MEASURING THE SOLUTION SET OF A SYSTEM OF POLYNOMIAL EQUATIONS

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BGSMATH MDM CLOSING WORKSHOP

BARCELONA, JUNE 6TH 2019

BGSMath BARCELONA GRADUATE SCHOOL OF MATHEMATICS

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How many solutions?

Let $\underline{f} = (f_1, \dots, f_n)$

with $f_i \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ a Laurent polynomial

Set $Z(\underline{f}) = \{ p \in (\mathbb{C}^\times)^n \mid f_1(p) = \cdots = f_n(p) = 0 \}$

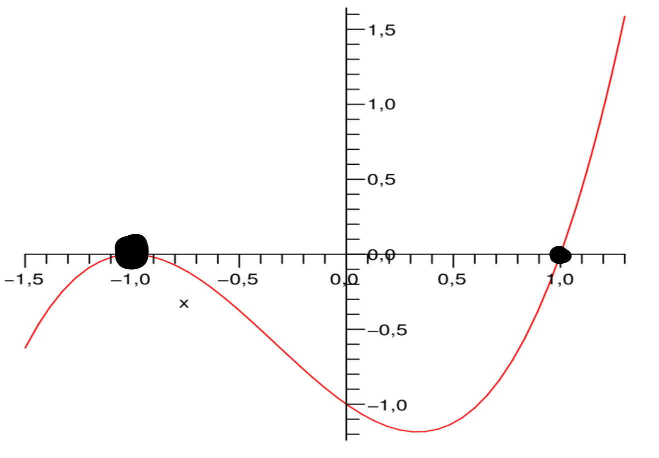
How **large** can $Z(\underline{f})$ be?

THE "FUNDAMENTAL THEOREM OF ALGEBRA"

Let $f = \alpha_d x^d + \dots + \alpha_0$ with $\alpha_d \cdot \alpha_0 \neq 0$, then $\# Z(f) = d$

Ex: $f = x^3 + x^2 - x - 1$

$Z(f) = \{-1, 1\}$



~~THE "FUNDAMENTAL THEOREM OF ALGEBRA"~~

THE "NICE COLLABORATION BETWEEN ALGEBRA AND ANALYSIS TO A RESULT ABOUT POLYNOMIALS"

Let $f = \alpha_d x^d + \cdots + \alpha_0$ with $\alpha_d \cdot \alpha_0 \neq 0$, then $\# Z(f) = d$

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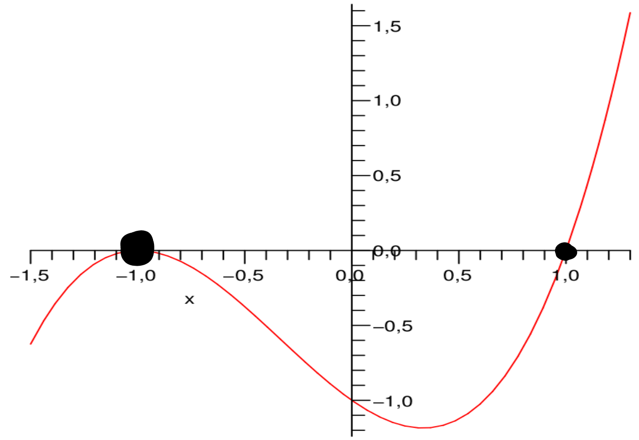
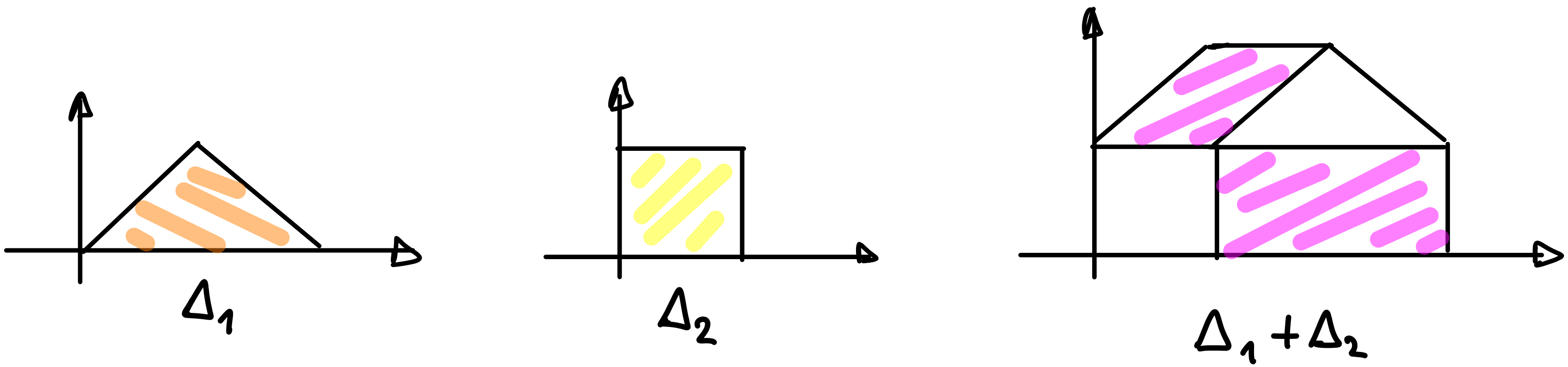


