Bounds for the Hilbert Function of Polynomial Ideals and for the Degrees in the Nullstellensatz

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Abstract. We present a new effective Nullstellensatz with bounds for the degrees which depend not only on the number of variables and on the degrees of the input polynomials but also on an additional parameter called the *geometric degree of the system of equations.* The obtained bound is polynomial in these parameters. It is essentially optimal in the general case, and it substantially improves the existent bounds in some special cases.

The proof of this result is combinatorial, and it relies on global estimations for the Hilbert function of homogeneous polynomial ideals.

In this direction, we obtain a lower bound for the Hilbert function of an arbitrary homogeneous polynomial ideal, and an upper bound for the Hilbert function of a generic hypersurface section of an unmixed radical polynomial ideal.

Introduction

Let k be a field with an algebraic closure denoted by $\bar{k}$, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials which have no common zero in $\bar{k}^n$. Classical Hilbert's Nullstellensatz ensures then that there exist polynomials $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that

$$1 = a_1 f_1 + \ldots + a_s f_s$$

An *effective Nullstellensatz* amounts to estimate the degrees of the polynomials $a_1, \ldots, a_s$ in one such a representation. An explicit bound for the degrees reduces the problem of effectively finding the polynomials $a_1, \ldots, a_s$ to the solving of a system of linear equations.

The effective Nullstellensatz has been the object of much research during the last ten years because of both its theoretical and practical interest. The most precise bound obtained up to now for this problem in terms of the number of variables n and the maximum degree d of the polynomials $f_1, \ldots, f_s$ is

$$\deg a_i \leq \max\{3, d\}^n \qquad 1 \leq i \leq s$$

This bound is due to J. Kollar [22], and it is essentially optimal for $d \geq 3$; in the case when $d = 2$ a sharper estimate can be given (see [30]).

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Related results can be found in the research papers [7], [8], [12], [31], [29], [4], [2], [23], also there are extensive discussions and bibliography about the effective Nullstellensatz in the surveys [33], [3].

Because of its exponential nature, this bound is hopeless for most practical applications. This behavior is in general unavoidable for polynomial elimination problems when only the number of variables and the degrees of the input polynomials are considered.

However, it has been observed that there are many particular cases in which this bound can be notably improved. This fact has motivated the introduction of new parameters which enable to differenciate special families of systems of polynomial equations whose behavior for the problem in question is, say, polynomial instead of exponential (see [15], [14]).

In this spirit, we consider an additional parameter associated to the input polynomials $f_1, \ldots, f_s$, called the *geometric degree of the system of equations*, which is defined as follows.

Let k be a zero characteristic field and $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ polynomials such that $1 \in (f_1, \ldots, f_s)$. Then there exist $t \leq s$ and $g_1, \ldots, g_t$ $\bar{k}$–linear combinations of $f_1, \ldots, f_s$ such that $1 \in (g_1, \ldots, g_t)$, $g_1, \ldots, g_{t-1}$ is a regular sequence, and $(g_1, \ldots, g_{t-1}) \subseteq k[x_1, \ldots, x_n]$ is a radical ideal for $1 \leq i \leq t-1$. Let $V_i \subseteq \mathbb{A}^n(\bar{k})$ be the affine variety defined by $g_1, \ldots, g_i$ for $1 \leq i \leq s$, and set

$$\delta_{g_1,\ldots,g_s} := \max_{1 \leq i \leq \min\{t,n\}-1} \deg V_i$$

where $\deg V_i$ stands for the degree of the affine variety V_i. Then the geometric degree of the system of equations $\delta(f_1, \ldots, f_s)$ is defined as the minimum of the $\delta_{g_1,\ldots,g_s}$ through all linear combinations of $f_1, \ldots, f_s$ satisfying the stated conditions.

In the case when k is a field of positive characteristic, the degree of the system of equations $f_1, \ldots, f_s$ is defined in an analogous way by considering $\bar{k}$–linear combinations of the polynomials $\{f_i, x_j f_i : 1 \leq i \leq s, 1 \leq j \leq n\}$.

In both cases, the existence of $g_1, \ldots, g_s$ satisfying these properties is a consequence of Bertini's theorem.

We obtain (Theorem 4.40):

Theorem. *Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials such that $1 \in (f_1, \ldots, f_s)$. Let $d := \max_{1 \leq i \leq s} \deg f_i$, and let δ be the geometric degree of the system of equations $f_1, \ldots, f_s$. Then there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that*

$$1 = a_1 f_1 + \ldots + a_s f_s$$

with $\deg a_i f_i \leq \min\{n, s\}^2 \, (d + 3n) \, \delta$ *for $i = 1, \ldots, s$.*

We also obtain a similar bound for the representation problem in complete intersections (Theorem 4.39).

Let $d := \max_{1 \leq i \leq s} \deg f_i$. Then we have

$$\delta(f_1, \ldots, f_s) \leq (d+1)^{\min\{s,n\}-1}$$

and so our bounds for the effective Nullstellensatz and for the representation problem in complete intersections are essentially sharp in the general case. We remark however that they can substantially improve the usual estimates in some special cases (see Example 4.42).

Similar bounds for the effective Nullstellensatz have also been recently obtained by algorithmic tools [15, Th. 19], [14, §4.2] and by duality methods [24].

The proof of these theorems are combinatorial, and rely on *global* estimations for the Hilbert function of certain polynomials ideals.

The study of the global behavior of the Hilbert function of homogeneous ideals is of independent interest. It is related to several questions of effective commutative algebra, mainly in connexion with the construction of regular sequences of maximal length with polynomials of controlled degree lying in a given ideal [10, §2], and to trascendental number theory, in the context of the so–called zero lemmas [5].

Let $I \subseteq k[x_0, \dots, x_n]$ be an homogeneous ideal. We understand for the dimension of I the dimension of the projective variety that it defines, and we denote by h_I its Hilbert function.

The problem of estimating h_I was first considered by Y. Nesterenko [28], who proved that for a zero characteristic field k and an homogeneous prime ideal $P \subseteq k[x_0, \dots, x_n]$ of dimension $d \geq 0$ it holds

$$\tbinom{m+d+1}{d+1} - \tbinom{m-\deg P+d+1}{d+1} \leq h_P(m) \leq \deg P\ (4\,m)^d \qquad m \geq 1$$

Later on, M. Chardin [10] improved Nesterenko's upper bound by simplifying his proof, and obtained that for a perfect field k and an homogeneous unmixed radical ideal $I \subseteq k[x_0, \dots, x_n]$ of dimension $d \geq 0$ it holds

$$h_I(m) \leq \deg I\ \tbinom{m+d}{d} \qquad m \geq 1$$

This estimation has also been obtained by J. Kollar (see [10, Note]) by cohomological arguments.

In this direction, we obtain a lower bound for the Hilbert function of an arbitrary homogeneous polynomial ideal of dimension $d \geq 0$ (Theorem 2.4). We have:

$$h_I(m) \geq \tbinom{m+d+1}{d+1} - \tbinom{m-\deg I+d+1}{d+1} \qquad m \geq 1$$

This result generalizes the bound of Y. Nesterenko [28] for the case of an homogeneous prime ideal $P \subseteq k[x_0, \dots, x_n]$. It is optimal in terms of the dimension and the degree of the ideal I.

We present also an upper bound for the Hilbert function of a generic hypersurface section f of an homogeneous unmixed radical ideal $I \subseteq k[x_0, \dots, x_n]$ of dimension $d \geq 1$ (Theorem 2.23). We have:

$$h_{(I,f)}(m) \leq 3\ \deg f\ \deg I\, \tbinom{m+d-1}{d-1}$$

for $m \geq 5\,d\ \deg I$.

Our approach to the Hilbert function is elementary, and yields a new point of view into the subject which is clearer than that of the previous works. We hope that our techniques would also be useful for treating arithmetic Hilbert functions (see [28]).

We shall briefly sketch the relationship between these bounds for the Hilbert function, and the effective Nullstellensatz and the representation problem in complete intersections.

Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be a regular sequence. There are several effectivity questions about this set of polynomials which can be easily solved in the case when the homogenization of these polynomials $\tilde{f}_1, \ldots, \tilde{f}_s \in k[x_0, \ldots, x_n]$ is again a regular sequence. An example of this situation is the effective Nullstellensatz, for which there exists a simple and well–known proof in this conditions (see for instance [26, §1, Th. 1, Cor.]).

The central point in our proof of the effective Nullstellensatz consists then in showing that the regular sequence $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ can in fact be replaced by polynomials $p_1, \ldots, p_s \in k[x_1, \ldots, x_n]$ of controlled degrees such that $(f_1, \ldots, f_i) = (p_1, \ldots, p_i)$ for $1 \leq i \leq s$, and such that the homogenizated polynomials $\tilde{p}_1, \ldots, \tilde{p}_s$ define a regular sequence in $k[x_0, \ldots, x_n]$. The proof of this result proceeds by induction, and the bounds for the Hilbert function allows us to controlle at each step $1 \leq i \leq s$ the degree of the polynomial p_i.

The spirit of our proof follows T. Dubé's paper on the effective Nullstellensatz [11]. We remark here that there are many errors in Dubé's argument, and a serious gap, for it relies on an assumption on the Hilbert function of certain class of homogeneous polynomial ideals [11, §2.1] which is unproved in his paper and which is neither in the literature, as it was noted by M. Almeida [1, §3.1], and thus this proof should be considered incomplete as it stands.

Our approach allows us not only to avoid Dubé's assumption and prove the results stated in his paper, but also to obtain our more refined bounds.

Finally, we want to remark that our exposition is elementary and essentially self–contained.

The exposition is divided in four parts. In the first we state some well–known features of degree of projective varieties and Hilbert function that will be needed in the subsequent parts, and prove some of them when suitable reference is lacking. In the second part we prove the lower and upper bounds for the Hilbert function and analize the extremal cases. In the third part, we apply the obtained results to the construction of regular sequences. In the fourth part we consider the consequences for the effective Nullstellensatz and for the representation problem in complete intersections.

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§ 0. Notations and Conventions

We work over an arbitrary field k with algebraic closure $\bar{k}$. As usual, $\mathbb{P}^n$ and $\mathbb{A}^n$ denote the projective space and the affine space of dimension n over $\bar{k}$. A variety is not necessarily irreducible.

The ring $k[x_0, \ldots, x_n]$ will be denoted alternatively by R or R_k.

Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal. We understand for the dimension of I the dimension of the projective variety that it defines and we shall denote it by $\dim I$, so that $\dim I = \dim_{\text{krull}} I - 1$.

Let $J \subseteq k[x_1, \ldots, x_n]$ be an affine ideal. We shall understand for the dimension of J its Krull dimension. In each appereance, it will be clear from the context to which notion we are refering to.

An ideal $I \subseteq k[x_0, \ldots, x_n]$ is *unmixed* if its associated prime ideals have all the same dimension. In particular, I has not imbedded associated primes and its primary decomposition is unique.

Given an ideal $I \subseteq k[x_0, \ldots, x_n]$, then $I^e := \bar{k} \otimes_k I \subseteq \bar{k}[x_0, \ldots, x_n]$ is the extended ideal of I in $\bar{k}[x_0, \ldots, x_n]$.

Given $I \subseteq R_k$ an homogeneous ideal,

$$V(I) := \{x \in \mathbb{P}^n : f(x) = 0 \ \ \forall f \in I\} \subseteq \mathbb{P}^n$$

is the projective variety defined by I. Conversely, given $V \subseteq \mathbb{P}^n$ a variety,

$$I_k(V) := \{f \in R_k : f|_V \equiv 0\} \subseteq R_k$$

and $I(V) := I_{\bar{k}}(V) \subseteq \bar{k}[x_0, \ldots, x_n]$ is the defining ideal of V.

Given a graded R–module M and $m \in \mathbb{Z}$, M_m is the homogeneous part of degree m.

Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal. The *Hilbert function* or *characteristic function* of the ideal I is

$$\begin{aligned} h_I : \ & \mathbb{Z} \to \mathbb{Z} \\ & m \mapsto \dim_k(k[x_0, \ldots, x_n]/I)_m \end{aligned}$$

Given $V \subseteq \mathbb{P}^n$ a variety, then h_V is the Hilbert function of $I(V)$.

Given $f \in k[x_0, \ldots, x_n]$ an homogeneous polynomial, $f^a \in k[x_1, \ldots, x_n]$ is its affinization and given $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous ideal, $I^a \subseteq k[x_1, \ldots, x_n]$ is its affinization.

Conversely, given an affine polynomial $g \in k[x_1, \dots, x_n]$, $\tilde{g} \in k[x_0, \dots, x_n]$ is its homogenization, and given $J \subseteq k[x_1, \dots, x_n]$ an affine ideal, we denote by $\tilde{J} \subseteq k[x_0, \dots, x_n]$ its homogenization.

§ 1. Preliminaries on Degree and Hilbert Function

In this section we state some well–known properties concerning the degree of a variety and the Hilbert function of an homogeneous polynomial ideal which will be needed in the sequel. Also we shall prove some of them when suitable reference is lacking.

Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety of dimension d. The *degree* of V is defined as

$$\deg V := \sup \begin{array}{l} \{ \#(V \cap H_1 \cap \ldots \cap H_d) : H_1, \dots, H_d \subseteq \mathbb{P}^n \text{ hyperplanes} \\ \text{and } \dim(V \cap H_1 \cap \ldots \cap H_d) = 0 \} \end{array}$$

This number is finite, and it realizes generically, if we think the set

$$\{(H_1, \dots, H_d) : H_1, \dots, H_d \subseteq \mathbb{P}^n \text{ hyperplanes } \}$$

as parametrized by a non empty set of $\mathbb{A}^{(n+1)d}$ [17, Lecture 18]. We agree that $\deg \emptyset = 1$.

The notion of degree can be extended to possible reducible projective varieties following [19]. Let $V \subseteq \mathbb{P}^n$, and let $V = \cup C$ be the minimal decomposition of V in irreducible varieties. Then the *(geometric) degree* of V is defined as

$$\deg V := \sum_C \deg C$$

For this notion of degree it holds the following Bézout's inequality without multiplicities for the degree of the intersection of two varieties. Let $V, W \subseteq \mathbb{P}^n$ be varieties. Then we have

$$\deg(V \cap W) \leq \deg V \deg W$$

This is a consequence of Bézout's inequality for affine varieties [19, Th. 1]. The details can be found in [9, Prop. 1.11]. This result can also be deduced from the Bézout's theorems [13, Th. 12.3], [34, Th. 2.1].

Also the notion of degree and the Bézout's theorem traslate to the affine context.

We turn our attention to the Hilbert function of an homogeneous ideal. Let $I \subseteq k[x_0, \dots, x_n]$ be an homogeneous ideal of dimension d. There exists a polynomial $p_I \in \mathbb{Q}[t]$ of degree d, and $m_0 \in \mathbb{Z}$ such that

$$h_I(m) = p_I(m)$$

for $m \geq m_0$. The polynomial p_I is called the *Hilbert polynomial* of the ideal I.

The degree of an homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$ can be defined through its Hilbert polynomial.

Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal of dimension d, with $d \geq 0$. Let $p_I = a_d\ t^d + \ldots + a_0 \in \mathbb{Q}[t]$ be its Hilbert polynomial. Then the *(algebraic) degree* of the ideal I is defined as

$$\deg I := d!\ a_d \quad \in \mathbb{N}$$

If $I \subseteq k[x_0, \ldots, x_n]$ is an homogeneous ideal of dimension -1, then I is a $(x_0, \ldots, x_n)$–primary ideal, and the *degree* of I is defined as the length of the k–module $k[x_0, \ldots, x_n]/I$, which equals its dimension as a k–linear space. We also agree that deg $k[x_0, \ldots, x_n] = 0$.

Given $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous ideal, we denote by $\operatorname{irr} I$ the number of irreducible components of $V(I) \subseteq \mathbb{P}^n$.

Let $I,\ J \subseteq k[x_0, \ldots, x_n]$ be homogeneous ideals. Then we have the following exact sequence of graded k–algebras

$$\begin{array}{ccccc} 0 \to R/I \cap J \to & R/I \oplus R/J & \to & R/I + J & \to 0 \\ & (f, g) & \longmapsto & f - g & \end{array}$$

from where we get

$$h_{I \cap J}(m) = h_I(m) + h_J(m) - h_{I+J}(m) \qquad\qquad m \geq 1$$

In particular, if $\dim I > \dim J$, then $\deg I \cap J = \deg I$.

Let k be a perfect field, $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous radical ideal, and let $I = \cap_P P$ be the minimal primary decomposition of I. In this situation we have

$$\deg I = \sum_{P:\ \dim P = \dim I} \deg V(P)$$

(see [34, Prop. 1.49] [17, Prop. 13.6]), and thus the degree of the ideal I may be calculated from the degrees of the varieties defined by its associated prime ideals of maximal dimension.

Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous radical ideal, and $I = \cap_P P$ the minimal primary decomposition of I. From the canonical inclusion of graded modules

$$R_k/I \hookrightarrow \bigoplus_P R_k/P$$

we deduce that

$$h_I(m) \leq \sum_P h_P(m) \qquad\qquad m \geq 1$$

Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal, and $I^e \subseteq \bar{k}[x_0, \ldots, x_n]$ be the extended ideal. Let

$$k[x_0, \ldots, x_n]/I = \bigoplus_m (k[x_0, \ldots, x_n]/I)_m$$

be the decomposition of $k[x_0,\ldots,x_n]/I$ into homogeneous parts. Then

$$(\bar{k}[x_0,\ldots,x_n]/I^e)_m = \bar{k}\otimes_k (k[x_0,\ldots,x_n]/I)_m \qquad m\in\mathbb{Z}$$

and so $h_{I^e}(m) = h_I(m)$, i.e. the Hilbert function is invariant under change of the base field. In particular

$$\deg I^e = \deg I$$

We have also that there exist $y_0,\ldots,y_d\in\bar{k}[x_0,\ldots,x_n]$ algebraically independent linear forms such that

$$\bar{k}[y_0,\ldots,y_d]\hookrightarrow \bar{k}[x_0,\ldots,x_n]/I^e$$

is an inclusion of $\bar{k}$–algebras, and so

$$h_I(m) = h_{I^e}(m) \geq \dim_{\bar{k}}(\bar{k}[y_0,\ldots,y_d])_m = \tbinom{m+d}{d}$$

We shall need the following identity for the combinatorial numbers.

Lemma 1.1 *Let $d\geq 0$, $D\geq 1$, $m\in\mathbb{Z}$. Then*

$$\tbinom{m+d+1+D}{d+1} - \tbinom{m+d+1}{d+1} = \sum_{i=1}^{D}\tbinom{m+d+i}{d}$$

Proof. The case $D=1$ is easy. In the case when $D>1$, we have

$$\tbinom{m+d+1+D}{d+1} - \tbinom{m+d+1}{d+1} = \sum_{i=1}^{D}\{\tbinom{m+d+i+1}{d+1} - \tbinom{m+d+i}{i}\} = \sum_{i=1}^{D}\tbinom{m+d+i}{d}$$

□

We shall make appeal also to Macaulay's characterization of the Hilbert function of an homogeneous polinomial ideal.

Given positive integers i, c, the *i–binomial expansion* of r is the unique expression

$$c = \tbinom{c(i)}{i} + \ldots + \tbinom{c(j)}{j}$$

with $c(i) > \ldots > c(j) \geq j \geq 1$.

Let $c = \binom{c(i)}{i} + \ldots + \binom{c(j)}{j}$ be the i–binomial expansion of c. Then we set

$$c^{\langle i\rangle} := \tbinom{c(i)+1}{i+1} + \ldots + \tbinom{c(j)+1}{j+1}$$

We note that this expression is the $i+1$–binomial expansion of $c^{\langle i\rangle}$.

Remark 1.2 Let b, c, $i\in\mathbb{Z}_{>0}$. Then it is easily seen that $b\geq c$ if and only if $(b(i),\ldots,b(j))$ is greater or equal that $(c(i),\ldots,c(j))$ in the lexicographic order, and thus and thus $b\geq c$ if and only if $b^{\langle i\rangle}\geq c^{\langle i\rangle}$.

We recall that a sequence of nonnegative integers $(c_i)_{i\in\mathbb{Z}_{\geq 0}}$ is called an *O–sequence* if

$$c_0 = 1 \qquad\qquad c_{i+1} \leq c_i^{\langle i\rangle} \qquad\qquad i \geq 1$$

We have then

Theorem*(Macaulay, [16])***.** *Let $h : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. Then h is the Hilbert function of an homogeneous polynomial ideal if and only if*

$$(h(i))_{i\in\mathbb{Z}_{\geq 0}}$$

is an O–sequence.

□

§ 2. Bounds for the Hilbert Function

In this section we shall derive both lower and upper bounds for the Hilbert function of homogeneous polynomial ideals. These estimations depend on the dimension and on the degree of the ideal in question, and eventually on its length.

We derive first a lower bound for the Hilbert function of an arbitrary homogeneous polynomial ideal.

We shall consider separately the case when $\dim I = 0$.

Lemma 2.3 *Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous unmixed ideal of dimension 0. Then*

$$\begin{array}{rclr} h_I(m) & \geq & m+1 & \qquad \deg I - 2 \geq m \geq 0 \\ h_I(m) & = & \deg\ I & \qquad m \geq \deg I - 1 \end{array}$$

Proof. We have that $I^e \subseteq \bar{k}[x_0, \ldots, x_n]$ is an unmixed ideal of dimension 0 [35, Ch. VII, §31, Th. 36, Cor.1]. As $\bar{k}$ is an infinite field, there exist a linear form $u \in \bar{k}[x_0, \ldots, x_n]$ which is a non zero divisor modulo I^e. Then

$$h_I(m) - h_I(m-1) = h_{I^e}(m) - h_{I^e}(m-1) = h_{(I^e,u)}(m)$$

Let m_0 be minimum such that

$$h_{I^e}(m) = \deg\ I^e = \deg\ I$$

for $m \geq m_0$. Then $h_{(I^e,u)}(m) \geq 1$ for $0 \leq m \leq m_0 - 1$ and $h_{(I^e,u)}(m) = 0$ for $m \geq m_0$, and thus we have

$$h_I(m) = h_{I^e}(m) \geq m+1 \qquad\qquad \deg I - 2 \geq m \geq 0$$

and also $h_I(m) = h_{I^e}(m) = \deg\ I$ for $m \geq \deg I - 1$. □

Theorem 2.4 *Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal of dimension d, with $d \geq 0$. Then*

$$h_I(m) \geq \tbinom{m+d+1}{d+1} - \tbinom{m-\deg I+d+1}{d+1} \qquad m \geq 1$$

Proof. Let $I^e = \bigcap_P Q_P$ be a minimal primary decomposition of I^e, and let

$$I^* = \cap_{\dim P = \dim I^e} Q_P$$

be the intersection of the primary components of I^e of maximal dimension, which is an unmixed ideal of dimension d. Then $h_I(m) = h_{I^e}(m) \geq h_{I^*}(m)$ for $m \geq 1$, and we have that

$$\deg I = \deg I^*$$

We shall proceed by induction on d. Consider first $d = 0$. We have then

$$h_I(m) = h_{I^e}(m) \geq h_{I^*}(m) \geq \tbinom{m+1}{1} - \tbinom{m-\deg I+1}{1} \qquad m \geq 1$$

by Lemma 2.3 applied to I^*.

