Intrinsic palindromes

Antonio J. Di Scala[1] and Martín Sombra[2]

To our friend Cristian Jansenson

Abstract. We introduce a notion of palindromicity of a natural number which is independent of the base. We study the existence and density of palindromic and multiple palindromic numbers, and we raise several related questions.

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Natural numbers occur everywhere in our daily life: a bus ticket, a car plate, an id number, a timetable, etc.. These numbers are mostly expressed in the decimal system. That is, for a natural number $n \in \mathbb{N}$ we write $n = (a_{k-1} \cdots a_0)_{10}$ for some $0 \le a_i \le 9$ and $a_{k-1} \ne 0$, which means that

$$n = a_{k-1} \, 10^{k-1} + a_{k-2} \, 10^{k-2} + \cdots + a_0.$$

Of particular attraction are the so-called palindromic numbers. These are the numbers whose decimal expansion is the same when read from left to right, and from right to left, that is $(a_{k-1} \cdots a_0)_{10} = (a_0 \cdots a_{k-1})_{10}$.
This kind of numbers appears already in the Ganitasârasamgraha, a sanskrit manuscript dated around 850 AD. Therein the Indian mathematician Mahâvîrâchârya described the 12345654321 as the quantity which "beginning with one until it reaches six, then decreases in reverse order" [1, p.399]. This is a curious palindromic number, and in particular it is the square of another palindrome: $12345654321 = 111111^2$. There are many ways of generating palindromic numbers (see for instance [2]) including an interesting conjecture [3].

Crossing by chance with one such a number is a rare occurrence: for instance, the probability of picking at random a number $10^4 \le n < 10^5$ and it resulting a palindrome is $1/10^2$, a fact well-known by the collectors of palindromic bus tickets. Hence we tend to feel pretty lucky when one of this rare numbers crosses our path. And the more figures it has, the luckier we feel, and the luckier the number itself seems to be.

But, is this feeling really justified? The truth has to be said: this property is *not* intrinsic to the number, but also depends on the base used to express it. So the number $n := (894111498)_{10}$ is lucky (or palindromic) in base 10, but is not in base 13 as $n = (113314377)_{13}$. Hence each time we encounter the 894111498 we have to — at least in principle — thank heaven for this occurrence *together* with the fact that the human race has five fingers in each hand.

However we can easily get independent of the base, defining a number to be intrinsically palindromic if is palindromic in *some* base. A minute reflection shows that this definition is meaningless as it stands: every number is palindromic in any base $m > n$, as $n = (n)_m$.

Indeed it is much more natural to take into account the number of figures. So we define a number $n \in \mathbb{N}$ to be *k-palindromic* if there is a base b such that the b-expansion of n is palindromic of length k.

The previous observation shows that every number is 1-palindromic. Also note that for all $n \geq 2$ it holds $n = (11)_{n-1}$, that is every $n \geq 2$ is 2-palindromic.

What about k-palindromic numbers for $k \geq 3$? Our thesis is that very few numbers are k-palindromic, at least for $k \geq 4$: the probability of a number in the appropriate range being, say, intrinsically 9-palindromic is small, and indeed quite close to the probability of being 9-palindromic in base 10. This justifies our first impression that the 894111498 is lucky, regardless the base chosen to represent it.

To write down our results we first have to introduce the following counting functions. Take $k, N, b \in \mathbb{N}$ and set

$$\Phi_k(N, b) := \#\{\ n \leq N\ ;\ n \text{ is } k\text{-palindromic in base } b\ \}.$$

Also set

$$\Phi_k(N) := \#\{\ n \leq N\ ;\ n \text{ is } k\text{-palindromic}\ \}.$$

Then $\Phi_k(N, b)/N$ and $\Phi_k(N)/N$ stand for the *density* (or probability) of numbers below N which are k-palindromic in base b and intrinsically k-palindromic, respectively.

Theorem 1 *Let $k \geq 4$, and write $k = 2\,i + r$ with $i \in \mathbb{N}$ and $r = 0, 1$. Then*

$$\Phi_k(N) \leq 4\,(N+1)^{\frac{i+r+1}{k}}.$$

Proof.– A base b contributes to $\Phi_k(N)$ if and only if $\Phi_k(N, b) > 0$, namely if and only if there exists a number

$$n = a_{k-1}\,b^{k-1} + \cdots + a_0\ \leq N$$

palindromic in base b of length k, that is such that $0 \leq a_j \leq b - 1$ and $a_{k-j} = a_j$ for $j = 0, \ldots, i + r$, and $a_{k-1} \neq 0$. Then $b^{k-1} + 1 \leq n \leq N$, and hence b contributes to $\Phi_k(N)$ if and only if $b \leq (N-1)^{\frac{1}{k-1}}$.