Mixed volumes

Let $\Delta_1, \dots, \Delta_n \subset \mathbb{R}^n$ polytopes. Their mixed volume is

$$MV(\Delta_1, \dots, \Delta_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathrm{vol}(\Delta_{i_1} + \dots + \Delta_{i_k})$$



$$\Longrightarrow \quad MV(\Delta_1, \Delta_2) = 3$$

- symmetric and multilinear w.r.t. +
- $MV(\Delta, \dots, \Delta) = n! \ \mathrm{vol}(\Delta)$ (Minkowski)

THE BKK THEOREM

For $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ let $N(f) \subset \mathbb{R}^n$ its Newton polytope

Thm (Bernstein, Kushnirenko, Khovanskii '1970)

Let $\underline{f} = (f_1, \dots, f_n)$ with $f_i \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ s.t. $Z(\underline{f})$ finite, then

$$\# Z(\underline{f}) \leq MV(N(f_1), \dots, N(f_n))$$

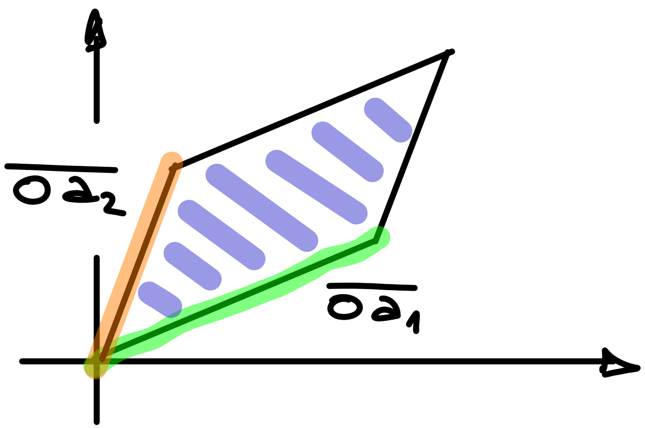
with $=$ for $\underline{f}$ generic

BINOMIALS

Let $f_i = x^{a_i} - 1$ with $a_i \in \mathbb{Z}^n$ l.i. and set

$$A = (a_1 \cdots a_n) \in \mathbb{Z}^{n \times n}$$

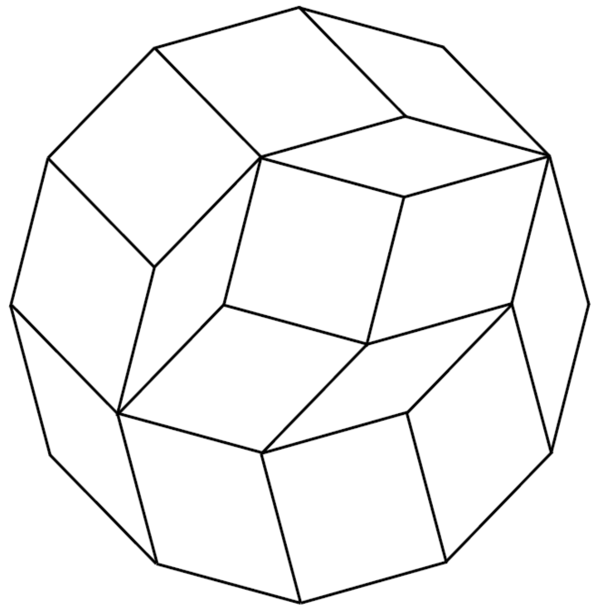
- $Z(f)$ subgroup of $(\mathbb{C}^\times)^n$ of cardinality $|\det(A)|$
- $N(f_i) = \overline{0\, a_i}$ and $MV(\overline{0a_1}, \ldots, \overline{0a_n}) = |\det(A)|$



THE BKK THEOREM FOR ZONOTOPES

- $\# Z(f_1 \cdot f_1', f_2, \dots, f_n) = \# Z(f_1, f_2, \dots, f_n) + \# Z(f_1', f_2, \dots, f_n)$

