THE CAYLEY-MENGER DETERMINANT IS IRREDUCIBLE FOR $n \geq 3$

CARLOS D'ANDREA AND MARTÍN SOMBRA

Abstract. We prove that the Cayley-Menger determinant of an n-dimensional simplex is an absolutely irreducible polynomial for $n \geq 3$. We also study the irreducibility of polynomials associated to related geometric constructions.

Let $\{d_{ij} \, : \, 0 \leq i < j \leq n\}$ be a set of $\frac{n\,(n+1)}{2}$ variables and consider the square $(n+2) \times (n+2)$ matrix

$$\mathrm{CM}_n := \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{01}^2 & d_{02}^2 & \cdots & d_{0n}^2 \\ 1 & d_{01}^2 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ 1 & d_{02}^2 & d_{12}^2 & 0 & \cdots & d_{2n}^2 \\ \vdots & & & & \ddots & \\ 1 & d_{0n}^2 & d_{1n}^2 & d_{2n}^2 & \cdots & 0 \end{bmatrix} \quad . \tag{1}$$

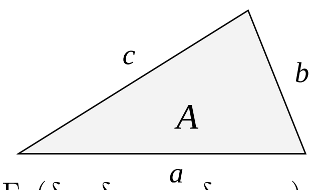
The multivariate polynomial $\Gamma_n := \det(\mathrm{CM}_n) \in \mathbb{Z}[d_{ij} : 0 \leq i < j \leq n]$ is the *Cayley-Menger determinant.*

Let $v_0, \ldots, v_n \in \mathbb{R}^n$ be $n+1$ points and denote by S its convex hull in $\mathbb{R}^n$. This determinant gives a formula for the n-dimensional volume of S in terms of the Euclidean distances $\{\delta_{ij} := \mathrm{dist}(v_i, v_j) \, : \, 0 \leq i < j \leq n\}$ among these points. We have [Blu53, Sec. IV.40], [Ber87, Sec. 9.7]

$$\mathrm{Vol}_n(S)^2 = \frac{(-1)^{n+1}}{2^n \, (n!)^2} \, \Gamma_n(\delta_{01}, \delta_{02}, \ldots, \delta_{(n-1)\,n}) \quad .$$

This formula shows that Γ_n is a homogeneous polynomial of degree $2\,n$. The second polynomial Γ_2 can be completely factorized, giving rise to the well-known *Heron's formula* for the area A of a triangle with edge lengths a, b, and c:

$$16\,A^2 = -\Gamma_2(a,b,c) = (a+b+c)(-a+b+c)(a-b+c)(a+b-c) \quad . \tag{2}$$



Note also that the equation $\Gamma_n(\delta_{01}, \delta_{02}, \ldots, \delta_{(n-1)\,n}) = 0$ gives a necessary and sufficient condition for the points $v_0, \ldots, v_n$ to lie in a proper affine subspace of $\mathbb{R}^n$.

Date: October 19, 2004.

2000 *Mathematics Subject Classification.* Primary 12E05; Secondary 52A38.

Key words and phrases. Volume of a simplex, Cayley-Menger determinant, irreducible polynomial.

C. D'Andrea was supported by a Miller Research Fellowship.

The Cayley-Menger determinant can be also used for deciding whether a set of positive real numbers $\{\delta_{ij} : 0 \le i < j \le n\}$ can be realized as the set of edge lengths of an n-dimensional simplex in $\mathbb{R}^n$: in [Ber87, Sec. 9.7.3] it is shown that this condition is equivalent to $(-1)^{h+1}\,\Gamma_h(\delta_{01}, \delta_{02}, \ldots, \delta_{(h-1)\,h}) > 0$, for $h = 1, 2, \ldots, n$.

The matrix CM_n also gives a criterion to determine if $n+2$ points in $\mathbb{R}^n$ lie in an $(n-1)$-dimensional sphere, and to solve the related problem of computing the radius of the sphere circumscribed around a simplex. To do this, consider the $(1,1)$-minor

$$\Delta_n := \det \begin{bmatrix} 0 & d_{01}^2 & d_{02}^2 & \cdots & d_{0n}^2 \\ d_{01}^2 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\ d_{02}^2 & d_{12}^2 & 0 & \cdots & d_{2n}^2 \\ \vdots & & & \ddots & \\ d_{0n}^2 & d_{1n}^2 & d_{2n}^2 & \cdots & 0 \end{bmatrix} \in \mathbb{Z}[d_{ij} : 0 \le i < j \le n] \ .$$