We consider separately the cases $b \leq (N+1)^{\frac{1}{k}}$ and $(N+1)^{\frac{1}{k}} < b \leq (N-1)^{\frac{1}{k-1}}$. In the first case, the largest k-palindrome in base b is

$$(b - 1)\,(b^{k-1} + \cdots + 1) = b^k - 1 \leq N,$$

and so $\Phi_k(N, b) = \Phi_k(\infty, b) = (b - 1)\,b^{i+r-1}$.

Now for the second case we let $\theta(b) \in \mathbb{N}$ be the largest integer such that $\theta(b)\,(b^{k-1} + 1) \leq N$, that is $\theta(b) = [N/(b^{k-1} + 1)] \leq N/b^{k-1}$. Then $(\theta(b) + 1)\,(b^{k-1} + 1) > N$ and so every k-palindromic number in base b begins with $a_{k-1} \leq \theta(b)$. Hence

$$\Phi_k(N, b) \leq \theta(b)\,b^{i+r-1}\ \leq N/b^i.$$

Set
$$L := [(N+1)^{\frac{1}{k}}] \qquad , \qquad M := [(N-1)^{\frac{1}{k-1}}].$$

We split the sum $\Phi_k(N) = \varphi + \chi + \psi$ with $\varphi := \sum_{b=2}^{L-1} \Phi_k(N,b)$, $\psi := \sum_{b=L+2}^{M} \Phi_k(N,b)$, and $\chi := \Phi_k(N,L) + \Phi_k(N,L+1)$. From the previous considerations we deduce that

$$\varphi \leq \sum_{b=2}^{L-1} b^{i+r} \leq \int_2^L t^{i+r}\,dt \leq \frac{L^{i+r+1}}{i+r+1} \leq \frac{(N+1)^{\frac{i+r+1}{k}}}{i+r+1}.$$

On the other hand

$$\psi \leq \sum_{L+2}^{M} \frac{N}{b^i} \leq N \int_{L+1}^{M} \frac{dt}{t^i} \leq \frac{N}{(i-1)\,(L+1)^{i-1}} \leq \frac{(N+1)^{\frac{i+r+1}{k}}}{i-1}.$$

Finally $\chi \leq L^{i+r} + N/(L+1)^i \leq 2\,(N+1)^{\frac{i+r}{k}}$ and thus

$$\Phi_k(N) \leq \left(\frac{2}{(N+1)^{\frac{1}{k}}} + \frac{1}{i+r+1} + \frac{1}{i-1} \right) (N+1)^{\frac{i+r+1}{k}} \leq 4\,(N+1)^{\frac{i+r+1}{k}}.$$

$\square$

Let $k = 2\,i + r \geq 4$, and let $b \geq 2$ be a base. Set $N := b^k - 1$, so that N is larger than every number whose representation in base b has length k. Then

$$\Phi_k(N,b) = \Phi_k(\infty, b) = (b-1)\,b^{i+r-1},$$

and so the density of numbers below N which are k-palindromic in base b is $(b-1)\,b^{i+r-1}/N \sim 1/b^i$. On the other hand, the previous result shows that the density of intrinsic k-palindromes below N is bounded by $4\,(b^k)^{\frac{i+r+1}{k}}/(b^k-1) \leq 4/b^{i-1}$.

For instance, the probability of a number $n < 10^9$ being 9-palindromic in base 10 is 0.00009, while the probability of it being 9-palindromic in any base is below 0.004.

From the point of view of probability, the situation is then — in most cases — quite clear: for $k \leq 2$ every number is k-palindromic, while for $k \geq 4$ almost every number is not.

The critical case is $k := 3$. Consider the following table:

$\Phi_3(10^2 + 100) - \Phi_3(100)$	=	61
$\Phi_3(10^3 + 100) - \Phi_3(10^3)$	=	70
$\Phi_3(10^4 + 100) - \Phi_3(10^4)$	=	83
$\Phi_3(10^5 + 100) - \Phi_3(10^5)$	=	86
$\Phi_3(10^6 + 100) - \Phi_3(10^6)$	=	89
$\Phi_3(10^7 + 100) - \Phi_3(10^7)$	=	94

This suggests that almost every number is 3-palindromic, but *not* every sufficiently large number. To tackle this problem it might be worth considering the following

reformulation. We recall that $\{\xi\} := \xi - [\xi] \in [0,1)$ denotes the fractional part of a real number $\xi \in \mathbb{R}$.