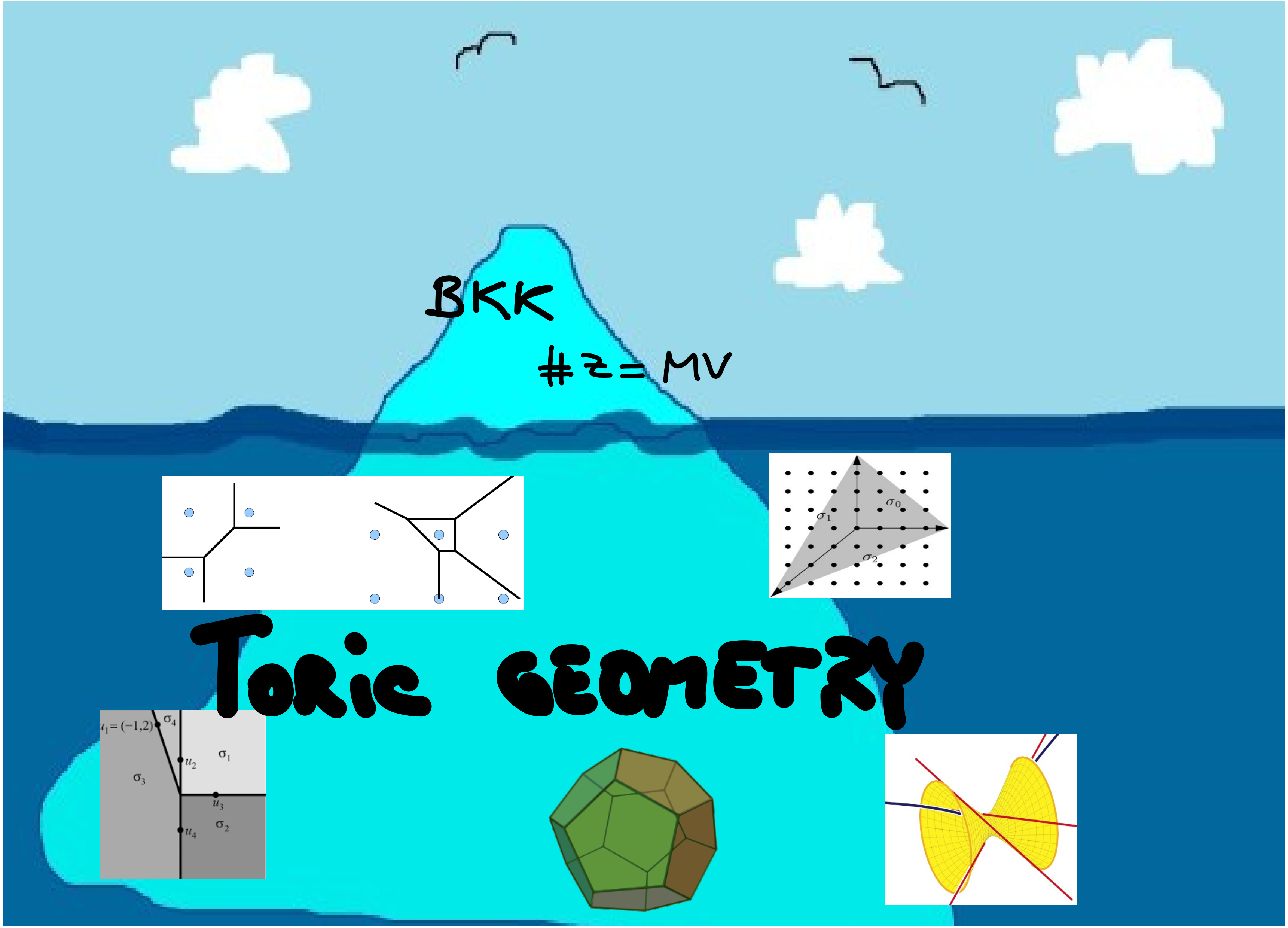
- $MV(N(f_1 \cdot f_1'), N(f_2), \dots, N(f_n)) = MV(N(f_1), N(f_2), \dots, N(f_n))$
$$+ MV(N(f_1'), N(f_2), \dots, N(f_n))$$



If $f = \prod_{a \in S} (x^a - 1)$

then $N(f) = \sum_{a \in S} \overline{0a}$ is a zonotope

The tip of an iceberg



Toric varieties

- $\mathbb{T} = (\mathbb{C}^\times)$ the algebraic torus
- toric variety with torus $\mathbb{T}$:= a (normal) algebraic variety X s.t.