From this expression we see that this is a homogeneous polynomial of degree $2\,n+2$. Assume now that $v_0, \ldots, v_n$ do not lie in a proper affine subspace, so that S is an n-dimensional simplex. The radius $\rho(S)$ of the sphere circumscribed around S is given by

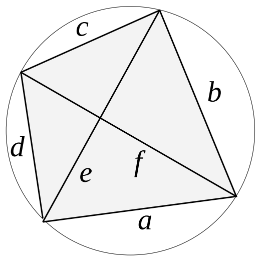
$$\rho(S)^2 = -\frac{1}{2}\frac{\Delta_n(\delta_{01}, \delta_{02}, \ldots, \delta_{(n-1)\,n})}{\Gamma_n(\delta_{01}, \delta_{02}, \ldots, \delta_{(n-1)\,n})} \ . \tag{3}$$

Also, the condition for $n+2$ points $v_0, \ldots, v_{n+1}$ in $\mathbb{R}^n$ to lie in the same sphere or hyperplane is given by the annihilation of the $(n+1)$-th polynomial $\Delta_{n+1}(\delta_{01}, \delta_{02}, \ldots, \delta_{n\,(n+1)}) = 0$, see [Ber87, Sec. 9.7.3.7].

The third polynomial Δ_3 factorizes as

$$\begin{aligned} \Delta_3 \ = \ & -(d_{01}\,d_{23} + d_{02}\,d_{13} + d_{03}\,d_{12})\,(d_{01}\,d_{23} + d_{02}\,d_{13} - d_{03}\,d_{12}) \\ & (d_{01}\,d_{23} - d_{02}\,d_{13} + d_{03}\,d_{12})\,(-d_{01}\,d_{23} + d_{02}\,d_{13} + d_{03}\,d_{12}) \ . \end{aligned} \tag{4}$$

This is equivalent to Ptolemy's theorem, which states that a convex quadrilateral with edge lengths a, b, c, d and diagonals e, f as in the picture, is circumscribed in a circle if and only if $a\,c + b\,d = e\,f$.



The key sources for the Cayley-Menger determinant are the classical books by L. Blumenthal [Blu53] and by M. Berger [Ber87].

This polynomial plays an important role in some problems of metric geometry. It was first applied by K. Menger in 1928, to characterize Euclidean spaces in metric terms alone [Blu53, Ch. IV]. It also appears in the metric characterization of Riemannian manifolds of constant sectional curvature obtained by M. Berger [Ber81].

Another important result based on the Cayley-Menger determinant is the proof of the invariance of the volume for flexible polyhedra in Euclidean 3-space (the "bellows" conjecture), see [Sab96, CSW97, Sab98]. There is also a huge literature about applications to the study of spatial shape of molecules (stereochemistry), see e.g. [KD80, EM99, DM00].

It is natural to ask whether Heron's formula (2) generalizes to higher dimensions, that is whether Γ_n splits as a product of linear forms. Note also that $\Gamma_1 = 2\, d_{01}^2$. The purpose of this paper is to prove that this is not possible for $n \geq 3$. Moreover, we show that for $n \geq 3$ the only factors of Γ_n in $\mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$ are the trivial ones, that is either a constant or a constant multiple of Γ_n. In other words Γ_n is *absolutely irreducible.*

Theorem 1.1. *The polynomial* Γ_n *is irreducible over* $\mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$ *for* $n \geq 3$.

In a similar way, one may wonder whether Δ_n splits as a product of simpler expressions, as in (4). Note that $\Delta_1 = -d_{01}^4$ and $\Delta_2 = 2\, d_{01}^2\, d_{02}^2\, d_{12}^2$. Again we can show that this is not possible for $n \geq 4$.

Theorem 1.2. *The polynomial* Δ_n *is irreducible over* $\mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$ *for* $n \geq 4$.

As a straightforward consequence of this, we find that the determinant of the general symmetric $n \times n$ matrix with zeros in the diagonal is an absolutely irreducible polynomial for $n \geq 4$, see Remark 1.7.

We can verify that Γ_3 is *twice* an integral polynomial and the same holds for Δ_4. This does not affect their irreducibility over $\mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$: 2 is trivial factor as it is a unit of $\mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$. Nevertheless it is interesting to determine how they split over $\mathbb{Z}[d_{ij} : 0 \leq i < j \leq n]$. Recall that the *content* of an integral polynomial is defined as the gcd of its coefficients.