Now let $d \geq 1$. Let $u \in \bar{k}[x_0, \ldots, x_n]$ be a linear form which is a non zero divisor modulo I^*. Then we have

$$h_{I^*}(m) - h_{I^*}(m-1) = h_{(I^*,u)}(m)$$

Then $\dim(I, u) = d - 1$ and deg $(I^*, u) =$ deg $I^* =$ deg I. By the inductive hypothesis we have that

$$h_{I^*}(m) - h_{I^*}(m-1) = h_{(I^*,u)}(m) \geq \tbinom{m+d}{d} - \tbinom{m-\deg I+d}{d} \qquad m \geq 1$$

Then

$$\begin{aligned} h_I(m) &\geq h_{I^*}(m) = \textstyle\sum_{j=0}^m h_{(I^*,u)}(j) \geq \\ &\geq \textstyle\sum_{j=0}^m \{\tbinom{j+d}{d} - \tbinom{j-\deg I+d}{d}\} = \tbinom{m+d+1}{d+1} - \tbinom{m-\deg I+d+1}{d+1} \qquad m \geq 1 \end{aligned}$$

by Lemma 1.1. □

This inequality extends Nesterenko's estimate for the case of a prime ideal [28, §6, Prop. 1] to the case of an arbitrary ideal.

Remark 2.5 By Gotzmann's persistence theorem [16] we have that for an homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$ of dimension d there exists $m_0 \in \mathbb{Z}$ such that

$$h_I(m) \geq \tbinom{m+d+1}{d+1} - \tbinom{m-\deg I+d+1}{d+1} \qquad m \geq m_0$$

as it is noted in [6, Rem. 0.6]. Our theorem shows that this inequality holds globally, not only for big values of m.

Given $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous ideal of dimension $d \geq 0$, let

$$H_I(t) := \sum_{m=0}^{\infty} h_I(m)\, t^m$$

be its Hilbert–Poincaré series. Then the previous result states that

$$H_I(t) \geq \frac{1 - t^{\deg I}}{(1-t)^{d+2}}$$

in the sense that the inequality holds for each term of the power series.

This estimate is optimal in terms of the dimension and the degree of the ideal I. The extremal cases correspond to hypersurfaces of linear subspaces of $\mathbb{P}^n$. This can be deduced from [6, Cor. 2.8], which in turn depends on Gotzmann's theorem. We give here a self–contained proof of this fact.

Proposition 2.6 *Let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal. Then*

$$h_I(m) = \binom{m+d+1}{d+1} - \binom{m-\deg I+d+1}{d+1} \qquad m \geq 1$$

if and only if there exist $u_1, \ldots, u_{n-d-1} \in k[x_0, \ldots, x_n]$ linearly independent linear forms and $f \notin (u_1, \ldots, u_{n-d-1})$ such that $I = (u_1, \ldots, u_{n-d-1}, f)$.

Proof. Let $u_1, \ldots, u_{n-d-1} \in k[x_0, \ldots, x_n]$ be linearly independent linear forms and $f \notin (u_1, \ldots, u_{n-d-1})$. Let $I := (u_1, \ldots, u_{n-d-1}, f)$. Then f is a non zero divisor modulo $(u_1, \ldots, u_{n-d-1})$, and so

$$\begin{aligned} h_I(m) &= h_{(u_1,\ldots,u_{n-d-1})}(m) - h_{(u_1,\ldots,u_{n-d-1})}(m - \deg f) = \\ &= \binom{m+d+1}{d+1} - \binom{m-\deg f+d+1}{d+1} = \binom{m+d+1}{d+1} - \binom{m-\deg I+d+1}{d+1} \end{aligned}$$

Conversely, let $I \subseteq k[x_0, \ldots, x_n]$ be an homogeneous ideal such that

$$h_I(m) = \binom{m+d+1}{d+1} - \binom{m-\deg I+d+1}{d+1} \qquad m \geq 1$$

Then $h_I(1) \leq d+2$, i. e. $\dim_k I_1 \geq n-d-1$. Let $u_1, \ldots, u_{n-d-1} \in I_1$ be linearly independent linear forms. We have

$$\begin{aligned} h_I(m) &= \binom{m+d+1}{d+1} & \deg\ I - 1 \geq m \geq 1 \\ h_I(\deg\ I) &= \binom{\deg I+d+1}{d+1} - 1 \end{aligned}$$

Thus

$$\begin{aligned} h_I(m) &= h_{(u_1,\ldots,u_{n-d-1})}(m) & \deg I - 1 \geq m \geq 1 \\ h_I(\deg\ I) &< h_{(u_1,\ldots,u_{n-d-1})}(\deg\ I) \end{aligned}$$

and so there exist $f \in I - (u_1, \ldots, u_{n-d-1})$ with $\deg f = \deg\ I$. Let $J := (u_1, \ldots, u_{n-d-1}, f) \subseteq k[x_0, \ldots, x_n]$. Then $J \subseteq I$ and $h_J(m) = h_I(m)$ for all $m \geq 0$, and thus we have $J = I$. $\square$

We devote now to the upper bounds. In this respect we have two different estimates. The first bound is sharp for small values and the second for big ones.

The first upper bound will be deduced from a series of results and observations.

Definition 2.7 Let $V \subseteq \mathbb{P}^n$ be a variety. Then the *linear closure* of V is the smallest linear subspace of $\mathbb{P}^n$ which contains V, and it is denoted by $L(V)$.

Remark 2.8 Let $E \subseteq \mathbb{P}^n$ be a linear space. Then its defining ideal $I(E) \subseteq R_{\bar{k}}$ is generated by linear forms, and it is easy to see that

$$\dim E = n - \dim_{\bar{k}} I(E)$$

Let $V \subseteq \mathbb{P}^n$ be a variety, and let $L \in R_{\bar{k}}$ linear form. Then $L|_V \equiv 0$ if and only if $L|_{L(V)} \equiv 0$, and thus

$$I(L(V)) = (I(V)_1) \subseteq \bar{k}[x_0, \ldots, x_n]$$

In particular we have

$$h_V(1) = n + 1 - \dim_{\bar{k}} \ I_{\bar{k}}(V)_1 = \dim L(V) + 1$$

The following proposition shows that the dimension of the linear closure is bounded in terms of the dimension and the degree of the variety. It is a consequence of Bertini's theorem [21, Th. 6.3]. A proof can be found in [17, Cor. 18.12].

Proposition 2.9 *Let $V \subseteq \mathbb{P}^n$ be an irreducible variety. Then*

$$\dim L(V) + 1 \leq \deg V + \dim V$$

□

The following is an estimation of the degree of the image of a variety under a regular map. It is a variant of [20, Lemma 1] and [30, Prop. 1].

Proposition 2.10 *Let $V \subseteq \mathbb{P}^n$ be a variety, $f_0, \ldots, f_N \in \bar{k}[x_0, \ldots, x_n]$ homogeneous polynomials of degree D which define a regular map*

$$\begin{array}{rl} \varphi: & \mathbb{P}^n \longrightarrow \mathbb{P}^N \\ & x := (x_0 : \ldots : x_n) \mapsto (f_0(x) : \ldots : f_N(x)) \end{array}$$

Then $\deg \varphi(V) \leq \deg V \ D^{\dim V}$.

Proof. We can suppose without loss of generality that V is irreducible. Let $d := \dim \ \varphi(V)$, and let $H_1, \ldots, H_d \subseteq \mathbb{P}^n$ be hyperplanes such that

$$\#(\varphi(V) \cap H_1 \cap \ldots \cap H_d) = \deg \ \varphi(V)$$

For $i = 1, \ldots, d$, let $L_i \in R_{\bar{k}}$ be linear forms such that $H_i = \{L_i = 0\}$. Then

$$\#(\varphi(V) \cap H_1 \cap \ldots \cap H_d)$$

is bounded by the number of irreducible components of $\varphi^{-1}(\varphi(V) \cap H_1 \cap \ldots \cap H_d)$ and so we have

$$\begin{aligned} \#(\varphi(V) \cap H_1 \cap \ldots \cap H_d) &\leq \deg\ \varphi^{-1}(\varphi(V) \cap H_1 \cap \ldots \cap H_d) = \\ &= \deg\ (V \cap \textstyle\bigcap_{i=1}^{d} V(L_i(f_0, \ldots, f_N))) \leq \deg V\ D^d \end{aligned}$$

by Bézout's inequality. We have then

$$\deg \varphi(V) \leq \deg V\ D^{\ \dim V}$$

as $\dim\ \varphi(V) \leq \dim\ V$. □

Now it follows easily the desired inequality for the case of an irreducible variety.

Proposition 2.11 *Let $V \subseteq \mathbb{P}^n$ be an irreducible variety of dimension d, with $d \geq 0$. Then*

$$h_V(m) \leq \deg V\ m^d + d \qquad\qquad m \geq 1$$

Proof. For $n, m \in \mathbb{N}$, let

$$\begin{aligned} v_m :\ \ & \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+m}{n}} \\ & (x_0 : \ldots : x_n) \mapsto (x^{(i)})_{|i|=m} \end{aligned}$$

be the Veronese map of degree m. Then $v_m|_V : V \mapsto v_m(V)$ is a birregular morphism of degree m, and so we have that

$$h_{v_m(V)}\ (k) = h_V(mk) \qquad\qquad k \geq 1$$

In particular we have that

$$h_V(m) = h_{v_m(V)}\ (1) = \dim L(V) + 1$$

by Remark 2.8, and so

$$h_V(m) \leq \deg v_m(V) + \dim v_m(V) \leq \deg V\ m^d + d$$

by application of Propositions 2.9 and 2.10. □

We can extend this bound to the more general case of an unmixed radical ideal in $k[x_0, \ldots, x_n]$.

Theorem 2.12 *Let k be a perfect field and $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous unmixed radical ideal of dimension d, with $d \geq 0$. Then*

$$h_I(m) \leq \deg I\ m^d + \operatorname{irr} I\ d \qquad\qquad m \geq 1$$

Proof. Let $I^e \subseteq R_{\bar{k}}$ be the extended ideal of I in $R_{\bar{k}}$. Then I^e is an unmixed radical ideal of dimension d [35, Ch. VII, §31, Th. 36, Cor. 1] [27, Th. 26.3]. Let $I^e = \cap_P P$ be the minimal primary decomposition of I^e. Then we have

$$h_I(m) \leq \sum_P h_P(m)$$

from where

$$h_I(m) \leq \sum_P (degV(P)\ m^d + d) = \deg I\ m^d + \operatorname{irr} I\ d \qquad\qquad m \geq 1$$

by Proposition 2.11. □

This inequality has the same order of growth of h_I. We see also that it does not improve the estimate

$$h_I(m) \leq \deg I\,\tbinom{m+d-1}{d} + \operatorname{irr} I\,\tbinom{m+d-1}{d-1} \qquad\qquad m \geq 1$$

which follows from Chardin's arguments [10].

From the asymptotic behavior $h_I(m) \sim \frac{\deg I}{d!} m^d$ we see that this inequality is sharp for big values of m only when $d = 1$. In this case, the inequality is optimal in terms of the degree and the length of the ideal, and we determine the extremal cases.

Definition 2.13 Let $V, W \subseteq \mathbb{P}^n$ be varieties. Then V, W are *projectively equivalent* if there exist an automorphism $A \in PGL_{n+1}(\bar{k})$ such that $W = A(V)$ [17, p. 22].

Remark 2.14 Let $V, W \subseteq \mathbb{P}^n$ be varieties. Then V, W are projectively equivalent if and only if its coordinated rings $\bar{k}[V]$, $\bar{k}[W]$ are isomorphic as graded $\bar{k}$–algebras. In particular their Hilbert function coincide.

A curve $C \subseteq \mathbb{P}^n$ is called a *rational normal curve* if it is projectively equivalent to $v_n(\mathbb{P}^1)$. Then C is non degenerated, i. e. $L(C) = \mathbb{P}^n$ [17, Example 1.14], and its degree is n [17, Exerc. 18.8]. By Proposition 2.9 the degree of C is minimum with the condition of being non degenerated. In fact, rational normal curves are characterized by these two properties [17, Prop. 18.9].