$$\mathbb{T} \underset{\text{open}}{\subset} X \quad \text{and} \quad \mathbb{T} \circlearrowright X$$

For instance: $\mathbb{T}$, $\mathbb{A}^n$, $\mathbb{P}^n$, $\mathbb{P}^n \times \mathbb{P}^m$,

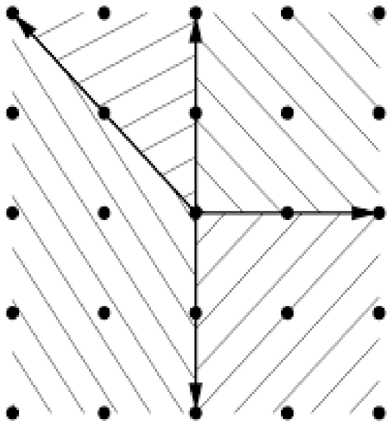
weighted projective spaces

Hirzebruch surfaces

projective bundles

- etc

CLASSIFYING TORIC VARIETIES



Σ fan in $\mathbb{R}^n$ $\xrightarrow[1:1]{}$ X_Σ toric variety

(Demazure)

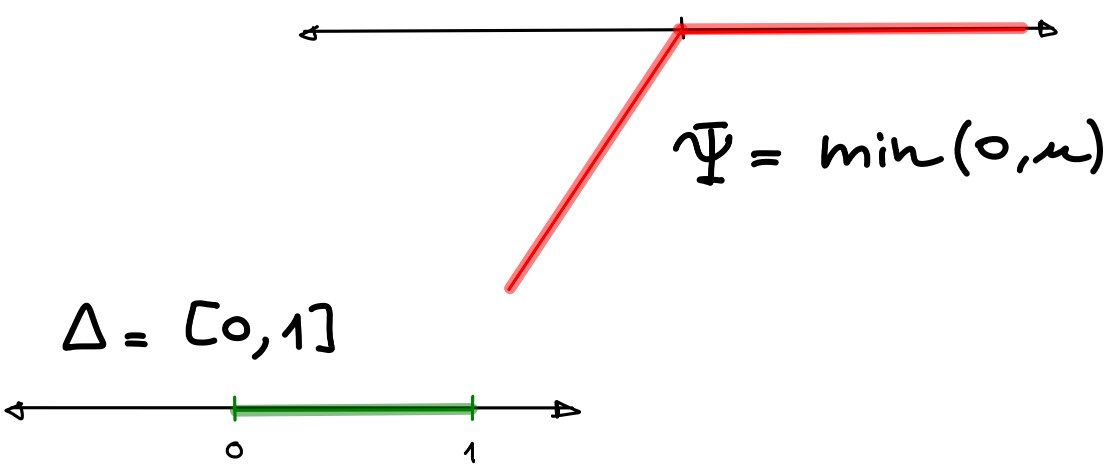
- X_Σ compact $\Leftrightarrow$ Σ complete: $\bigcup_{\sigma\in\Sigma}\sigma = \mathbb{R}^n$

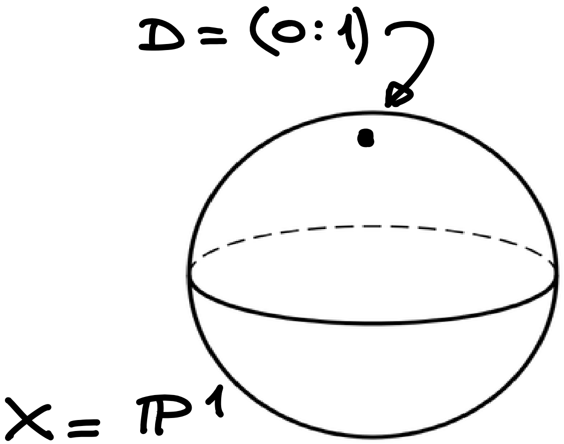
Toric Divisors

- toric divisor on X_Σ := an equivariant Cartier divisor
- virtual support function on Σ := a piecewise linear function $\Psi : \mathbb{R}^n \to \mathbb{R}$ defined on Σ

Ψ vsf $\longrightarrow D_\Psi$ toric divisor is also 1:1 !

