THE HEIGHT OF THE MIXED SPARSE RESULTANT

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Abstract. We present an upper bound for the height of the mixed sparse resultant, defined as the logarithm of the maximum modulus of its coefficients. We obtain a similar estimate for its Mahler measure.

Let $\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$ be finite sets of integer vectors and let $\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n} \in \mathbb{Z}[U_0, \dots, U_n]$ be the associated mixed sparse resultant — or $(\mathcal{A}_0, \dots, \mathcal{A}_n)$-resultant — which is a polynomial in $n+1$ groups $U_i := \{U_{i\,a}\,;\, a \in \mathcal{A}_i\}$ of $m_i := \#\mathcal{A}_i$ variables each. We refer to [Stu94] and [CLO98, Chapter 7] for the definitions and basic facts.
This resultant is widely used as a tool for polynomial equation solving, a fact that has sparked a lot interest in its computation, see e.g. [CLO98, Sec. 7.6], [EM99], [D'An02], [JKSS04], while it is also studied from a more theoretical point of view because of its connections with toric varieties and hypergeometric functions, see e.g. [GKZ94], [CDS98].

We assume for the sequel that the family of supports $\mathcal{A}_0, \dots, \mathcal{A}_n$ is essential (see [Stu94, Sec. 1]) which does not represent any loss of generality, by [Stu94, Cor. 1.1].
Set $\mathcal{A} := (\mathcal{A}_0, \dots, \mathcal{A}_n)$, and let $L_{\mathcal{A}} \subset \mathbb{Z}^n$ denote the $\mathbb{Z}$-module affinely spanned by the pointwise sum $\sum_{i=0}^n \mathcal{A}_i$. This is a subgroup of $\mathbb{Z}^n$ of finite index

$$[\mathbb{Z}^n : L_{\mathcal{A}}] := \#(\mathbb{Z}^n / L_{\mathcal{A}})$$

because we assumed that the family $\mathcal{A}$ is essential. Also set $Q_i := \mathrm{Conv}(\mathcal{A}_i) \subset \mathbb{R}^n$ for the convex hull of $\mathcal{A}_i$ for $i = 0, \dots n$.
We note by MV the mixed volume function as defined in e.g. [CLO98, Sec. 7.4]: this is normalized so that for a polytope $P \subset \mathbb{R}^n$, the mixed volume $\mathrm{MV}(P, \dots, P)$ equals $n!$ times its Euclidean volume $\mathrm{Vol}_{\mathbb{R}^n}(P)$. We also set $\mathrm{Vol}(P) := \mathrm{MV}(P, \dots, P) = n!\,\mathrm{Vol}_{\mathbb{R}^n}(P)$.
Under this notation and assumption, the resultant is a multihomogeneous polynomial of degree

$$\deg_{U_i}(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}) \;=\; \frac{1}{[\mathbb{Z}^n : L_{\mathcal{A}}]}\,\mathrm{MV}(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n) \;>0$$

with respect to each group of variables U_i, see [PS93, Cor. 2.4].
The *absolute height* of a polynomial $g = \sum_a c_a\, x^a \in \mathbb{C}[x_1, \dots, x_n]$ is defined as $H(g) := \max\{|c_a|\,;\, a \in \mathbb{N}^n\}$. Hereby we will be mainly concerned with its *(logarithmic) height*:

$$h(g) := \log H(g) = \log \max\{|c_a|\,;\, a \in \mathbb{N}^n\}.$$

Date: April 15, 2004.
2000 *Mathematics Subject Classification.* Primary 11G50; Secondary 13P99.
Key words and phrases. Mixed sparse resultant, height of polynomials, Mahler measure.
Supported by a Marie Curie post-doctoral fellowship of the European Community Program *Improving Human Research Potential and the Socio-economic Knowledge Base*, contract n° HPMFCT-2000-00709.