Theorem 1.3. *Let* $n \in \mathbb{N}$, *then both* Γ_n *and* Δ_{n+1} *have content 1 for even* n *and 2 for odd* n.

Let us denote $\overline{\mathbb{Z}}$ the ring of algebraic integers, that is the ring formed by elements in the algebraic closure $\overline{\mathbb{Q}}$ satisfying a *monic* integral equation. It is well-known that an integral polynomial is irreducible over $\overline{\mathbb{Z}}[d_{ij} : 0 \leq i < j \leq n]$ if and only if it is irreducible over $\mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$ and has content 1. Set

$$I_n := \begin{cases} \Gamma_n & \text{for } n \text{ even} \\ \Gamma_n/2 & \text{for } n \text{ odd} \end{cases} , \qquad J_n := \begin{cases} \Delta_n/2 & \text{for } n \text{ even} \\ \Delta_n & \text{for } n \text{ odd} \end{cases} .$$

Hence Theorems 1.1, 1.2 and 1.3 can be equivalently rephrased as the fact that I_n and J_n are irreducible over $\overline{\mathbb{Z}}[d_{ij} : 0 \leq i < j \leq n]$ (and in particular over $\mathbb{Z}[d_{ij} : 0 \leq i < j \leq n]$) for $n \geq 3$ and for $n \geq 4$, respectively.

Let t_n be a new variable and set

$$\Lambda_{n,n-1} := \Gamma_n\big(d_{in} \mapsto t_n : 0 \leq i \leq n-1\big) \in \mathbb{Z}[d_{ij} : 0 \leq i < j \leq n-1][t_n] .$$

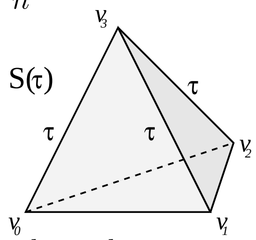
Up to a scalar factor, $\sqrt{\Lambda_{n,n-1}}$ is the formula for the volume of an isosceles simplex $S(\tau) \subset \mathbb{R}^n$ with base $B := \operatorname{Conv}(v_0, \ldots, v_{n-1})$ and vertex v_n equidistant at distance τ to the other vertices.

In [Ber87, Sec. 9.7.3.7] it is mentioned that

$$\Lambda_{n,n-1} = -2\,\Gamma_{n-1}\, t_n^2 - \Delta_{n-1} \; ; \tag{5}$$

this can be easily derived from the determinant defining $\Lambda_{n,n-1}$. The dominant term in this expression corresponds with the geometric intuition

$$\mathrm{Vol}_n(S(\tau)) \sim \frac{\tau}{n}\,\mathrm{Vol}_{n-1}(B) \qquad \text{for } \tau \to \infty \ .$$



Assuming $\dim(B) = n-1$, note that when $\tau = \rho(B)$ is the radius of the circle circumscribing B we have $\Lambda_{n,n-1} = 0$ and thus we recover (3).
More generally, let $1 \le p \le n$ and set

$$\Lambda_{n,p} := \begin{cases} \Gamma_n \ , & \text{if } p = n \\ \Gamma_n\big(d_{i\ell} \mapsto t_\ell \ : \ p+1 \le \ell \le n,\ 0 \le i \le \ell - 1\big) \ , & \text{if } p \le n-1 \ . \end{cases}$$

Here, $\{t_2, \dots, t_n\}$ denotes a further group of variables. If $p < n$ it turns out that $\Lambda_{n,p}$ is a homogeneous evaluation of Γ_n, and so $\Lambda_{n,p}$ is a homogeneous polynomial of degree $2\,n$, with respect to the whole set variables $\{d_{ij} \ : \ 0 \le i < j \le p\} \cup \{t_{p+1}, \dots, t_n\}$.
Let $B_p := \mathrm{Conv}(v_0, \dots, v_p)$ be a p-dimensional simplex with edge lengths $\{\delta_{ij} \ : 0 \le i < j \le p\}$ and $0 \ll \tau_{p+1} \ll \cdots \ll \tau_n$, meaning that τ_ℓ is sufficiently big with respect to $\tau_{p+1}, \dots, \tau_{\ell-1}$ for $\ell = p+1, \dots, n$. We set $S(\tau_{p+1}, \dots, \tau_n) \subset \mathbb{R}^n$ the n-dimensional simplex built from B_p by successively adjoining a vertex v_ℓ equidistant at distance τ_ℓ to the previous vertices $v_0, \dots, v_{\ell-1}$. Up to a scalar factor, $\sqrt{\Lambda_{n,p}}$ is the formula for the volume of $S(\tau_{p+1}, \dots, \tau_n)$. We have the recursive relation:

Lemma 1.4. $\Lambda_{n,p} = -2\,\Lambda_{n-1,p}\,t_n^2 - \Lambda_{n-2,p}\,t_{n-1}^4$ *for* $n \ge p+2$.