$\exists$ polytope Δ_Ψ := sup-differential at 0 $\subset (\mathbb{R}^n)^\vee$





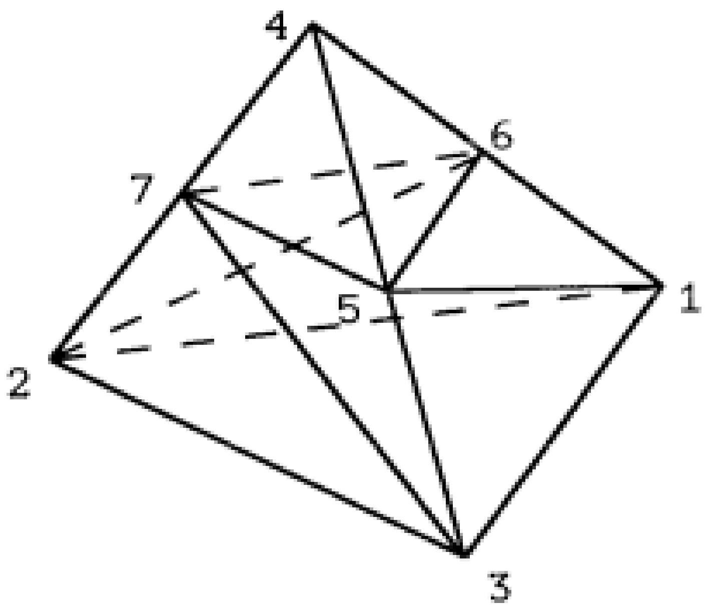
Positivity

- $\mathcal{O}(D_\Psi)$ globally generated (gg) $\Leftrightarrow$ Ψ concave

  associated line bundle

- $\mathcal{O}(D_\Psi)$ ample $\Leftrightarrow$ Ψ "strictly" concave

A non-projective toric threefold



INTERSECTION THEORY

Let D_i Cartier divisor on X, $i = 1, \dots, n = \dim(X)$

$$\deg_{D_1, \dots, D_n}(X) := \#\left(X \cap Z(s_1, \dots, s_n)\right)$$

with s_i (sufficiently general) rtl section of $\mathcal{O}(D_i)$

Thm (BKK revisited, Teissier 1979)

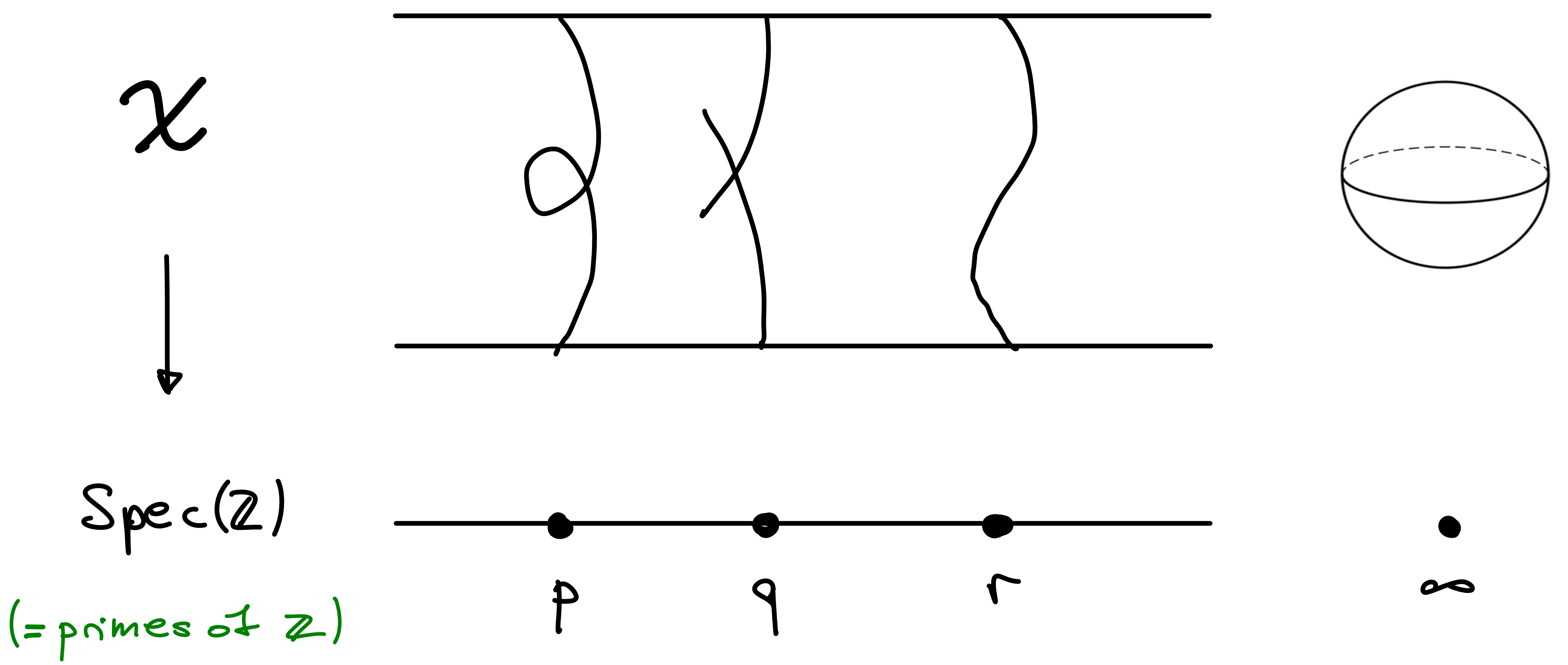
Let D_i gg toric divisor on a toric variety X_Σ given by a vsf Ψ_i, $i = 1, \dots, n$. Then