The main result of this paper is the following upper bound for the height of the resultant:

Theorem 1.1.

$$h\big(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}\big) \;\le\; \frac{1}{[\mathbb{Z}^n : L_{\mathcal{A}}]} \sum_{i=0}^{n} \mathrm{MV}(Q_0,\dots,Q_{i-1},Q_{i+1},\dots,Q_n)\,\log(\#\mathcal{A}_i).$$

We write for short $\mathrm{Res}_{\mathcal{A}} := \mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}$ and $\mathrm{MV}_i(\mathcal{A}) := \frac{1}{[\mathbb{Z}^n:L_{\mathcal{A}}]}\,\mathrm{MV}(Q_0,\dots,Q_{i-1},Q_{i+1},\dots,Q_n)$ for $i=0,\dots,n$. The previous result can thus be rephrased as

$$H\big(\mathrm{Res}_{\mathcal{A}}\big) \;\le\; \prod_{i=0}^{n} (\#\mathcal{A}_i)^{\mathrm{MV}_i(\mathcal{A})}\;.$$

This improves our previous bound for the unmixed case [Som02, Cor. 2.5] and extends it to the general case. We remark that the obtained upper bound is *polynomial* in the size of the input family of supports $\mathcal{A}$ and in the mixed volumes $\mathrm{MV}_i(\mathcal{A})$, and hence it represents a truly substantial improvement over all previous general estimates. These are the ones which follow either from the Canny-Emiris type formulas (Inequality (4) in the appendix, see also [KPS01, Prop. 1.7] or [Roj00, Thm. 23]) or from direct application of the unmixed case (see the inequality (3) below for $k=1$).

We also consider the Mahler measure, which is another usual notion for the size of a n-variate polynomial. The *Mahler measure* of $g \in \mathbb{C}[x_1,\dots,x_n]$ is defined as

$$m(g) := \int_{S_1^n} \log|g|\; d\mu^n,$$

where $S_1 \subset \mathbb{C}$ is the unit circle and $d\mu$ is the Haar measure over S_1 of total mass 1. This can be compared with the height: in our case

$$(1)\qquad -\sum_{i=0}^{n} \mathrm{MV}_i(\mathcal{A})\,\log(m_i) \;\le\; m(\mathrm{Res}_{\mathcal{A}}) - h(\mathrm{Res}_{\mathcal{A}}) \;\le\; \sum_{i=0}^{n} \mathrm{MV}_i(\mathcal{A})\,\log(m_i)$$

by [KPS01, Lem. 1.1]. We refer to [KPS01, Sec. 1.1.1] for an account on some of the notions of height of complex polynomials: just note that the height $h(g)$ here coincides with $\log|g|_\infty$ in that reference.

We obtain the same estimate as before for the Mahler measure of the resultant.

Theorem 1.2.

$$m\big(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}\big) \;\le\; \frac{1}{[\mathbb{Z}^n : L_{\mathcal{A}}]} \sum_{i=0}^{n} \mathrm{MV}(Q_0,\dots,Q_{i-1},Q_{i+1},\dots,Q_n)\,\log(\#\mathcal{A}_i).$$

Note that this improves by a factor of 2 the estimate which would derive from direct application of Theorem 1.1 and the inequalities (1) above.

Both estimates are a consequence of the following:

Lemma 1.3. *Let $f_0 \in \mathbb{C}^{\mathcal{A}_0},\dots,f_n \in \mathbb{C}^{\mathcal{A}_n}$. Then*

$$\log\big|\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}(f_0,\dots,f_n)\big| \;\le\; \frac{1}{[\mathbb{Z}^n : L_{\mathcal{A}}]} \sum_{i=0}^{n} \mathrm{MV}(Q_0,\dots,Q_{i-1},Q_{i+1},\dots,Q_n)\,\log||f_i||_1,$$

where $||f_i||_1 := \sum_{a\in\mathcal{A}_i} |f_{i\,a}|$ denotes the ℓ^1-norm of the vector $f_i = (f_{i\,a}\,;\, a \in \mathcal{A}_i)$.