Lemma 2 *Let $n, b \in \mathbb{N}$ such that the b-expansion of n has length 3. Then n is 3-palindromic in base b if and only if*

$$\left\{ (n+1)\frac{b}{b^2+1} \right\} < \frac{b}{b^2+1}.$$

Proof.– First note that the hypothesis that the b-expansion of n has length 3 is equivalent to the fact that $b^2 + 1 \leq n \leq b^3 - 1$. Now n is 3-palindromic in base b if and only if there exists $0 < e < b$ and $0 \leq f < b$ such that

$$n = e\,(b^2+1) + f\,b.$$

Solving the associated Diophantine linear equation $n = x\,(b^2+1) + y\,b$ with respect to x, y we see that the above representation is equivalent to the existence of $\ell \in \mathbb{Z}$ satisfying

$$0 < n - \ell\,b < b \qquad , \qquad 0 \leq \ell\,(b^2+1) - n\,b < b.$$

The second pair of inequalities is equivalent to $n\,b/(b^2+1) \leq \ell < (n+1)\,b/(b^2+1)$, and so it implies that $\{\,(n+1)\,b/(b^2+1)\,\} < b/(b^2+1)$. Then this condition is necessary for n to be 3-palindromic in base b.

Let's check that it is also sufficient: the integer $\ell := [(n+1)\,b/(b^2+1)]$ satisfies the second pair of inequalities. Then it only remains to prove that it also satisfies the first pair, which is equivalent to $\ell < n/b < \ell + 1$. This follows from the inequalities

$$(n+1)\frac{b}{b^2+1} < \frac{n}{b} < n\frac{b}{b^2+1} + 1,$$

which are in term a consequence of the hypothesis $b^2 + 1 \leq n \leq b^3 - 1$.

$\square$

Now we begin to look at palindromicity as an intrinsic property — not attached to any particular base — nothing stop us from considering the fact that a given number can be palindromic in several different basis. For instance

$$3074 = (44244)_5 = (22122)_6.$$

Common sense dictates that multiple palindromicity should be a much more rare occurrence than simple one, which is also rare as we have already shown. In fact it even seems unclear whether there are numbers which are k-palindromic in as many basis as desired. We formalize this: let

$$\mu_k(n) := \#\{\ b\,;\ n \text{ is } k\text{-palindromic in base } b\ \}.$$

So in first instance, we propose the problem of determining whether μ_k is unbounded or not. Again the cases $k = 1, 2$ are easy. In the first case $n = (n)_m$ for any base

$m > n$, and so $\mu_1(n) = \infty$ for every n. In the second case, set $n := 2^{2\,u+1}$ for some $u \in \mathbb{N}$. Then $n = 2^v\,(2^w - 1) + 2^v = (2^v, 2^v)_{2^w - 1}$ for $v < w$ such that $v + w = 2\,u + 1$. Then $\mu_2(n) \geq u$.

The following solves the case $k = 3$:

Theorem 3 *There exists an infinite sequence $n_1 < n_2 < n_3 < \cdots$ such that*

$$\mu_3(n_j) \geq \frac{1}{7}\,\log(n_j + 1).$$

Proof.– Take $N \gg 0$ and assume that $\mu_3(n) < (1/7)\,\log(N + 1)$ for all $n \leq N$, so that

$$\sum_b \Phi_3(N, b) = \sum_{n=1}^N \mu_3(n) < \frac{1}{7}\,N\,\log(N + 1). \tag{1}$$