Now let $l, n \in \mathbb{N}$, $\delta = (\delta_1, \dots, \delta_l) \in \mathbb{N}^l$ such that $|\delta| := \delta_1 + \dots + \delta_l \leq n + 1 - l$. For $1 \leq j \leq l$, let $n_j := \delta_1 + \dots + \delta_j + j$, and consider the inclusion of linear spaces given by

$$\begin{aligned} i_j :\ & \mathbb{P}^{\delta_j} \hookrightarrow \mathbb{P}^n \\ & (x_0 : \dots : x_{\delta_j}) \mapsto (\overbrace{0 : \dots : 0}^{n_{j-1}} : x_0 : \dots : x_{\delta_j} : 0 : \dots : 0) \end{aligned}$$

The linear subspaces $i_j(\mathbb{P}^{\delta_j}) \subseteq \mathbb{P}^n$ are disjoint one from each other. Let

$$C(n, \delta) := \bigcup_{j=1}^{l} i_j(v_{\delta_j}(\mathbb{P}^1)) \subseteq \mathbb{P}^n$$

A curve $C \subseteq \mathbb{P}^n$ is projectively equivalent to $C(n,\delta)$ if and only if there exist $E_1,\ldots,E_l \subseteq \mathbb{P}^n$ disjoint linear subspaces such that $\dim E_j = \delta_j$, $C \subseteq \cup_j E_j$, and

$$C_j := C \cap E_j \subseteq E_j$$

is a rational normal curve for $1 \le j \le l$.

Definition 2.15 Let $V \subseteq \mathbb{P}^n$ be a variety. Then V is *defined over* k if $I_{\bar{k}}(V) = \bar{k} \otimes_k I_k(V) \subseteq \bar{k}[x_0,\ldots,x_n]$, i. e. if its defining ideal is generated over k.

The following lemma is well–known, we prove it here for lack of suitable reference.

Lemma 2.16 *Let $\varphi : \mathbb{P}^n \to \mathbb{P}^N$ be a regular map defined over k, $V \subseteq \mathbb{P}^n$ be a variety defined over k. Then $\varphi(V) \subseteq \mathbb{P}^N$ is defined over k.*

Proof. We have the following conmutative diagram

$$\begin{array}{ccc} k[x_0,\ldots,x_N] & \xrightarrow{\varphi_k^*} & k[V] \\ \downarrow & & \downarrow \\ \bar{k}[x_0,\ldots,x_N] & \xrightarrow{\varphi_{\bar{k}}^*} & \bar{k}[V] \end{array}$$

with $\ker \varphi_k^* = I_k(W)$ and $\ker \varphi_{\bar{k}}^* = I_{\bar{k}}(W)$. We have $\bar{k} \otimes_k k[V] \cong \bar{k}[V]$ as V is defined over k, and tensoring with $\bar{k}$ we get

$$\begin{array}{ccc} \bar{k}[x_0,\ldots,x_N] & \xrightarrow{\bar{k}\otimes_k \varphi_k^*} & \bar{k} \otimes_k k[V] \\ \| & & \| \\ \bar{k}[x_0,\ldots,x_N] & \xrightarrow{\varphi_{\bar{k}}^*} & \bar{k}[V] \end{array}$$

with $\ker \bar{k} \otimes_k \varphi_k^* = \bar{k} \otimes_k I_k(W)$, from where we deduce $I_{\bar{k}}(W) = \bar{k} \otimes_k I_k(W)$, i. e. $I_{\bar{k}}(W)$ is defined over k. $\square$

Let $v_n : \mathbb{P}^1 \to \mathbb{P}^n$ be the Veronese map of degree n and let $C_n := v_n(\mathbb{P}^1)$ be its image. By the preceeding lemma, C_n is defined over k.

For $1 \le j \le l$, let $C_j := i_j(C_{\delta_j}) \subseteq \mathbb{P}^n$. Then

$$C(n,\delta) = \bigcup_{j=1}^{l} i_j(v_{\delta_j}(\mathbb{P}^1))$$

is the minimal decomposition of $C(n,\delta)$ in irreducible curves. Thus $C(n,\delta)$ is also defined over k, and so

$$\operatorname{irr} I_k(C(n,\delta)) = l, \qquad \deg I_k(C(n,\delta)) = |\delta|$$

Lemma 2.17 *Let $V, W \subseteq \mathbb{P}^n$ be varieties. Then*

$$I(V) + I(W) = (x_0,\ldots,x_n)$$

if and only if V, W lie in disjoint linear subspaces of $\mathbb{P}^n$.

Proof. Given $V, W \subseteq \mathbb{P}^n$ varieties, they lie in disjoint linear subspaces if and only if

$$L(V) \cap L(W) = \emptyset$$

Let $L_V := I(L(V))$, $L_W := I(L(W))$. By Remark 2.8 we have

$$\begin{array}{c} L_V = (I(V)_1) \subseteq I(V) \\ L_W = (I(W)_1) \subseteq I(W) \end{array}$$

In particular L_V, L_W are generated by linear forms, and so

$$L_V + L_W = I(L(V) \cap L(W))$$

Let $V, W \subseteq \mathbb{P}^n$ such that $L(V) \cap L(W) = \emptyset$. Then

$$L_V + L_W = (x_0, \ldots, x_n)$$

and so $I(V) + I(W) = (x_0, \ldots, x_n)$. Conversely, suppose that $I(V) + I(W) = (x_0, \ldots, x_n)$. Then

$$x_0, \ldots, x_n \in I(V)_1 + I(W)_1$$

Thus $L_V + L_W = (x_0, \ldots, x_n)$ and so $L(V) \cap L(W) = \emptyset$ $\square$

Proposition 2.18 *Let k be a perfect field and $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous unmixed radical ideal of dimension 1. Then*

$$h_I(m) = \deg I\ m + \operatorname{irr} I \qquad\qquad m \geq 1$$

if and only if there exist $\delta \in \mathbb{N}^l$ with $l := \operatorname{irr} I$, such that $|\delta| = \deg I$, and a curve $C \subseteq \mathbb{P}^n$ defined over k projectively equivalent to $C(n, \delta)$ such that $I = I_k(C)$.

Proof. Let $C \subseteq \mathbb{P}^n$ be a curve defined over k projectively equivalent to $C(n, \delta)$ for some $\delta \in \mathbb{N}^l$ and $l = \operatorname{irr} I$. Then $\bar{k} \otimes_k I_k(C) = I_{\bar{k}} \subseteq R_{\bar{k}}$ and so

$$\operatorname{irr} I_k(C) = \operatorname{irr} I(C(n, \delta)) = l, \qquad\qquad \deg\ I_k(C) = \deg\ I(C(n, \delta)) = |\delta|$$

We aim at proving that

$$h_{I_k(C)}(m) = |\delta|\ m + l \qquad\qquad m \geq 1$$

We have that $h_{I_k(C)}(m) = h_C(m) = h_{C(n,\delta)}(m)$ and so it suffices to prove that

$$h_{C(n,\delta)}(m) = |\delta| m + l \qquad\qquad m \geq 1$$

We shall proceed by induction on l. Let $C_d := v_d(\mathbb{P}^1)$. We have the inclusion of graded $\bar{k}$–algebras

$$\begin{array}{rl} \bar{k}[C_d] \cong & \bar{k}[x_0, \ldots, x_d]/I_{\bar{k}}(C_d) \stackrel{v_d^*}{\hookrightarrow} \bar{k}[x, y] \\ & x_i \longmapsto x^i y^{d-i} \end{array}$$

We have then
$$\bar{k}[C_d] \cong \bigoplus_{j=0}^{\infty} \bar{k}[x,y]_{d\,j}$$
from where $h_{C_d}(m) = d\,m+1$ for $m \geq 1$, and so the assertion holds for $l=1$. Let $l>1$, and let
$$C(n,\delta) = \bigcup_j C_j$$
be the minimal decomposition of $C(n,\delta)$ in irreducible curves. Then $C_1 \cup \ldots \cup C_{l-1}$, C_l lie in disjoint linear spaces and so
$$I(C_1 \cup \ldots \cup C_{l-1}) + I(C_l) = (x_0, \ldots, x_n)$$
by Lemma 2.17. We have then
$$h_C(m) = h_{C_1 \cup \ldots \cup C_{l-1}}(m) + h_{C_l}(m) \qquad m \geq 1$$
and from the inductive hypothesis we get
$$h_C(m) = \{(\delta_1 + \ldots + \delta_{l-1})\,m + (l-1)\} + \{\delta_l\,m + 1\} = |\delta|\,m + l \qquad m \geq 1$$

Now we shall prove the converse. We have that I^e is a radical ideal, and so I^e is the ideal of some curve $C \subseteq \mathbb{P}^n$ defined over k.

We shall proceed by induction on $l := \operatorname{irr} I$. Let $l=1$, i. e. $C \subseteq \mathbb{P}^n$ irreducible. Then
$$\dim L(C) = h_C(1) - 1 = \deg C$$
and so $C \subseteq L(C)$ is a non degenerated irreducible curve of minimal degree. We have then that $C \subseteq L(C)$ is a rational normal curve [17, Prop. 18.9].

Let $l>1$, and suppose that the assertion is proved for $l(I) \leq l-1$ and K an arbitrary field. In particular it is proved for $\bar{k}$, the algebraic closure of k. Let $C = C_1 \cup \ldots \cup C_l$ be the minimal decomposition of C in irreducible curves. Then
$$h_C(m) = h_{C_1 \cup \ldots \cup C_{l-1}}(m) + h_{C_l}(m) - h_{I(C_1 \cup \ldots \cup C_{l-1}) + I(C_l)}(m) \qquad m \geq 1$$

We deduce from theorem 2.12 that
$$\begin{aligned} h_{C_l}(m) &= \ \delta_l\ m + 1 \\ h_{C_1 \cup \ldots \cup C_{l-1}}(m) &= \ (\delta_1 + \ldots + \delta_{l-1})\ m + (l-1) \end{aligned}$$
and so $C_l \subseteq L(C_l)$ is a rational normal curve, and by the inductive hypothesis $C_1 \cup \ldots \cup C_{l-1}$ is projectively equivalent to $C(n, (\deg C_1, \ldots, \deg C_{l-1}))$. Thus
$$h_C(m) = |\delta| m + l - h_{I(C_1 \cup \ldots \cup C_{l-1}) + I(C_l)}(m) \qquad m \geq 1$$
and so
$$I(C_1 \cup \ldots \cup C_{l-1}) + I(C_l) = (x_0, \ldots, x_n)$$

Then $C_1 \cup \ldots \cup C_{l-1}$, C_l lie in disjoint linear spaces by Lemma 2.17, and so C is projectively equivalent to $C(n, (\deg C_1, \ldots, \deg C_l))$. □

Now we shall derive another upper bound for the Hilbert function of an unmixed radical ideal. The following lemma is well–known, we prove it here for lack of suitable reference.

Lemma 2.19 *Let A be an integrally closed domain, K its quotient field, L a finite separable extension of K, B the integral closure of A in L. Let $\eta \in B$ such that $L = K[\eta]$, and let $f \in K[t]$ be its minimal polinomial. Then*

$$f'(\eta)B \subseteq A[\eta]$$

Proof. Let $M \subseteq L$ be an A–module. Then

$$M' := \{x \in L : \mathrm{Tr}_K^L(xM) \subseteq A\}$$

is called the *complementary module* (relative to the trace) of M [25, Ch. III, §1].

It is straightfoward that if $M \subseteq B$ then $M' \supseteq B$. We have that

$$A[\eta]' = \frac{A[\eta]}{f'(\eta)}$$

(see [25, Ch. III, Prop. 2, Cor.]) and so

$$B \subseteq A[\eta]' = \frac{A[\eta]}{f'(\eta)}$$

□

In the languaje of integral dependence theory, the last assertion says that in this case $f'(\eta)$ lies in the conductor of B in $A[\eta]$.