Let $g = \sum_a c_a \, x^a \in \mathbb{C}[x_1, \dots, x_n]$. Then for $a \in \mathbb{N}^n$ we have that

$$c_a = \int_{S_1^n} \frac{g(z_1, \dots, z_n)}{z_1^{a_1+1} \cdots z_n^{a_n+1}} \, d\mu^n$$

by Cauchy's formula and so $h(g) \le \sup \big\{ \log |g(\xi)| \, ; \, \xi \in S_1^n \big\}$. Thus Theorem 1.1 is a consequence of this inequality applied to $g := \mathrm{Res}_{\mathcal{A}}$, together with Lemma 1.3.
On the other hand, Theorem 1.2 follows from Lemma 1.3 by a straightforward estimation of the integral in the definition of the Mahler measure.

Proof of Lemma 1.3.– Let $k \in \mathbb{N}$. Then let $k \, \mathcal{A}_i \subset \mathbb{Z}^n$ denote the pointwise sum of k copies of $\mathcal{A}_i$, and set $k \, \mathcal{A} := (k \, \mathcal{A}_0, \dots, k \, \mathcal{A}_n)$. It is easy to verify that $k \, \mathcal{A}$ is also essential, $L_{k\,\mathcal{A}} = L_{\mathcal{A}}$ and $\mathrm{Conv}(k \, \mathcal{A}_i) = k \, Q_i$.
We identify each $f_i \in \mathbb{C}^{\mathcal{A}_i}$ with the corresponding Laurent polynomial $f_i = \sum_{a \in \mathcal{A}_i} f_{i\,a} \, x^a$, and we set $f_i^k \in \mathbb{C}^{k\mathcal{A}_i}$ for the vector which corresponds to the k-th power of f_i. By the factorization formula for resultants [PS93, Prop. 7.1] we get that

$$\mathrm{Res}_{k\mathcal{A}}(f_0^k, \dots, f_n^k) = \mathrm{Res}_{\mathcal{A}}(f_0, \dots, f_n)^{k^{n+1}}$$

and so

$$\begin{aligned} k^{n+1} \, \log \big| \mathrm{Res}_{\mathcal{A}}(f_0, \dots, f_n) \big| \; & \le \; h(\mathrm{Res}_{k\mathcal{A}}) + \sum_{i=0}^{n} \mathrm{MV}_i(k \, \mathcal{A}) \, \log ||f_i^k||_1 \\ & \le \; h(\mathrm{Res}_{k\mathcal{A}}) + k^{n+1} \sum_{i=0}^{n} \mathrm{MV}_i(\mathcal{A}) \; \log ||f_i||_1. \end{aligned} \tag{2}$$

The first inequality follows from the straightforward estimate $|G(u_0, \dots, u_n)| \le H(G) \prod_{i=0}^n ||u_i||_1^{d_i}$ for a multihomogeneous polynomial G of degree d_i in each group of variables, applied to $G := \mathrm{Res}_{k\,\mathcal{A}}$ and $u_i := f_i^k$. The second one follows from the linearity of the mixed volume, and the sub-additivity of the ℓ^1-norm with respect to polynomial multiplication (which implies that $\log ||f_i^k||_1 \le k \, \log ||f_i||_1$).

Now let $\mathcal{B} \subset \mathbb{Z}^n$ be any finite set such that $L_{\mathcal{B}} = \mathbb{Z}^n$ and such that $\mathcal{A}_0, \dots, \mathcal{A}_n \subset \mathcal{B}$. Set $n(k) := \# k \, \mathcal{B}$ and $P := \mathrm{Conv}(\mathcal{B}) \subset \mathbb{R}^n$. Then the (unmixed) resultant $\mathrm{Res}_{k\mathcal{B}}$ is a polynomial in $(n+1) \, n(k)$ variables and total degree $(n+1) \, \mathrm{Vol}(k \, P) = (n+1) \, k^n \, \mathrm{Vol}(P)$. We have also that $L_{k\mathcal{B}} = \mathbb{Z}^n$ and so we are in the hypothesis of [Som02, Cor. 2.5], which gives the height estimate

$$h(\mathrm{Res}_{k\mathcal{B}}) \le 2 \, (n+1) \, \log\big(n(k)\big) \, \mathrm{Vol}(k \, P) = 2 \, (n+1) \, \log\big(n(k)\big) \, k^n \, \mathrm{Vol}(P).$$