$$\deg_{D_1, \dots, D_n}(X) = MV(\Delta_{\Psi_1}, \dots, \Delta_{\Psi_n})$$

Arakelov geometry

Arakelov 1975, Faltings 1983, Gillet-Soulé 1990,

Zhang 1995, Chambert-Loir 2004, Gubler 2005, ...



Analytifications

- $\mathcal{M}_{\mathbb{Q}} := \{\text{absolute values on } \mathbb{Q}\}/_{\sim} = \{\infty\} \cup \{\text{primes of } \mathbb{Z}\}$

  places of $\mathbb{Q}$

- For X proper algebraic variety$/\mathbb{Q}$ and $v \in \mathcal{M}_{\mathbb{Q}}$ let

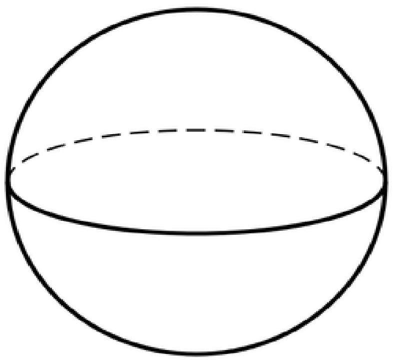
X_v v-adic analytification

$X(\mathbb{C})$

$v = \infty$

Berkovich space

$v \neq \infty$



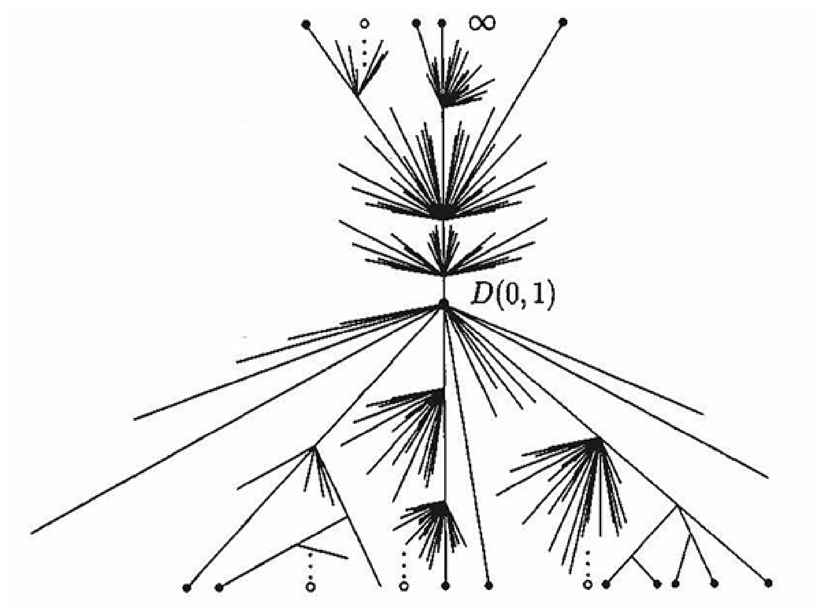


Figure by Joe Silverman

METRIZED DIVISORS AND HEIGHTS

Metrized divisor on X := $\overline{D} = (D, (\|\cdot\|_v)_{v \in \mathfrak{M}_{\mathbb{Q}}})$ with

- D Cartier divisor on X
- $\|\cdot\|_v$ metric on $\mathcal{O}(D)_v$

The height of $p \in X(\overline{\mathbb{Q}})$ w.r. to $\overline{D}$ is

$$h_{\overline{D}}(p) = -\sum_v \log \|s(p)\|_v$$

for a rtl section s of $\mathcal{O}(D)$ regular and $\neq 0$ at p

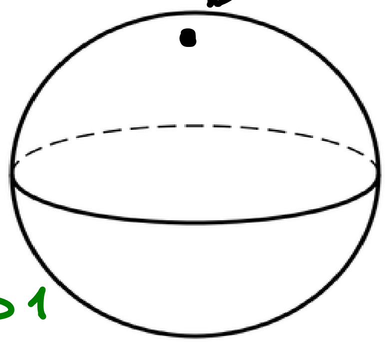
Extends to subvarieties and semipositive metrized divisors

$$h_{\overline{D}_0, \dots, \overline{D}_r}(Y)$$

satisfying the arithmetic Bézout theorem

Some examples

$D = (0:1)$



$X = \mathbb{P}^1$

ℓ linear form in x_0, x_1

$\updownarrow$ 1:1

s_ℓ global section of $\mathcal{O}(D)$

- $\|s_\ell(p)\|_v = \dfrac{|\ell(p)|}{\max(|p_0|, |p_1|)}$ v-adic canonical metric