Theorem 2.20 *Let k be an perfect field and $I \subseteq k[x_0, \dots, x_n]$ an homogeneous unmixed radical ideal of dimension d, with $d \geq 0$. Then*

$$h_I(m) \leq \binom{m+\deg I+d}{d+1} - \binom{m+d}{d+1} \qquad m \geq 1$$

Proof. We shall consider first the case when $P \subseteq \bar{k}[x_0, \dots, x_n]$ is an homogeneous prime ideal.

The field $\bar{k}$ is algebraically closed, and so it is both infinite and perfect. Let $y_0, \dots, y_d, \eta \in \bar{k}[x_0, \dots, x_n]$ be linear forms such that

$$\bar{k}[y_0, \dots, y_d] \hookrightarrow \bar{k}[x_0, \dots, x_n]/P$$

is an integral inclusion of graded $\bar{k}$–algebras, and such that if K, L are the quotient fields of $\bar{k}[y_0, \dots, y_d]$, $\bar{k}[x_0, \dots, x_n]/P$ respectively, then $K \hookrightarrow L$ is separable algebraic and $L = K[\eta]$.

Let $A := \bar{k}[y_0, \dots, y_d]$, $B := \bar{k}[x_0, \dots, x_n]/P$. As a consequence of Krull's Hauptidealsatz we have that

$$A[\eta] \cong A[t]/(F)$$

where $F \in \bar{k}[y_0, \ldots, y_d][t]$ is a non zero homogeneous polynomial. We have then

$$\dim_{\bar{k}}(A[\eta])_m = h_{(F)}(m) = \binom{m+d+1}{d+1} - \binom{m-\deg F+d+1}{d+1}$$

We have also

$$A[\eta] \hookrightarrow B \hookrightarrow \frac{A[\eta]}{F'(\eta)}$$

by Lemma 2.19, and thus

$$\binom{m+d+1}{d+1} - \binom{m-\deg F+d+1}{d+1} \leq h_P(m) \leq \binom{m+\deg F+d}{d+1} - \binom{m+d}{d+1} \qquad m \geq 1$$

We deduce that $\deg F = \deg P$, and so

$$h_P(m) \leq \binom{m+\deg P+d}{d+1} - \binom{m+d}{d+1} \qquad m \geq 1$$

Now we extend this bound to the case of an unmixed ideal. We have that I^e is and unmixed radical ideal. Let $I^e = \cap_P P$ be the primary decomposition of I^e. We have

$$h_I(m) \leq \sum_P \left\{\binom{m+\deg P+d}{d+1} - \binom{m+d}{d+1}\right\} \qquad m \geq 1$$

Then we have

$$h_I(m) \leq \sum_P \sum_{i=0}^{\deg P-1} \binom{m+d+i}{d} \leq \sum_{i=0}^{\deg I-1} \binom{m+d+i}{d} = \binom{m+\deg I+d}{d+1} - \binom{m+d}{d+1}$$

$\square$

Remark 2.21 This inequality is sharp for big values of m, as it is seen by comparing it with the principal term of the Hilbert polynomial of I

From the expression

$$h_I(m) \leq \binom{m+\deg I+i}{d+1} - \binom{m+d}{d+1} = \sum_{i=0}^{\deg I-1} \binom{m+d+i}{d}$$

we see that it does not improve Chardin's estimate [10, Th.]

$$h_I(m) \leq \deg I \binom{m+d}{d} = \sum_{i=0}^{\deg I-1} \binom{m+d}{d}$$

in any case. However we remark that the proof is simpler and that we can use it in our aplications instead of Chardin's estimate obtaining very similar results.

Let k be a perfect field, $I \subseteq k[x_0, \ldots, x_n]$ an homogeneous unmixed radical ideal of dimension $d \geq 0$, and let H_I be its Hilbert–Poincaré series. Then the previous result states that

$$t^{\deg I-1} H_I(t) \leq \frac{1 - t^{\deg I}}{(1-t)^{d+2}}$$

in the sense that this inequality holds for each term of the power series.

We derive an upper bound for the Hilbert function of a generic hypersurface section of an unmixed radical ideal, which need not be unmixed nor radical. This result is an application of both our upper and lower bounds for the Hilbert function. The use of our upper bound (Theorem 2.20) can be replaced by Chardin's estimate [10, Th.] but the bound so obtained is essentially the same. In this way we keep our exposition self–contained.

Lemma 2.22 *Let k be a perfect field and $I \subseteq k[x_0, \dots, x_n]$ an homogeneous unmixed radical ideal of dimension d, with $d \geq 1$, and let $\eta \in k[x_0, \dots, x_n]$ be a linear from which is a non zero divisor modulo I. Then there exists m_0 such that*

$$h_{(I,\eta)}(m_0) \leq \binom{m_0+d}{d} - \binom{m_0+d-3\deg I}{d}$$

and $3 \deg I \leq m_0 \leq 5\, d \deg I$.

Proof. Let $\delta := \deg I$, $k := 3\,\delta$, $l := 2\,\delta$, $m := 5\,d\,\delta$. We aim at proving that

$$\sum_{j=0}^{l-1} \left\{ \binom{m-j+d}{d} - \binom{m-j+d-k}{d} \right\} \geq \sum_{j=0}^{l-1} h_{(I,\eta)}(m-j)$$

We have that

$$\sum_{j=0}^{l-1} \left\{ \binom{m+d-j}{d} - \binom{m+d-k-j}{d} \right\} = \left\{ \binom{m+d+1}{d+1} - \binom{m+d+1-l}{d+1} \right\} - \left\{ \binom{m+d+1-k}{d+1} - \binom{m+d+1-k-l}{d+1} \right\}$$

We have also that

$$\sum_{j=0}^{l-1} h_{(I,\eta)}(m-j) = h_{(I,\eta^r)}(m) \leq \left\{ \binom{m+d+\delta}{d+1} - \binom{m+d}{d+1} \right\} - \left\{ \binom{m+d+1-l}{d+1} - \binom{m+d+1-\delta-l)}{d+1} \right\}$$

by application of Theorems 2.20 and 2.4. Thus it suffices to prove that

$$\begin{aligned} \left\{ \binom{m+d+1-\delta}{d+1} - \binom{m+d+1-\delta-l}{d+1} \right\} & -\left\{ \binom{m+d+1-k}{d+1} - \binom{m+d+1-k-l}{d+1} \right\} \geq \\ & \geq \left\{ \binom{m+d+\delta}{d+1} - \binom{m+d}{d+1} \right\} - \left\{ \binom{m+d+1}{d+1} - \binom{m+d+1-\delta}{d+1} \right\} \end{aligned}$$

We have that

$$\begin{aligned} \left\{ \binom{m+d+1-\delta}{d+1} - \binom{m+d+1-\delta-l}{d+1} \right\} & -\left\{ \binom{m+d+1-k}{d+1} - \binom{m+d+1-k-l}{d+1} \right\} = \\ & = \textstyle\sum_{i=1}^{l} \left\{ \binom{m+d+1-\delta-i}{d} - \binom{m+d+1-k-i}{d} \right\} = \\ & = \textstyle\sum_{i=1}^{l} \sum_{j=1}^{k-\delta} \binom{m+d+1-\delta-i-j}{d-1} \geq l\,(k-\delta) \binom{m+d-1-k-l}{d-1} \end{aligned}$$

and

$$\begin{aligned} \left\{ \binom{m+d+\delta}{d+1} - \binom{m+d}{d+1} \right\} - \left\{ \binom{m+d+1}{d+1} \right. & - \left. \binom{m+d+1-\delta}{d+1} \right\} = \textstyle\sum_{i=1}^{\delta} \left\{ \binom{m+d+\delta-i}{d} - \binom{m+d+1-i}{d} \right\} = \\ & = \textstyle\sum_{i=1}^{\delta} \sum_{j=1}^{\delta} \binom{m+d+\delta-i-j}{d-1} \leq \delta^2 \binom{m+d-1+\delta}{d-1} \end{aligned}$$

and thus it suffices to prove that

$$4=\frac{l\,(k-\delta)}{\delta^2}\geq\frac{\binom{m+d-1+\delta}{d-1}}{\binom{m+d-1-k-l}{d-1}}$$

This is clear when $d=1$, as in this case the right side of this expression equals 1. When $d\geq 2$ we have that

$$\frac{\binom{m+d-1+\delta}{d-1}}{\binom{m+d-1-k-l}{d-1}}=\prod_{j=1}^{d-1}\frac{m+\delta+j}{m-k-l+j}\leq(1+\frac{6/5}{d-1})^{d-1}\leq e^{\frac{6}{5}}$$

and so our claim follows, and we conclude that

$$h_{(I,\eta)}(m_0)\leq\binom{m_0+d}{d}-\binom{m_0+d-3\,\deg I}{d}$$

for some m_0 such that $5\,d\,\delta-2\,\delta+1\leq m_0\leq 5\,d\,\delta$.

$\square$

Theorem 2.23 *Let k be a perfect field and $I\subseteq k[x_0,\ldots,x_n]$ be an homogeneous unmixed radical ideal of dimension d, with $d\geq 0$, and let $f\in k[x_0,\ldots,x_n]$ be a non zero divisor modulo I. Then*

$$\begin{array}{lcll} h_{(I,f)}(m) & \leq & \deg\ I & m\geq 1\\ h_{(I,f)}(m) & = & 0 & m\geq \deg\ I+\deg f-1\end{array}$$

if $d=0$ and

$$h_{(I,f)}(m)\leq 3\,\deg f\,\deg I\,\binom{m+d-1}{d-1}$$

if $d\geq 1$ and $m\geq 5\,d\,\deg I$.

Proof. Let $\delta:=\deg I$, $d_0:=\deg f$. We have

$$h_{(I,f)}(m)=h_I(m)-h_I(m-d_0)$$

Consider first the case $d=0$. Then $h_I(m)\leq\delta$ for $m\geq 1$ and $h_I(m)=\delta$ for $m\geq\delta-1$ by Lemma 2.3, and thus

$$h_{(I,f)}(m)=0\qquad\qquad m\geq\delta+d_0-1$$

Now let $d\geq 1$. We have that I^e is an unmixed radical ideal, and so there exists a linear form $\eta\in k[x_0,\ldots,x_n]$ which is a non zero divisor modulo I^e. By Lemma 2.22

$$h_{(I^e,\eta)}(m_0)\leq\binom{m+d}{d}-\binom{m+d-3\,\deg I}{d}$$

for some $3\,\deg I\leq m_0\leq 5\,d\,\deg I$.

Let $m\geq 3\,\delta$. We have then that

$$\binom{m+d}{d}-\binom{m+d-3\,\deg I}{d}=\sum_{j=1}^{3\,\delta}\binom{m+d-j}{m-j+1}$$

is the m–binomial expansion of $\binom{m+d}{d} - \binom{m+d-3\deg I}{d}$, and so

$$h_{(I,\eta)}(m) \leq \binom{m+d}{d} - \binom{m+d-3\,\delta}{d}$$

for $m \geq m_0$ by Macaulay's theorem and Remark 1.2. We have then

$$h_{(I,f)}(m) = h_{(I^e,f)}(m) = \sum_{j=0}^{d_0-1} h_{(I^e,\eta)}(m-j) \leq 3\,d_0\,\delta\,\binom{m+d-1}{d-1}$$

for $m \geq 5\,d\,\delta$.

$\square$

§ 3. Construction of Regular Sequences

In this section we devote to the construction of regular sequences with polynomials of controlled degrees satisfying different conditions. Throughout this section k will denote an infinite perfect field

An upper bound for the Hilbert function implies in rather a direct way the existence of regular sequences in the ideal I with polynomials of bounded degree. The estimations we get are somewhat worse than [10, §2, Cor. 2 and Cor. 4]. Compare also with [28, §1, Cor. 1] and [19, Prop. 3].

