q.

THEOREM 4.1 ([4]). *M_Q is a parametric variety of dimension $\#(\partial Q \cap \mathbb{Z}^2)$.*

In particular, $\dim(M_Q) \geq 3$ as Q must have at least three edges. It turns out that any lattice polygon with non empty interior and supported in the coordinate axes is the Newton polygon of a parametric curve.

A further consequence of this result is that the codimension of M_Q equals the number of lattice points in the interior of Q. This is interesting for the inverse problem: given a polynomial $E(x,y) \in \mathbb{C}[x,y]$, decide whether it defines a parametric curve or not and if it is the case, compute a parametrization.

If the Newton polygon of the equation has a lot of points in its interior, then the probability that E defines a parametric curve is low. If nevertheless this is the case, the corresponding parametrization will be defined by $\#(\partial Q\cap\mathbb{Z}^2)$ degrees of freedom, and hence the efficiency of the computation of such a parametrization should be correlated with the number of lattice points in ∂Q and not with the number of lattice points in the whole of Q.

Some pointers to the literature:

- Parametric curves in general: [1, 14, 21]
- Numerical interpolation methods: [2, 10, 19]
- Newton polytopes and specialized resultants: [7, 8]
- Newton polytopes and Tropical Geometry: [5, 12, 15, 16, 17]
- Newton polytopes and mixed fiber polytopes: [9, 11, 17]
- Newton polytopes and Intersection Theory: [4, 13]

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