$\rightsquigarrow \overline{D}^{can}$

- $\|s_\ell(p)\|_v = \begin{cases} \dfrac{|\ell(p)|_v}{(|p_0|_v^2 + |p_1|^2)^{1/2}} & v = \infty \\ \text{canonical} & v \neq \infty \end{cases}$ Fubini-Study metric

$\rightsquigarrow \overline{D}^{FS}$

Height of points

Classically

height = arithmetic complexity (bit length)

Northcott, Weil ~1940

For $p = \left(1 : \frac{a}{b}\right) \in \mathbb{P}^1(\mathbb{Q})$ with $a, b \in \mathbb{Z}$ coprime

$$h_{\overline{D}^{can}}(p) = \sum_{v} \log \max(|a|_v, |b|_v)$$

$$= \log \max(|a|_\infty, |b|_\infty) = h_{\mathrm{Weil}}(p) \quad \text{Weil height}$$

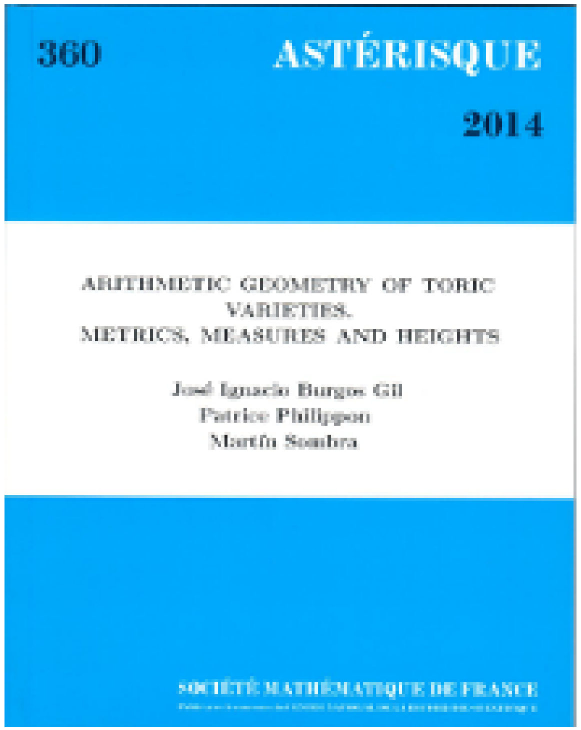
Ex: $\dfrac{\text{Start of BGSMath}}{\text{This workshop}} \quad \dfrac{2012}{2019} =: \xi$

then $|\xi|_\infty \approx 1$ but $|\xi|_{673} = 673$

Arakelov geometry of toric varieties

Pb: extend the toric dictionnary to Arakelov geometry

The fan and the vsf do not characterize all possible structures (e.g. $\overline{D}^{can}$ vs $\overline{D}^{FS}$)



Toric metrics

Let X toric variety and D toric divisor

- $\|\cdot\|_v$ v-adic toric metric on D := $\mathbb{S}_v$-invariant

compact torus of $\mathbb{T}_v$

$$\begin{tikzcd}
p \in \mathbb{T}_v \arrow[d, mapsto] \arrow[drr, "-\log\|s_D\|_v"] & & \\
\operatorname{Log}|p|_v \in \mathbb{R}^n \arrow[rr, "f_v"'] \arrow[ur, phantom, "\circlearrowright"] & & \mathbb{R}
\end{tikzcd}$$

v-adic metric function

Classifying toric metrics

Thm (BPS 2014)

$|f_v - \Psi|$ bounded and $\|\cdot\|_v$ is semipositive $\Longleftrightarrow$ f_v concave

If $\|\cdot\|_v$ is SP, its v-adic roof function is the Legendre-Fenchel dual of f_v

$$\vartheta_v = f_v^\vee : \Delta_\Psi \longrightarrow \mathbb{R}$$

Thm (BPS 2014)

$$\left\{\begin{array}{l} v\text{-adic toric} \\ \text{SP metrics on } \mathcal{O}(D)_v \end{array}\right\}$$