Lemma 3.24 *Let $I, P \subseteq k[x_0, \ldots, x_n]$ be homogeneous ideals, I unmixed radical of dimension d, $d \geq 0$, P prime of dimension e, with $e > d$. Then there exists $f \in I - P$ such that*

$$\deg f \leq (e!\,\deg I)^{\frac{1}{e-d}}$$

Proof. We have

$$h_P(m) \geq \binom{m+e}{e}$$

Let $\delta := \deg I$, $m_0 := [(e!\,\delta)^{\frac{1}{e-d}}]$. Then

$$\begin{aligned} e!\,\binom{m_0+e}{e} &\geq (m_0+1)(m_0+2)^{e-1} > \\ &> (e!\ \delta)^{\frac{e}{e-d}} + e!\ \delta d \geq e!\ \delta\ (m_0^d + d) \end{aligned}$$

as $e \geq 1$. By Theorem 2.12 we have $h_P(m_0) > h_I(m_0)$, and so there exists $f \in I - P$ such that $\deg f = m_0$. $\square$

Proposition 3.25 *Let $I, J \subseteq k[x_0, \ldots, x_n]$ be homogeneous ideals, I unmixed radical of dimension d, $d \geq 0$, J Cohen–Macaulay of dimension e, with $e \geq d$. Then there exist homogeneous polynomials $f_1, \ldots, f_{e-d} \in I$ such that*

$$\deg f_j \leq ((d+j)!\,\deg I)^{\frac{1}{j}}$$

for $j = 1, \ldots, e-d$, whose image in $k[x_0, \ldots, x_n]/J$ form a regular sequence.

Proof. We proceed by induction on the dimension of J. If $\dim J = \dim I = d$ there is nothing to prove.

Let $\dim J = e$ with $d+1 \leq e \leq n$. Let P be an associated prime of J. The ideal J is unmixed, and so $\dim P = e$, and thus by Lemma 3.24 there exists $f_P \in I - P$ such that

$$\deg f_P \leq (e! \deg I)^{\frac{1}{e-d}}$$

By eventually multiplying each f_P by a linear form which is a non zero divisor modulo P we can suppose that

$$\deg f_P = [(e!\ \deg I)^{\frac{1}{e-d}}]$$

for each associated prime ideal P of J. As the field is infinite, there exists a linear combination

$$f := \sum_{P \in \text{ Ass}(J)} \lambda_P f_P$$

such that $f \in I - P$ for every associated prime ideal of J. Then f is homogeneous,

$$\deg f = [(e!\ \deg I)^{\frac{1}{e-d}}]$$

and it is a non divisor of zero modulo J. Thus (J, f) is a Cohen–Macaulay ideal of dimension $e-1$, and we can apply the inductive hypothesis to get homogeneous polynomials $f_1, \ldots, f_{e-d-1} \in I$ which form a regular sequence in $R/(J, f)$ with $\deg f_j \leq ((d+j)! \deg I)^{\frac{1}{j}}$ for $j = 1, \ldots, e-d-1$. Thus $f, f_1, \ldots, f_{e-d-1}$ are homogeneous polynomials which form a regular sequence in R/J, and so

$$f_1, \ldots, f_{e-d-1}, f \in R/J$$

is a regular sequence for which it holds the stated bounds on the degrees. □

Consider an homogeneous unmixed radical ideal $I \subseteq k[x_0, \ldots, x_n]$ of dimension $d \geq 0$, and an homogeneous polynomial $F \in k[x_0, \ldots, x_n]$ which is a nonzero divisor modulo I. We shall show first that there exist homogeneous polynomials of controlled degrees $f_1, \ldots, f_{n-d} \in I$ which form a regular sequence which avoids the hypersurface $\{F = 0\}$, i. e. such that no associated prime ideal of $(f_1, \ldots, f_i)$ lies in $\{F = 0\}$ for $1 \leq i \leq n-d$. This result is an application of our bound for the Hilbert function of a generic hypersurface section of an unmixed radical ideal (Theorem 2.23).

Lemma 3.26 *Let $I, P \subseteq k[x_0, \ldots, x_n]$ be homogeneous ideals, I unmixed radical of dimension d, with $d \geq 0$, P prime of dimension e, with $e \geq d$. Let $F \in k[x_0, \ldots, x_n]$ be an homogeneous polynomial which is a non zero divisor modulo I. Then there exists $g \in (I, F) - P$ such that*

$$\deg g \leq \deg I + \deg F - 1 \qquad \textit{if } d{=}0$$

$$\deg g \leq 5\, d\, \deg F\, \deg I \qquad \textit{if } d \geq 1.$$

Proof. Let $\delta := \deg\ I$, $d_0 := \deg F$, $J := (I, F)$. Consider first the case $d = 0$. Then

$$h_J(m) = 0 \qquad\qquad m \geq \delta + d_0 - 1$$

by Theorem 2.23, and so there exist $g \in J - P$ with $\deg g \leq \delta + d_0 - 1$.

Now let $d \geq 1$. We have that $P \subseteq k[x_0, \ldots, x_n]$ is an homogeneous prime ideal of dimension $e \geq d$, and so

$$h_P(m) \geq \binom{m+e}{e} \geq \binom{m+d}{d}$$

Let $m_0 := 5\, d\, d_0\, \delta$. We have then

$$h_{(I,f)}(m_0) \leq 3\, d_0\, \delta \binom{m_0+d-1}{d-1} < \binom{m_0+d}{d} \leq h_P(m)$$

by Theorem 2.23, and so there exists $g \in J - P$ such that $\deg g \leq m_0$. $\square$

Theorem 3.27 *Let $I \subseteq k[x_0, \ldots, x_n]$ be an unmixed radical ideal of dimension d, with $d \geq 0$, and let $F \in k[x_0, \ldots, x_n]$ be an homogeneous polynomial which is a non zero divisor modulo I. Then there exist homogeneous polynomials $f_1, \ldots, f_{n-d} \in I$ such that $F, f_1, \ldots, f_{n-d} \in k[x_0, \ldots, x_n]$ is a regular sequence and*

$$\deg f_i \leq \deg I + \deg F - 1 \qquad\qquad if\ \ d = 0$$
$$\deg f_i \leq 5\ d \deg F \deg I \qquad\qquad if\ \ d \geq 1.$$

Proof. Applying Lemma 3.26 it follows that there exist homogeneous polynomials $g_1, \ldots, g_{n-d} \in (I, F)$ such that $F, g_1, \ldots, g_{n-d}$ is a regular sequence and

$$\deg g_i \leq \deg I + \deg F - 1 \qquad\qquad \text{if } d = 0$$
$$\deg g_i \leq 5\ d \deg F \deg I \qquad\qquad \text{if } d \geq 1.$$

The proof this statement is similar to that of Proposition 3.25, we omit it here in order to avoid repetitive arguments.

Let $g_i = f_i + F\, h_i$ with $f_i \in I$ for $1 \leq i \leq n - d$. Then $\deg f_i = \deg g_i$, and $g_i \equiv f_i \mod (F)$, and so $F, f_1, \ldots, f_{n-d}$ is a regular sequence for which it holds the announced bounds on the degrees. $\square$

We observe that in the case when $\deg F = 1$, Lemma 3.26 can be deduced from Lemma 2.22, and so both Lemma 3.26 and Theorem 3.27 do not depend on Macaulay theorem. It can also be shown in the case when $\deg F \geq 2$ that they do not depend on Macaulay theorem altogether [32, Th. 3.40].

Definition 3.28 Let A be a ring. Then $f_1, \ldots, f_s \in A$ is a *weak regular sequence* if $\bar{f}_i$ is a non zero divisor in $A/(f_1, \ldots, f_{i-1})$ for $1 \leq i \leq s$.

This definition differs from the definition of regular sequence in that we allow $\bar{f}_s \in A/(f_1, \ldots, f_{s-1})$ to be a unit.

Let $F, f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ be homogeneous polynomials such that $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]_F$ is a weak regular sequence. Then it is not always the case that $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ is a regular sequence, as some components of high dimension may appear in the hypersurface $\{F = 0\}$. Consider the following example:

Example 3.29 Let $F \in k[x_0, x_1, x_2, x_3]$ be an homogeneous polynomial of degree $d \geq 1$ such that $F \notin (x_1, x_2)$. Let

$$f_1 := x_1, \quad f_2 := x_1^{d+1} + x_2 F, \quad f_3 := x_1^{d+1} + x_3 F \ \in k[x_0, x_1, x_2, x_3]$$

Then $f_2 \equiv x_2 F, f_3 \equiv x_3 F \mod (f_1)$, and so they form a regular sequence in $k[x_0, x_1, x_2, x_3]_F$. We have that

$$\{(x_0 : \ldots : x_4) \in \mathbb{P}^3 : F = 0, x_1 = 0\} \subseteq V(f_1, f_2, f_3) \subseteq \mathbb{P}^3$$

and so $f_1, f_2, f_3 \in k[x_0, x_1, x_2, x_3]$ cannot be a regular sequence.

We shall show that the weak regular sequence $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]_F$ can in fact be replaced by polynomials $p_1, \ldots, p_s \in k[x_0, \ldots, x_n]$ of controlled degrees such that $(f_1, \ldots, f_i) = (p_1, \ldots, p_i) \subseteq k[x_0, \ldots, x_n]_F$ for $1 \leq i \leq s$ and such that $p_1, \ldots, p_s \in k[x_0, \ldots, x_n]$ is a regular sequence. Our proof follows T. Dubé's arguments, who gave an incomplete proof of a similar statement [11, Lemma 4.1] under an unproved assumption on the Hilbert function of a certain class of ideals [11, §2.1].

Proposition 3.30 *Let $s \leq n+1$, and let $F, f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ be homogeneous polynomials, with $\deg F \geq 1$, such that $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]_F$ is a weak regular sequence and such that $(f_1, \ldots, f_i) \in k[x_0, \ldots, x_n]_F$ is a radical ideal for $1 \leq i \leq s-1$. Let $I_i := (f_1, \ldots, f_i) \subseteq k[x_0, \ldots, x_n]_F$ and let $I_i^c := I_i \cap k[x_0, \ldots, x_n]$ for $1 \leq i \leq s$. Then there exist homogeneous polynomials $p_1, \ldots, p_s \in k[x_0, \ldots, x_n]$ which satify the following conditions:*

i) $p_1 = F^{c_1} f_1, \ \ p_2 = F^{c_2} f_2, \ \ p_i \equiv F^{c_i} f_i \mod I_{i-1}$ *with* $c_i \in \mathbb{Z}$, *for* $i = 1, \ldots, s$.

ii) $p_1, \ldots, p_s \in k[x_0, \ldots, x_n]$ *is a regular sequence.*

iii) $\deg p_i \leq \max\{\deg f_i, 5(n+1-i) \deg F \deg I_{i-1}^c\}$ *if* $i \leq n$

$\deg p_{n+1} = \max\{\deg f_{n+1}, \deg I_n^c + \deg F - 1\}$

Proof. We shall proceed by induction. Let $f_1 = F^{e_1} a_1$, $f_2 = F^{e_2} a_2$, with $F \not| a_1$, $F \not| a_2$. Then f_1, f_2 is a regular sequence in $k[x_0, \ldots, x_n]_F$ if and only if $(a_1 : a_2) = 1$, and thus

$$p_1 := F^{-e_1} f_1, \qquad p_2 := F^{-e_2} f_2$$

is a regular sequence in $k[x_0, \ldots, x_n]$. Now let $i \geq 3$, and suppose that $p_1, \ldots, p_{i-1}$ are already constructed with the stated properties. Let

$$L_{i-1} := (p_1, \ldots, p_{i-1}) \subseteq k[x_0, \ldots, x_n]$$

and let $L_{i-1} = \cap_{j=1}^t Q_j$ be an irredundant primary decomposition of L_{i-1} such that

$$F \notin \sqrt{Q_j} \qquad \text{for } 1 \leq j \leq r$$
$$F \in \sqrt{Q_j} \qquad \text{for } r+1 \leq j \leq t$$

Let $(L_{i-1}) \subseteq k[x_0, \ldots, x_n]_F$. Then $(L_{i-1}) = I_{i-1}$ and so

$$I_{i-1}^c = \cap_{j=1}^r Q_j \subseteq k[x_0, \ldots, x_n]$$

is a primary decomposition of I_{i-1}^c. We have $I_{i-1}^c = (1)$ or $\dim I_{i-1}^c = n-i+1$. Then there exist $b_1, \ldots, b_{i-1} \in I_{i-1}^c$ homogeneous polynomials such that $F, b_1, \ldots, b_{i-1}$ is a regular sequence and such that

$$\deg b_j = \max\{\deg f_i, 5\ (n+1-i)\ \deg F\ \ \deg I_{i-1}^c\} \qquad 1 \leq j \leq i-1$$

if $i \leq n$ and

$$\deg b_j = \max\{\deg f_{n+1},\ \deg I_n^c + \deg F - 1\} \qquad 1 \leq j \leq n$$

if $i = n+1$, by application of Theorem 3.27 and by eventually multiplying each b_j by an appropiated linear form. Let

$$u_i := \sum_{j=1}^{i-1} \lambda_j\ b_j \in I_{i-1}^c$$

be a linear combination of the b_j. We shall prove that a generic choice of $\lambda_1, \ldots, \lambda_{i-1}$ makes $p_i := F^{c_i} f_i + u_i$ with $c_i := \deg u_i - \deg f_i \geq 0$ satisfy the stated conditions. We have

$$\deg p_i = \max\{\deg f_i, 5\ (n+1-i)\ \deg F\ \ \deg I_{i-1}^c\}, \qquad \text{if } i \leq n \text{ and}$$

$$\deg p_{n+1} = \max\{\deg f_{n+1}, \deg I_n^c + \deg F - 1\}$$

We aim at proving that p_i does not belong to any of the associated prime ideals of L_{i-1}.