We have that $k \, \mathcal{A}_i \subset k \, \mathcal{B}$ for $i = 0, \dots, n$ and so by [Stu94, Cor. 4.2] there exists a monomial order $\prec$ such that $\mathrm{Res}_{k\,\mathcal{A}}$ divides the initial form $\mathrm{init}_{\prec}(\mathrm{Res}_{k\,\mathcal{B}})$. This is a polynomial in $(n+1) \, n(k)$ variables of degree and height bounded by those of $\mathrm{Res}_{k\mathcal{B}}$, and so

$$\begin{aligned} h(\mathrm{Res}_{k\mathcal{A}}) \; & \le \; h(\mathrm{Res}_{k\mathcal{B}}) + 2 \, \log \big((n+1) \, n(k) + 1 \big) \, (n+1) \, k^n \, \mathrm{Vol}(P) \\ & \le \; 4 \, (n+1) \, \log \big((n+1) \, n(k) + 1 \big) \, k^n \, \mathrm{Vol}(P) \end{aligned} \tag{3}$$

by the inequality $h(f) \le h(g) + 2 \, \deg(g) \, \log(n+1)$, which holds for $f, g \in \mathbb{Z}[x_1, \dots, x_n]$ such that $f | g$ (see [KPS01, Lem. 1.2(1.d)]) applied to $f := \mathrm{Res}_{k\mathcal{A}}$ and $g := \mathrm{init}_{\prec}(\mathrm{Res}_{k\,\mathcal{B}})$.

Finally we set $\mathcal{B} := b + d \, [0,1]^n \subset \mathbb{R}^n$ where $[0,1]$ denotes the unit interval of $\mathbb{R}$, for some $b \in \mathbb{Z}^n$ and $d \in \mathbb{N}$ such that $\mathcal{A}_0, \dots, \mathcal{A}_n \subset b + d \, [0,1]$. Then $n(k) =$

$\log\big(\#(k\,b+k\,d\,[0,1]^n\cap\mathbb{Z}^n)\big)=\log(k\,d+1)^n=O_k(\log k)$ (here the notation O_k refers to the dependence on k) and so

$$h(\mathrm{Res}_{k\mathcal{A}})=O_k(k^n\log k).$$

Note that alternatively, we could have obtained this from the inequality (4) in the appendix.

Toghether with the inequality (2) this implies that

$$\log\big|\mathrm{Res}_{\mathcal{A}}(f_0,\dots,f_n)\big|\ \le\ \sum_{i=0}^{n}\mathrm{MV}_i(\mathcal{A})\ \log||f_i||_1\ +O_k\Big(\frac{\log k}{k}\Big),$$

from where we conclude by letting $k\to\infty$. □

Let us consider some examples. For short we set $H(\mathcal{A}):=H(\mathrm{Res}_{\mathcal{A}})$ and $E(\mathcal{A}):=\prod_{i=0}^{n}(\#\mathcal{A}_i)^{\mathrm{MV}_i(\mathcal{A})}$; we also set

$$q(\mathcal{A}):=\frac{\log E(\mathcal{A})}{\log H(\mathcal{A})}$$

for the quotient between the height of the resultant and the estimate from Theorem 1.1.

Example 1.1. *Sylvester resultants.* For $d\in\mathbb{N}$ we let

$$\mathcal{A}_0(d)=\mathcal{A}_1(d):=\{0,1,2,\dots,d\}\ \subset\mathbb{Z}.$$

The corresponding resultant coincides with the Sylvester resultant of two univariate polynomials of the same degree d. In this case $\mathrm{MV}_0(d)=\mathrm{MV}_1(d)=d$ and $\#\mathcal{A}_0(d)=\#\mathcal{A}_1(d)=d+1$, and so $E(d):=E\big(\mathcal{A}_0(d),\mathcal{A}_1(d)\big)=(d+1)^{2\,d}$.