Proof. From the determinantal expression of Δ_n we get

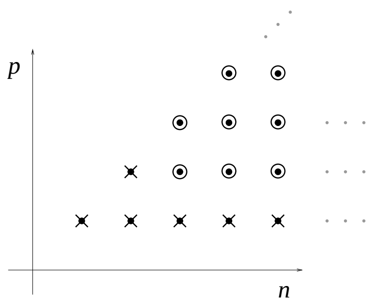
$$\Delta_{n-1}(d_{i\,(n-1)} \mapsto t_{n-1} \ : \ 0 \le i \le n-2) = t_{n-1}^4\,\Gamma_{n-2} \ , \tag{6}$$

and so by (5) we have $\Lambda_{n,n-2} = -2\,\Lambda_{n-1,n-2}\,t_n^2 - \Lambda_{n-2,n-2}\,t_{n-1}^4$ for $n \ge 2$. The general case follows by evaluating $d_{i\ell} \mapsto t_\ell$ for $p+1 \le \ell \le n-2$ and $0 \le i \le \ell-1$ in both sides of this identity. □

Theorem 1.1 is a particular case of the following:

Proposition 1.5. *The polynomial* $\Lambda_{n,p}$ *is irreducible over* $\mathbb{C}[d_{ij} : 0 \le i < j \le n]$ *if and only if* $n \ge 3$ *and* $2 \le p \le n$.

The following is a graphical visualization of this proposition. We encircle the integral points (n, p) such that $\Lambda_{n,p}$ is absolutely irreducible, and we mark with a cross the points where it is not. The behavior of Γ_n is read from the diagonal.



Proof. First we will prove by induction that $\Lambda_{n,2}$ is absolutely irreducible for $n \geq 3$. Let $n = 3$. Identity (5) and Heron's formula imply

$$\begin{aligned} \frac{\Lambda_{3,2}}{2} &= -\Gamma_2\, t_3^2 - \frac{\Delta_2}{2} \\ &= (d_{01} + d_{12} + d_{02})(-d_{01} + d_{12} + d_{02})(d_{01} - d_{12} + d_{02})(d_{01} + d_{12} - d_{02})\, t_3^2 \\ &\quad -d_{01}^2\, d_{12}^2\, d_{02}^2 \ . \end{aligned} \tag{7}$$

The polynomials $f := (d_{01}+d_{12}+d_{02})(-d_{01}+d_{12}+d_{02})(d_{01}-d_{12}+d_{02})(d_{01}+d_{12}-d_{02})$ and $g := -d_{01}^2\, d_{12}^2\, d_{02}^2$ have no common non constant factor and so a (non trivial) factorization of $\Lambda_{3,2}$ should be of the form

$$\Lambda_{3,2} = (\alpha\, t_3 + \beta)(\gamma\, t_3 + \delta)$$

with $\alpha\,\gamma = 2\,f$, $\beta\,\delta = 2\,g$ and $\alpha\,\delta + \beta\,\gamma = 0$. But this is impossible since that pairwise, $\alpha, \gamma, \beta \cdot \delta$ have no common (non constant) factor over $\mathbb{C}[d_{01}, d_{02}, d_{12}]$; we conclude that $\Lambda_{3,2}$ is irreducible.