Consider first $1 \leq j \leq r$. Then $f_i \notin \sqrt{Q_j}$ as f_i is a non zero divisor modulo I_{i-1}. We have that $u_i \in I_{i-1}$, and so $p_i = F^{c_i} f_i + u_i \notin \sqrt{Q_j}$.

Now let $r+1 \leq j \leq t$. Then $\dim Q_j = n-i+1$ as L_{i-1} is an unmixed ideal of dimension $n-i+1$, and we have also $F \in \sqrt{Q}_j$. Thus there exist $1 \leq l \leq i-1$ such that $b_l \notin \sqrt{Q}_j$, and so $p_i \notin \sqrt{Q}_j$ for a generic choice of the $\lambda_1, \ldots, \lambda_{i-1}$. □

As a corollary, we deduce that if we have a weak regular sequence $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ of afine polynomials, we can replace it by another weak regular sequence $p_1, \ldots, p_s \in k[x_1, \ldots, x_n]$ with polynomials of controlled degrees such that $(f_1, \ldots, f_i) = (p_1, \ldots, p_i)$ for $1 \leq i \leq s$ and such that the homogenizated polynomials $\tilde{p}_1, \ldots, \tilde{p}_s \in k[x_0, \ldots, x_n]$ form a regular sequence.

Corollary 3.31 *Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be a weak regular sequence of affine polynomials such that $(f_1, \ldots, f_i) \subseteq k[x_1, \ldots, x_n]$ is a radical ideal for $1 \leq i \leq s-1$. Let $I_i := (f_1, \ldots, f_i) \subseteq k[x_1, \ldots, x_n]$ for $1 \leq i \leq s$. Then there exist $p_1, \ldots, p_s \in k[x_1, \ldots, x_n]$ which satisfy the following conditions:*

i) $p_1 = f_1,\ p_2 = f_2,\ p_i \equiv f_i \mod I_{i-1}$ *for* $i = 1, \ldots, s$.

ii) $\tilde{p}_1, \ldots, \tilde{p}_s \in k[x_0, \ldots, x_n]$ *is a regular sequence.*

iii) $\deg p_i \leq \max\{\deg f_i, 5(n+1-i)\deg \tilde{I}_{i-1}\}$ *if* $i \leq n$

$\deg p_{n+1} = \max\{\deg f_{n+1}, \deg \tilde{I}_n\}$

Proof. We have that $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ is a weak regular sequence, and so $\tilde{f}_1, \ldots, \tilde{f}_s \in k[x_0, \ldots, x_n]_{x_0}$ is also a weak regular sequence, and also $(\tilde{f}_1, \ldots, \tilde{f}_i) \subseteq k[x_0, \ldots, x_n]_{x_0}$ is radical for $1 \leq i \leq s-1$. Let $r_1, \ldots, r_s \in k[x_0, \ldots, x_n]$ be the homogeneous polynomials obtained by aplying Proposition 3.30 to $\tilde{f}_1, \ldots, \tilde{f}_s$. Let

$$p_i := r_i^a \qquad\qquad 1 \leq i \leq s$$

Thus $\deg p_i \leq \deg r_i$, and $x_0^{e_i}\tilde{p}_i = r_i$ for some $e_i \geq 0$. Then $\tilde{p}_1, \ldots, \tilde{p}_s \in k[x_0, \ldots, x_n]$ is a regular sequence, and $p_1, \ldots, p_s$ satisfy the stated conditions. □

Our bounds for the degrees in the preceding propositions depend on the degree of certain ideals associated to $f_1, \ldots, f_s$. The following is a Bézout–type lemma which shows that these bounds can also be expressed in terms of the degrees of the polynomials $f_1, \ldots, f_s$.

Lemma 3.32 *Let $s \leq n$, and let $F, f_1, \ldots, f_s \in k[x_0, \ldots, x_n]$ be homogeneous polynomials, with $\deg F \geq 1$, such that $f_1, \ldots, f_s \in k[x_0, \ldots, x_n]_F$ is a weak regular sequence. Let $I := (f_1, \ldots, f_s) \subseteq k[x_0, \ldots, x_n]_F$, and let $I^c := I \cap k[x_0, \ldots, x_n]$. Then*

$$\deg I^c \leq \prod_{i=1}^{s} \deg f_i$$

Proof. If $I^c = (1)$ there is nothing to prove. Otherwise we have $\dim I^c \geq 0$.

Let $I_i := (f_1, \ldots, f_i) \subseteq k[x_0, \ldots, x_n]_F$, $J_i := (I_{i-1}^c, f_i) \subseteq k[x_0, \ldots, x_n]$ for $1 \leq i \leq s$. Then $\dim I_i^c = \dim J_i = n - i$ and $J_i \subseteq I_i^c$, and so $\deg I_i^c \leq \deg J_i$.

We shall proceed by induction on i. For $i = 1$ we have

$$\deg I_1^c \leq \deg J_1 = \deg f_1$$

Let $i \geq 2$. Then f_i is a non zero divisor modulo I_{i-1}^c and so

$$\deg I_i^c \leq \deg J_i = \deg f_i\ \deg I_{i-1}^c \leq \prod_{j=1}^{s} \deg f_j$$

by the inductive hypothesis. □

§ 4. The Effective Nullstellensatz and the Representation Problem in Complete Intersections

In this section we consider the problem of bounding the degrees of the polynomials in the Nullstellensatz and in the representation problem in complete intersections.

As a consequence of the results of the previous section we obtain bounds for these two problems which depend not only on the number of variables and on the degrees of the input polynomials but also on an additional parameter called the geometric degree of the system of equations. The bounds so obtained are more intrinsic and refined than the usual estimates, and we show that they are sharper in some special cases.

Our arguments at this point are essentially the same of T. Dubé [11].

The bound we obtain for the effective Nulltellensatz is similar to that announced in [15, Th. 19] and proved in [14] by algorithmic methods and to that obtained in [24] by duality methods.

Let $g, f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials such that $g \in (f_1, \ldots, f_s)$. Let $D \geq 0$. Then there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that

$$g = a_1 f_1 + \ldots + a_s f_s$$

with $\deg a_i f_i \leq \deg g + D$ for $i = 1, \ldots, s$ if and only if

$$x_0^D \tilde{g} \in (\tilde{f}_1, \ldots, \tilde{f}_s) \subseteq k[x_0, \ldots, x_n]$$

and so in this situation we aim at bounding D such that $x_0^D \tilde{g} \in (\tilde{f}_1, \ldots, \tilde{f}_s)$.

We shall suppose $n, s \geq 2$, as the cases $n = 1$ or $s = 1$ are well–known. Also we shall suppose without loose of generality that k is algebraically closed, and in particular infinite and perfect.

Let $h_1, \ldots, h_s \in k[x_1, \ldots, x_n]$ be a weak regular sequence of affine polynomials such that $(h_1, \ldots, h_i)$ is radical for $1 \leq i \leq s - 1$. In particular we have $s \leq n + 1$. We fix the following notation:

$$\begin{aligned} I_i &:= (h_1, \ldots, h_i) \subseteq k[x_1, \ldots, x_n] \\ J_i &:= (\tilde{I}_{i-1}, \tilde{h}_i) \subseteq k[x_0, \ldots, x_n] \\ H_i &:= (\tilde{h}_1, \ldots, \tilde{h}_s) \subseteq k[x_0, \ldots, x_n] \end{aligned}$$

for $1 \leq i \leq s$. Let $J_i = \cap_P Q_P$ be a primary decomposition of J_i, and let

$$J_i^* = \bigcap_{P : \dim P = \dim I} Q_P$$

be the intersection of the primary components of maximal dimension of J_i, which is well defined as the isolated components are unique. We have that $J_i \subseteq J_i^* \subseteq \tilde{I}_i$.

Let

$$\begin{aligned}
&\gamma_1 := 0 \\
&\gamma_i := \deg h_i \, \deg \tilde{I}_{i-1} - \deg \tilde{I}_i && 1 \leq i \leq n, \\
&\gamma_{n+1} := \deg h_{n+1} + \deg \tilde{I}_n - 1
\end{aligned}$$

Proposition 4.33 *Let $g \in \tilde{I}_i$ for some $1 \leq i \leq s$. Then*

$$x_0^{\gamma_i} g \in J_i^*$$

Proof. The case $1 \leq i \leq n$ is [11, Lemma 5.5].

We consider the case $i = n+1$. We have that $\tilde{I}_n \subseteq k[x_0, \ldots, x_n]$ is an unmixed radical ideal of dimension 0 and we have that h_{n+1} is a non zero divisor modulo I_n, and so by Theorem 2.23 $h_{J_{n+1}}(m) = 0$ for $m \geq \deg \tilde{I}_n + \deg h_{n+1} - 1$. Then $x_0^{\gamma_{n+1}} \in J_{n+1} \subseteq J_{n+1}^*$ and thus

$$x_0^{\gamma_{n+1}} g \in J_{n+1}^*$$

□

Then we apply Corollary 3.31 to the sequence $h_1, \ldots, h_s$, to obtain $p_1, \ldots, p_s \in k[x_1, \ldots, x_n]$ such that

i) $p_1 = h_1$, $p_2 = h_2$, $p_i = h_i + u_i$, with $u_i \in I_{i-1}$ for $1 \leq i \leq s$.

ii) $\tilde{p}_1, \ldots, \tilde{p}_s \in k[x_0, \ldots, x_n]$ is a regular sequence.

iii) $\deg p_i \leq \max\{\deg h_i, 5\,(n+1-i) \deg \tilde{I}_{i-1}\}$ for $1 \leq i \leq n$,

$\deg p_{n+1} \leq \max\{\deg h_{n+1},\ \deg \tilde{I}_n\}$

Then $\tilde{p}_i = x_0^{c_i} \tilde{f}_i + \tilde{u}_i$, with $c_1 = 0$, $c_2 = 0$, $c_i = \max\{0, 5\,(n+1-i) \deg \tilde{I}_{i-1} - \deg h_i\}$ for $1 \leq i \leq n$, and $c_{n+1} = \max\{0, \deg \tilde{I}_n - \deg h_{n+1}\}$. Let

$$D_i := \sum_{j=2}^{i} (i+1-j)\gamma_j + \sum_{j=3}^{i-1} (i-j) c_j$$

for $1 \leq i \leq s$.