We compute the height $H(d):=H\big(\mathcal{A}_0(d),\mathcal{A}_1(d)\big)$ for $2\le d\le 7$ with the aid of Maple and we collect the results in the following comparative table:

d	2	3	4	5	6	7
$H(d)$	2	3	10	23	78	274
$E(d)$	81	4,096	390,625	60,466,176	13,841,287,201	4,398,046,511,104
$q(d)$	6.33	7.57	5.59	5.71	5.35	5.18

Example 1.2. We take this example from [EM99, Example 3.5]. Let

$$\begin{aligned}\mathcal{A}_0 &:= \big\{(0,0),(1,1),(2,1),(1,0)\big\},\\ \mathcal{A}_1 &:= \big\{(0,1),(2,2),(2,1),(1,0)\big\},\\ \mathcal{A}_2 &:= \big\{(0,0),(0,1),(1,1),(1,0)\big\}.\end{aligned}$$

Then $\mathrm{MV}_0=4$, $\mathrm{MV}_1=3$ and $\mathrm{MV}_2=4$, so that $E(\mathcal{A})=4^4\,4^3\,4^4$. On the other hand, we can compute the resultant using its expression in [EM99, Example 3.19] as a quotient of determinants and we obtain that $H(\mathcal{A})=8$. Hence

$$H(\mathcal{A})=8 \qquad , \qquad E(\mathcal{A})=4{,}194{,}304 \qquad , \qquad q(\mathcal{A})=7.33\ .$$

For reference, the straightforward estimation of $H(\mathcal{A})$ via the Canny-Emiris formula (see the appendix below) gives:

$$H(\mathcal{A})\le 2^{82}=4{,}835{,}703{,}278{,}458{,}516{,}698{,}824{,}704\ .$$

Example 1.3. We take this example from [Stu94, Example 2.1]. Let

$$\begin{aligned}\mathcal{A}_0 &:= \big\{(0,0),(2,2),(1,3)\big\},\\ \mathcal{A}_1 &:= \big\{(0,1),(2,0),(1,2)\big\},\\ \mathcal{A}_2 &:= \big\{(3,0),(1,1)\big\}.\end{aligned}$$

Then $\mathrm{MV}_0 = 5$, $\mathrm{MV}_1 = 7$ and $\mathrm{MV}_2 = 7$, so that $E(\mathcal{A}) = 3^5\,3^7\,2^7$. From the explicit monomial expansion of the resultant (see [Stu94, Example 2.1]) we find that $H(\mathcal{A}) = 14$ and so

$$H(\mathcal{A}) = 14 \qquad , \qquad E(\mathcal{A}) = 68{,}024{,}448 \qquad , \qquad q(\mathcal{A}) = 6.83\ .$$

These examples show that there is still some room for improvement over Theorem 1.1. It is however possible that our estimate is quite sharp anyway: in spite of the large difference between $H(\mathcal{A})$ and $E(\mathcal{A})$ in the computed examples, the quotient $q(\mathcal{A})$ is quite small, and moreover it does not seem to grow when $E(\mathcal{A}) \to \infty$.
In any case, it would be very interesting to have an *exact* expression for $h(\mathrm{Res}_\mathcal{A})$ — as was remarked to me by B. Sturmfels — or at least a non trivial lower bound. Note that the only information that we dispose about the exact value of the coefficients of $\mathrm{Res}_\mathcal{A}$ is for the extremal ones, which are equal to ± 1 [Stu94, Cor. 3.1].

Remark 1.4. *After a first version of this paper was circulating, C. D'Andrea (personal communication) obtained a non trivial lower bound for the height of the Sylvester resultant, and an improvement of the upper bound for this case: in the notation of Example 1.1 above, he obtains that $H(d) \le d!$.*

Appendix: Estimation of the height via the Canny-Emiris formula

For purpose of easy reference, we establish herein the estimate for $h(\mathrm{Res}_\mathcal{A})$ which follows from the Canny-Emiris formula and the standard estimates for the behavior of the height of polynomials under addition, multiplication and division.