Now let $n \geq 4$ and assume that $\Lambda_{n-1,2}$ is irreducible. By Lemma 1.4

$$\Lambda_{n,2} = -2\,\Lambda_{n-1,2}\, t_n^2 - \Lambda_{n-2,2}\, t_{n-1}^4 \ .$$

The polynomials $\Lambda_{n-1,2}$ and $\Lambda_{n-2,2}\, t_{n-1}^4$ are coprime, since $\Lambda_{n-1,2}$ is irreducible of degree $2\,n-2$ and $\deg(\Lambda_{n-2,2}) = 2\,n-4$. As before, this implies that any non trivial factorization of $\Lambda_{n,2}$ should be of the form

$$\Lambda_{n,2} = (-2\,\Lambda_{n-1,2}\, t_n + \beta)(t_n + \delta)$$

with $\beta, \delta \in \mathbb{C}[d_{ij} \ : \ 0 \leq i < j \leq 2][t_3, \ldots, t_{n-1}]$ such that $\beta\,\delta = -\Lambda_{n-2,2}\, t_{n-1}^4$ and $-2\,\Lambda_{n-1,2}\,\delta + \beta = 0$. But this is impossible because $\Lambda_{n-1,2}$ and $\Lambda_{n-2,2}\, t_{n-1}^4$ are coprime. We conclude that $\Lambda_{n,2}$ is irreducible.

Now let $n \geq 3$ and $3 \leq p \leq n$. Suppose that we can write $\Lambda_{n,p} = F \cdot G$ with $F, G \in \mathbb{C}[d_{ij} \ : \ 0 \leq i < j \leq p][t_{p+1}, \ldots, t_n]$ homogeneous of degree ≥ 1.
The evaluation map $d_{i\ell} \mapsto t_\ell$ $(p+1 \leq \ell \leq n, 0 \leq i \leq \ell - 1)$ is homogeneous and so

$$\begin{aligned} F' &:= F\big(d_{i\ell} \mapsto t_\ell \ : \ p+1 \leq \ell \leq n, \ 0 \leq i \leq \ell - 1\big) \ , \\ G' &:= G\big(d_{i\ell} \mapsto t_\ell \ : \ p+1 \leq \ell \leq n, \ 0 \leq i \leq \ell - 1\big) \end{aligned}$$

are also homogeneous polynomials of degree ≥ 1, which would give a non trivial factorization of $\Lambda_{n,2}$. This shows that $\Lambda_{n,p}$ is also irreducible.

To conclude, we have to verify that $d_{01} | \Lambda_{n,1}$ for all n, which follows by checking that $\Lambda_{n,1}(d_{01} \mapsto 0) = 0$, due to the fact that the second and third rows in the matrix defining $\Lambda_{n,1}(d_{01} \mapsto 0)$ coincide. The remaining case $n = p = 2$ corresponds to Heron's formula. □

Proof of Theorem 1.2. Set

$$\Delta_n' := \Delta_n(d_{in} \mapsto 1 \ : \ 1 \leq i \leq n-1) \in \mathbb{Z}[d_{ij} \ : \ 0 \leq i < j \leq n-1] \ .$$

From the determinantal expression of Δ_n we get

$$\Delta_n' = d_{0n}^4\, \Gamma_{n-1}\Big(\frac{d_{01}}{d_{0n}}, \ldots, \frac{d_{0\,(n-1)}}{d_{0n}}, \ d_{12}, d_{13}, \ldots, d_{(n-2)\,(n-1)}\Big) \ . \tag{8}$$

Note that the partial degree of Γ_{n-1} in the group of variables

$$\{d_{0i} \ : \ 1 \leq i \leq n-1\} \tag{9}$$

is four. Hence Δ_n' is the homogenization of Γ_{n-1} with respect to these variables, with d_{0n} as the homogenization variable. This follows again from the same determinantal expression.
Now let $F, G \in \mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$ such that $\Delta_n' = F \cdot G$. Since Δ_n' is homogeneous with respect to the variables (9), we have that F and G are also homogeneous with respect to this group. Now we dehomogenize this identity by setting $d_{0n} \mapsto 1$ and we find

$$\Gamma_{n-1} = F(d_{0n} \mapsto 1) \cdot G(d_{0n} \mapsto 1).$$

By Theorem 1.1, Γ_{n-1} is irreducible for $n \geq 4$, which implies that either $F(d_{0n} \mapsto 1) \in \mathbb{C}$ or $G(d_{0n} \mapsto 1) \in \mathbb{C}$. This can only hold if F or G is a monomial in d_{0n}, but this is impossible since d_{0n} is the homogenization variable. We conclude that Δ_n' is irreducible.