Proposition 4.34 *Let $g \in \tilde{I}_i$ for some $1 \leq i \leq s$. Then*

$$x_0^{D_i} g \in H_i$$

Proof. This proposition follows from the proof of [11, Lemma 6.1] and [11, Lemma 6.2], applying Proposition 4.33 for the case $i = n+1$. □

Now the task consists in bounding D_s. Our bound will depend not only on the number of variables and on the degrees of the polynomials $h_1, \ldots, h_s$, but also on the degree of some homogeneous ideals associated to them.

Proposition 4.35 *Let* $d := \max_{1\le i\le s} \deg h_i$ *and* $\delta_i := \deg \tilde{I}_i$ *for* $1 \le i \le s$. *We have then*

i) $D_s \le s^2\ (d-1+3n)\ \max_{1\le i\le s-1} \delta_i$

for $s \le n$ *and*

ii) $D_{n+1} \le n^2(d-1+3n)\ \max_{1\le i\le s-1} \delta_i$

Proof. Let $d_i := \deg h_i$ for $1 \le i \le s$. We have

$$\begin{aligned}
\textstyle\sum_{j=3}^{s-1}(s-j)c_j &= \textstyle\sum_{j=3}^{s-1}(s-j)\max\{0, 5(n+1-j)\delta_{j-1} - d_j\} \le \\
&\le 5\,(\textstyle\sum_{j=3}^{s-1}(s-j)(n+1-j))\max_{1\le i\le s-2}\delta_i \le \\
&\le 5\ (n-2)(\textstyle\sum_{j=3}^{s-1}(s-j))\max_{1\le i\le s-2}\delta_i \le \\
&\le 3(n-2)(s-2)^2\max_{1\le i\le s-2}\delta_i
\end{aligned}$$

Let $s \le n$. We have then

$$\begin{aligned}
\textstyle\sum_{j=2}^{s}(s-j+1)\ \gamma_j &= \textstyle\sum_{j=2}^{s}(s-j+1)\,(d_j\delta_{j-1} - \delta_j) \le \\
&\le (\textstyle\sum_{j=1}^{s-1} j)\ \ d\ \max_{1\le i\le s-1}\delta_i \le s^2\ d\ \max_{1\le i\le s-1}\delta_i
\end{aligned}$$

Thus

$$\begin{aligned}
D_s &\le s^2\, d\ \max_{1\le i\le s-1}\delta_i + 3(n-2)(s-2)^2\max_{1\le i\le s-2}\delta_i \le \\
&\le s^2\,(d-1+3n)\ \max_{1\le i\le s-1}\delta_i
\end{aligned}$$

Also we have

$$\begin{aligned}
\textstyle\sum_{j=2}^{n+1}(n+2-j)\,\gamma_j &= \textstyle\sum_{j=2}^{n}(n+2-j)(d_j\delta_{j-1} - \delta_j)\ + d_{n+1} + \delta_n - 1 \le \\
&\le (\textstyle\sum_{j=1}^{n} j)\ \ d\max_{1\le i\le n-1}\delta_i \le n^2\, d\ \max_{1\le i\le n-1}\delta_i
\end{aligned}$$

and thus

$$\begin{aligned}
D_{n+1} &\le n^2\, d\ \max_{1\le i\le n-1}\delta_i + 3(n-2)(n-1)^2\max_{1\le i\le n-1}\delta_i \le \\
&\le n^2(d-1+3n)\ \max_{1\le i\le n-1}\delta_i
\end{aligned}$$

□

Given $f_1,\dots,f_s \in k[x_1,\dots,x_n]$ polynomials which define a proper ideal $(f_1,\dots,f_s) \subseteq k[x_1,\dots,x_n]$ of dimension $n-s$ or $1 \in (f_1,\dots,f_s)$, there exist $t \le s$ and $h_1,\dots,h_t \in k[x_1,\dots,x_n]$ linear combinations of the polynomials $\{f_i, x_jf_i : 1 \le i \le s, 1 \le j \le n\}$ such that

i) $(h_1,\dots,h_t) = (f_1,\dots,f_s)$

ii) $h_1,\dots,h_t$ is a weak regular sequence.

iii) $(h_1,\dots,h_i)\subseteq k[x_1,\dots,x_n]$ is a radical ideal for $1\le i\le t-1$.

In the case when k is a zero characteristic field, we can take $h_1,\dots,h_t$ linear combinations of $f_1,\dots,f_s$. In fact, in both cases a generic linear combination will satisfy the stated conditions. This result is a consequence of Bertini's theorem [21, Cor. 6.7] (see for instance [30, §5.2], [23, Prop. 37]), and allows us to reduce from the general situation to the previously considered one.

Let $d:=\max_{1\le i\le s}\deg f_i$, and suppose that $\deg f_i\le\deg f_{i+1}$ for $1\le i\le s-1$. Thus in the case when k is a zero characteristic field we can take $h_1,\dots,h_t$ such that

$$\deg h_i\le\deg f_i \qquad\qquad 1\le i\le t$$

and $\deg h_i\le d+1$ in the case when $\mathrm{char}\,(k)=p>0$.

Definition 4.36 Let k be a zero characteristic field and $f_1,\dots,f_s\in k[x_1,\dots,x_n]$ polynomials which define a proper ideal $(f_1,\dots,f_s)\subsetneq k[x_1,\dots,x_n]$ of dimension $n-s$ or such that $1\in(f_1,\dots,f_s)$. For $\lambda=(\lambda_{ij})_{ij}\in\bar{k}^{s\times s}$ and $1\le i\le s$ let

$$g_i(\lambda):=\sum_j\lambda_{ij}f_j\ \in\bar{k}[x_1,\dots,x_n]$$

be linear combinations of $f_1,\dots,f_s$. Consider the set of matrices $\Gamma\subseteq\bar{k}^{s\times s}$ such that for $\lambda\in\Gamma$ there exist $t=t(\lambda)\le s$ such that $(g_1,\dots,g_t)=(f_1,\dots,f_s)$, $g_1,\dots,g_t$ is a weak regular sequence and $(g_1,\dots,g_i)\subseteq\bar{k}[x_0,\dots,x_n]$ is a radical ideal for $1\le i\le t-1$. Then $\Gamma\neq\emptyset$, and in fact Γ contains a non empty open set $U\subseteq\bar{k}^{s\times s}$. Let $V_i(\lambda):=V(g_1,\dots,g_i)\subseteq\mathbb{A}^n$ be the affine variety defined by $g_1,\dots,g_i$ for $1\le i\le s$, and define

$$\delta(\lambda)=\max_{1\le i\le\min\{t(\lambda),n\}-1}\deg\,V_i(\lambda)$$

Then the *geometric degree of the system of equations* $f_1,\dots,f_s$ is defined as

$$\delta=\delta(f_1,\dots,f_s):=\min_{\lambda\in\Gamma}\,\delta(\lambda)$$

In the case when $\mathrm{char}(k)=p>0$ we define the degree of the system of equations $f_1,\dots,f_s$ in an analogous way by considering linear combinations of the polynomials $f_1,\dots,f_s,x_1f_1,\dots,x_nf_s$.

This definition extends [24, Def. 1] to the case of a complete intersection ideal. It is analogous to the definition of degree of a system of equations of [15], though this degree is not defined as a minimum through all the possible choices of $\lambda\in\Gamma$ but through a generic choice.

Remark 4.37 We see from the definition that the degree of a system of equations $f_1,\dots,f_s$ does not depend on inversible linear combinations, i. e. if $\mu=(\mu_{ij})_{ij}\in GL_s(k)$ and

$$g_i(\mu):=\sum_j\mu_{ij}f_j$$

for $1\le i\le s$, then $\delta(f_1,\dots,f_s)=\delta(g_1,\dots,g_s)$, and so this parameter is in some sense an invariant of the system.

The following lemma shows that $\delta(f_1, \ldots, f_s)$ can be bounded in terms of the degrees of the polynomials $f_1, \ldots, f_s$.

Lemma 4.38 *Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials which define a proper ideal $(f_1, \ldots, f_s) \subseteq k[x_1, \ldots, x_n]$ of dimension $n - s$, or $1 \in (f_1, \ldots, f_s)$ and such that $d_i \geq d_{i+1}$ for $1 \leq i \leq s-1$, with $d_i := \deg f_i$, and let $d := \max_{1 \leq i \leq s} d_i$. Then*

$$\delta(f_1, \ldots, f_s) \leq \prod_{i=1}^{\min\{s,n\}-1} d_i$$

in the case when k is a zero characteristic field, and

$$\delta(f_1, \ldots, f_s) \leq (d+1)^{\min\{s,n\}-1}$$

in the case when $\mathrm{char}(k) = p > 0$.

Proof. This follows at once from Lemma 3.32. □

We have the following bounds for the representation problem in complete intersections and for the effective Nullstellensatz in terms of this parameter.

Theorem 4.39 *(Representation Problem in Complete Intersections) Let $s \leq n$, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials which define a proper ideal $(f_1, \ldots, f_s) \subseteq k[x_1, \ldots, x_n]$ of dimension $n - s$. Let $d := \max_{1 \leq i \leq s} \deg f_i$, and let δ be the geometric degree of the system of equations $f_1, \ldots, f_s$. Let $g \in (f_1, \ldots, f_s)$. Then there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that*

$$g = a_1 f_1 + \ldots + a_s f_s$$

with $\deg a_i f_i \leq \deg g + s^2\,(d + 3n)\,\delta$ *for* $i = 1, \ldots, s$.

Proof. This follows from Proposition 4.35 (i). □

Theorem 4.40 *(Effective Nullstellensatz) Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials such that $1 \in (f_1, \ldots, f_s)$. Let $d := \max_{1 \leq i \leq s} \deg f_i$, and let δ be the geometric degree of the system of equations $f_1, \ldots, f_s$. Then there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that*

$$1 = a_1 f_1 + \ldots + a_s f_s$$

with $\deg a_i f_i \leq \min\{n, s\}^2\,(d + 3n)\,\delta$ *for* $i = 1, \ldots, s$.

Proof. This follows from Proposition 4.35. □

We can essentially recover from Theorem 4.39 and Theorem 4.40 the usual bounds for the representation problem in complete intersections and the effective Nullstellensatz. We have for instance:

Corollary 4.41 *Let k be a zero characteristic field and $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ polynomials such that $1 \in (f_1, \ldots, f_s)$, and such that $d_i \geq d_{i+1}$ for $1 \leq i \leq s-1$, with $d_i := \deg f_i$. Then there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that*

$$1 = a_1 f_1 + \ldots + a_s f_s$$

with $\deg a_i f_i \leq \min\{n, s\}^2 \, (d + 3n) \, \prod_{j=1}^{\min\{n,s\}-1} d_j$ *for* $i = 1, \ldots, s$.

$\square$

We remark that our results yield much sharper bounds for these two problems in some particular cases. Consider for instance the following example:

Example 4.42 Let k be a zero characteristic field and $h_1, \ldots, h_s \in k[x_1, \ldots, x_n]$ a weak regular sequence of polynomials such that $1 \in (h_1, \ldots, h_s)$. Let $d := \max_{1 \leq i \leq s} \deg h_i$, and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ such that

$$f_i = h_i + u_i$$

with $u_i \in (h_1, \ldots, h_{i-1})$ for $1 \leq i \leq s$. Then

$$\delta := \delta(f_1, \ldots, f_s) = \delta(h_1, \ldots, h_s) \leq d^{\,\min\{n,s\}-1}$$

Let $D := \max_{1 \leq i \leq s} \deg f_i$. By Theorem 4.40 there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that

$$1 = a_1 f_1 + \ldots, a_s f_s$$

with $\deg a_i f_i \leq \min\{n, s\}^2 \, (D + 3n) \, d^{\,\min\{n,s\}-1}$ for $i = 1, \ldots, s$. This estimate is sharper for big values of D than the bound

$$\deg a_i f_i \leq D^{\,\min\{n,s\}} \qquad\qquad i = 1, \ldots, s$$

for $D \geq 3$, which results from application of the bound of [22].

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