Assume that $L_\mathcal{A} = \mathbb{Z}^n$ and set $Q := \sum_{i=0}^n Q_i \subset \mathbb{R}^n$. Let $\mathcal{M}_0, \ldots, \mathcal{M}_n$ be a family of Canny-Emiris (square, non singular) matrices for $\mathcal{A}$; we refer to [CLO98, Sec. 7.6] for their precise definition. In the sequel we just describe the aspects needed for the height estimate.
A family of Canny-Emiris matrices is not unique, as their construction depends on a choice of a coherent mixed subdivision of Q and of a sufficiently small and generic vector $\delta \in \mathbb{Q}^n$. Set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n.$$

The procedure then uses the given subdivision of Q to split this set into a disjoint union $\mathcal{E} = \mathcal{E}_0(j) \cup \cdots \cup \mathcal{E}_n(j)$, for each $0 \le j \le n$. The elements in $\mathcal{E}$ are in bijection with the rows of $\mathcal{M}_j$, and to each $p \in \mathcal{E}_i(j)$ corresponds a row of $\mathcal{M}_j$ with exactly $m_i = \#\mathcal{A}_i$ non zero entries, which consist of the variables in $U_i := \{U_{i\,a}\,;\, a \in \mathcal{A}_i\}$.
Set $D_j := \det(\mathcal{M}_j) \in \mathbb{Z}[U_0, \ldots, U_n] \setminus \{0\}$. The *Canny-Emiris formula* [CLO98, Ch. 7, Thm. 6.12] states that $\mathrm{Res}_\mathcal{A} = \gcd(D_0, \ldots, D_n)$.

Then D_0 is a multihomogeneous polynomial of degree $N_i := \#\mathcal{E}_i(0)$ in each set of variables U_i and of height bounded by $\sum_{i=0}^n N_i\, \log(m_i)$. We have that $\mathrm{Res}_\mathcal{A} | D_0$ and so $m(\mathrm{Res}_{\mathcal{A}}) \le m(D_0)$, which combined with [KPS01, Lem. 1.1] gives

$$\begin{aligned} h(\mathrm{Res}_{\mathcal{A}}) &\le h(D_0) + \sum_{i=0}^{n} \big(N_i + \mathrm{MV}_i(\mathcal{A})\big) \log(\#\mathcal{A}_i) \\ &\le \sum_{i=0}^{n} \big(2\,N_i + \mathrm{MV}_i(\mathcal{A})\big) \log(\#\mathcal{A}_i). \end{aligned} \tag{4}$$

Applied to Example 1.2, this gives the stated estimate: $N_0 = N_1 = 4$ and $N_2 = 7$ (see [EM99, Example 3.5]) and so the previous estimate gives $H(\mathcal{A}) \le 4^{2\cdot 15+11} = 2^{82}$.

In general, the estimate so obtained is *much* worse than that of Theorem 1.1, especially for $n \gg 0$. Consider e.g. $\mathcal{A}_i := \{0, \ldots, d\}^n \subset \mathbb{Z}^n$ for $i = 0, \ldots, n$. Then it is easy to show that Inequality (4) gives

$$h(\mathrm{Res}_{\mathcal{A}}) \le \big(2\,((n+1)\,d)^n + (n+1)!\,d^n\big) \log(d+1)^n = n\,\big(2\,(n+1)^n + (n+1)!\big)\,d^n \log(d+1)$$

while Theorem 1.1 gives $h(\mathrm{Res}_{\mathcal{A}}) \le n\,(n+1)!\,d^n \log(d+1)$.

Acknowledgements. I thank Carlos D'Andrea for several helpful discussions, and for having communicated me his work on the height of the Sylvester resultant. I also thank Matilde Lalín and Carlos D'Andrea for disproving a conjecture I made in a previous version of this paper, on the value of $m(\mathrm{Res}_{\mathcal{A}})$,

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