Now suppose that Δ_n *can* be factorized, and let $P, Q \in \mathbb{C}[d_{ij} : 0 \leq i < j \leq n]$ be homogeneous polynomials of degree ≥ 1 such that $\Delta_n = P \cdot Q$. This implies that $\Delta_n' = P' \cdot Q'$ with

$$P' := P(d_{in} \mapsto 1 \, : \, 1 \leq i \leq n-1) \quad , \qquad Q' := Q(d_{in} \mapsto 1 \, : \, 1 \leq i \leq n-1) \ .$$

Note that $\deg(\Delta_n') = \deg(\Gamma_{n-1}) + 4 = 2\,n + 2$ and so $\deg(\Delta_n') = \deg(\Delta_n)$. This implies that both $\deg(P') = \deg(P) \geq 1$ and $\deg(Q') = \deg(Q) \geq 1$, which contradicts the irreducibility of Δ_n'. Hence Δ_n is irreducible. □

For the proof of Theorem 1.3 we need an auxiliary result. Let $n \in \mathbb{N}$ and $\{x_{ij} : 1 \leq \ i < j \leq n\}$ be a set of $(n-1)\,n/2$ variables. Then set

$$X_n := \begin{bmatrix} 0 & x_{12} & x_{13} & \dots & x_{1n} \\ x_{12} & 0 & x_{23} & \dots & x_{2n} \\ x_{13} & x_{23} & 0 & \dots & x_{3n} \\ \vdots & & & \ddots & \\ x_{1n} & x_{2n} & x_{3n} & \dots & 0 \end{bmatrix}$$

for the general symmetric matrix of order n with zeros in the diagonal.

Lemma 1.6. *For odd values of n, the content of* $\det(X_n)$ *is divisible by 2.*

Proof. Set

$$A_n := \begin{bmatrix} 0 & x_{12} & x_{13} & \dots & x_{1n} \\ -x_{12} & 0 & x_{23} & \dots & x_{2n} \\ -x_{13} & -x_{23} & 0 & \dots & x_{3n} \\ \vdots & & & \ddots & \\ -x_{1n} & -x_{2n} & -x_{3n} & \dots & 0 \end{bmatrix}$$

for the general *antisymmetric* matrix of order n. Then

$$\det(A_n) = \det(A_n^t) = (-1)^n \det(A_n) \in \mathbb{Z}[x_{ij} :: 1 \leq i < j \leq n] \ ,$$

which implies $\det(A_n) = 0$ because n is odd; here A_n^t denotes the transpose of A_n. On the other hand $X_n \equiv A_n \pmod 2$ and so we conclude

$$\det(X_n) \equiv \det(A_n) = 0 \pmod 2 \ .$$

□

Proof of Theorem 1.3. Let $c(n) \in \mathbb{N}$ be the content of Γ_n. Lemma 1.6 shows that $2|c(n)$ for odd n, as the Cayley-Menger matrix CM_n is symmetric of order $n+2$ with zeros in the diagonal. By Lemma 1.4

$$\Lambda_{n,n-2}(t_n \mapsto 0) = -\Gamma_{n-2}\, t_{n-1}^4 \ .$$

By definition $\Lambda_{n,n-2}(t_n \mapsto 0)$ is an integral evaluation of Γ_n and so $c(n)$ divides its content, that is $c(n)|c(n-2)$. We conclude by induction, checking the statement directly for $n = 1$ and $n = 2$.

Now let $c'(n) \in \mathbb{N}$ be the content of Δ_n. Lemma 1.6 shows that $2|c'(n)$ for even n, as the matrix in the definition of Δ_n is symmetric of order $n+1$ with zeros in the diagonal. Identity (8) implies that $c'(n)|c(n-1)$, that is $c'(n) = 1$ for n odd and $c'(n)|2$ for n even; we conclude that $c'(n) = 2$ in this case. □

Remark 1.7. *Set*

$$K_n := \begin{cases} \det(X_n) & \text{for } n \text{ even} \ , \\ \det(X_n)/2 & \text{for } n \text{ odd} \ . \end{cases}$$

As a byproduct of Theorems 1.2 and 1.3, we find that K_n is an irreducible polynomial over $\overline{\mathbb{Z}}[x_{ij} \,:\, 1 \leq i < j \leq n]$ for $n \geq 5$; a direct verification shows that this is also true for $n = 4$.

Acknowledgments. It is a pleasure to thank Victor Alexandrov from the Sobolev Institute of Mathematics at Novosibirsk, who posed us this problem and constantly encouraged us to solve it. We also thank him for pointers to the literature on the Cayley-Menger determinant. We are also grateful to Matías Graña for suggesting us a simplified proof of Lemma 1.6.

Part of this paper was written while Martín Sombra was visiting the University of California at Berkeley, supported by the Miller Institute for Basic Research in Science.

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