We will see in a minute that this is contradictory:
Set $L := [(N + 1)^{\frac{1}{3}}]$ and $M := [(N - 1)^{\frac{1}{2}}]$. For $L \leq b \leq M$ we let $\zeta(b) \in \mathbb{N}$ be the largest integer such that

$$\zeta(b)\,(b^2 + 1) + (b - 1)\,b \leq N.$$

Then $1 \leq \zeta(b) \leq b - 1$, and also every 3-palindromic number $n := e\,b^2 + f\,b + e$ with $e \leq \zeta(b)$ is less or equal than N. Hence $\Phi_3(N, b) \geq \zeta(b)\,b$ which implies that

$$\sum_b \Phi_3(N, b) \geq \sum_{b=L}^M \Phi_3(N, b) \geq \sum_{b=L}^M \zeta(b)\,b \geq \sum_{b=L}^M b\,\Big(\frac{N}{b^2 + 1} - 2\Big),$$

as $\zeta(b) + 2 \geq N/(b^2 + 1)$. We have that

$$\sum_{b=L}^M b\,\Big(\frac{N}{b^2 + 1} - 2\Big) \geq N\,\int_L^{M+1} \frac{t}{t^2 + 1}\,dt - 2\,M^2 \geq \frac{N}{2}\,\Big(\log((M + 1)^2 + 1) - \log(L^2 + 1)\Big) - 2\,N.$$

We have that $(M + 1)^2 + 1 \geq N$ and $L^2 + 1 \leq 2\,N^{2/3}$ and thus we conclude $\sum_p \Phi_3(N, p) > \frac{N}{6}\,\log N - 2\,N - \log 2$, which contradicts Inequality 1 for N large enough.

It follows that for each (sufficiently large) $N \in \mathbb{N}$ there exists $n \leq N$ such that

$$\mu_3(n) \geq \frac{1}{7}\,\log(N + 1) \geq \frac{1}{7}\,\log(n + 1).$$

The fact that $\mu_3(n) < \infty$ implies that the set of such n's is infinite. $\qquad\square$

Here is some sample data for the cases $k := 4, 5$:

$$\mu_4(624) = \mu_4(910) = 2 \qquad , \qquad \mu_4(19040) = 3 \qquad , \qquad \mu_5(2293) = 2.$$

For $k, \ell, N \in \mathbb{N}$, we let $\Phi_{k,\ell}(N)$ be the number of $n \leq N$ which are k-palindromes in ℓ different basis, that is

$$\Phi_{k,\ell}(N) := \#\{\ n \leq N\,;\ \mu_k(n) \geq \ell\ \}.$$

In particular $\Phi_{k,1} = \Phi_k$. The following table gives some more informative data:

k	ℓ	N	$\Phi_{k,\ell}(N)$
4	2	10^4	13
4	3	10^5	2
4	4	10^5	0
5	2	10^4	10
5	3	10^5	0
6	2	10^5	0

This suggest that for $k \geq 4$ and $k + \ell \geq 8$ there are no k-palindromic numbers with multiplicity ℓ at all.

Finally we can also consider

$$\mu_{\geq k}(n) := \#\{ b \,;\, n \text{ is } j\text{-palindromic in base } b \text{ for some } j \geq k \} = \sum_{j \geq k} \mu_j(n),$$

that is the number of different basis in which n is a palindrome of length *at least* k. It is easy to see that this function is unbounded: we have that

$$n_L := 2^{2^L} - 1 = (\overbrace{2^{2^\ell} - 1, \cdots, 2^{2^\ell} - 1}^{2^{L-\ell}})_{2^{2^\ell}}$$

and so n_L is $2^{L-\ell}$-palindromic in base 2^{2^ℓ} for $\ell = 0, \ldots, L$. Hence $\mu_{\geq k}(n_L) \geq L - \log_2 k$.

A further problem is to determine the *density* of k-palindromic numbers in ℓ different basis. From this point of view, Theorem 1 is an important advance towards the solution of the cases $k \geq 4$ and $\ell = 1$.

The cases when $\ell \geq 2$ seem to be much more elusive, but also interesting. A solution of them would allow you, for instance, to know how lucky you are when the number of the taxi-cab you are riding is the

$$19040 = (8888)_{13} = (5995)_{15} = (2, 14, 14, 2)_{19}.$$

References

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Post-Scriptum: After this paper was written, N.J.A. Sloane included the sequences of $k-$palindromic number for $k = 3, \ldots, 9$ in his on-line Encyclopædia of Integer Sequences (http://www.research.att.com/~njas/sequences).

Antonio J. Di Scala: Facultad de Matemática, Astronomía y Física (Fa.M.A.F.), Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

E-mail: discala@mate.uncor.edu

Martín Sombra: Université de Paris 7, Institut de Mathématiques, Équipe de Géométrie et Dynamique, Case 7012, 2 place Jussieu, 75251 Paris Cedex 05, France; and Departamento de Matemática, Universidad Nacional de La Plata, Calle 50 y 115, 1900 La Plata, Argentina.

E-mail: sombra@math.jussieu.fr