1:1

$$\left\{\begin{array}{l} f : \mathbb{R}^n \to \mathbb{R} \ \text{ concave} \\ \text{s.t. } |f - \Psi| \text{ bounded} \end{array}\right\}$$

1:1

1:1

$$\left\{\begin{array}{l} \vartheta : \Delta_\Psi \longrightarrow \mathbb{R} \\ \text{concave and continuous} \end{array}\right\}$$

THE HEIGHT OF A TORIC VARIETY

Thm (BPS 2014)

Let $\overline{D}_i$ toric SP metrized divisor on X, $i=0,\dots,n+1$, then

$$h_{\overline{D}_0,\dots,\overline{D}_n}(X) = \sum_v \mathrm{MI}(\vartheta_{0,v},\dots,\vartheta_{n,v})$$

mixed integrals of v-adic roof functions

If $\overline{D}_0 = \dots = \overline{D}_n = \overline{D}$ then

$$h_{\overline{D}}(X) = (n+1)! \sum_v \int_{\Delta_\Psi} \vartheta_v \, d\underline{x}$$

An abridged toric dictionary

X toric variety with torus $\mathbb{T}$	Σ fan on $\mathbb{R}^n$
D toric gg divisor	$\Psi : \mathbb{R}^n \to \mathbb{R}$ concave Σ-linear $\Delta \subset \mathbb{R}^n$ lattice polytope
$\deg_D(X)$	$n! \operatorname{vol}(\Delta)$
$\overline{D}$ toric SP metrized divisor	$(\psi_v : \mathbb{R}^n \to \mathbb{R})_{v \in \mathfrak{M}}$ concave s.t. $\lvert \psi_v - \Psi \rvert$ bounded $\forall v$ and $\psi_v = \Psi$ $\forall^0 v$ $(\vartheta_v : \Delta \to \mathbb{R})_{v \in \mathfrak{M}}$ concave s.t. $\vartheta_v \equiv 0$ $\forall^0 v$
$h_{\overline{D}}(X)$	$(n+1)! \sum_v \int_\Delta \vartheta_v \, dx$

Height of hypersurfaces

$$f \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad \begin{array}{l} \nearrow \; Z(f) \subset X \text{ hypersurface} \\ \searrow \; N(f) \subset \mathbb{R}^n \text{ Newton polytope} \end{array}$$

$$\Rightarrow \quad \deg_{D_1, \dots, D_{n-1}}(Z(f)) = MV(\Delta_1, \dots, \Delta_{n-1}, N(f))$$

For $v \in \mathfrak{M}$ we consider the v-adic Ronkin function and its dual

$$\rho_{f,v} : \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \rho^\vee_{f,v} : N(f) \longrightarrow \mathbb{R}$$

Thm (Gualdi 2018)

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z(f)) = \sum_v MI(\vartheta_{0,v}, \dots, \vartheta_{n-1,v}, \rho^\vee_{f,v})$$

Higher codimension

For $\zeta_3 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$ and $\zeta_4 = i$ set

$$f = 1 + x_1 + x_2, \qquad g = 1 + \zeta_3^2 x_1 + \zeta_3 x_2, \qquad g' = 1 - x_1 + \zeta_4 x_2$$

Then $\rho_{f,v} = \rho_{g,v} = \rho_{g',v}$ $\forall v$

but $h_{\overline{D}_{\mathrm{can}}}(Z(f,g)) = 0 \neq \frac{\log 2}{2} = h_{\overline{D}_{\mathrm{can}}}(Z(f,g'))$

A clue

"While the individual man is an insoluble puzzle, in the aggregate he becomes a mathematical certainty. You can, for example, never foretell what any one man will do, but you can say with precision what an average number will be up to"

— Arthur Conan Doyle, The sign of the Four

Average heights

- For $d \geq 1$ set $\tau_d := \{\omega \in \mathrm{Tors}(\mathbb{T}) \mid \omega^d = 1\}$
- For $\omega \in \mathrm{Tors}(\mathbb{T})$ set

$$\eta(\omega) = h_{\overline{\mathcal{D}}^{\mathrm{can}}}\big(Z(1 + x_1 + x_2,\ 1 + \omega_1 x_1 + \omega_2 x_2)\big)$$

Thm (Gualdi, S 2019)

expected value

$$\lim_{d \to \infty} \mathbb{E}(\eta \mid \tau_d) = \sum_v \mathrm{MI}\left(\vartheta_v, \rho^\vee_{f,v}, \rho^\vee_{g,v}\right)$$

$$= \frac{2\,\zeta(3)}{3\,\zeta(2)}$$

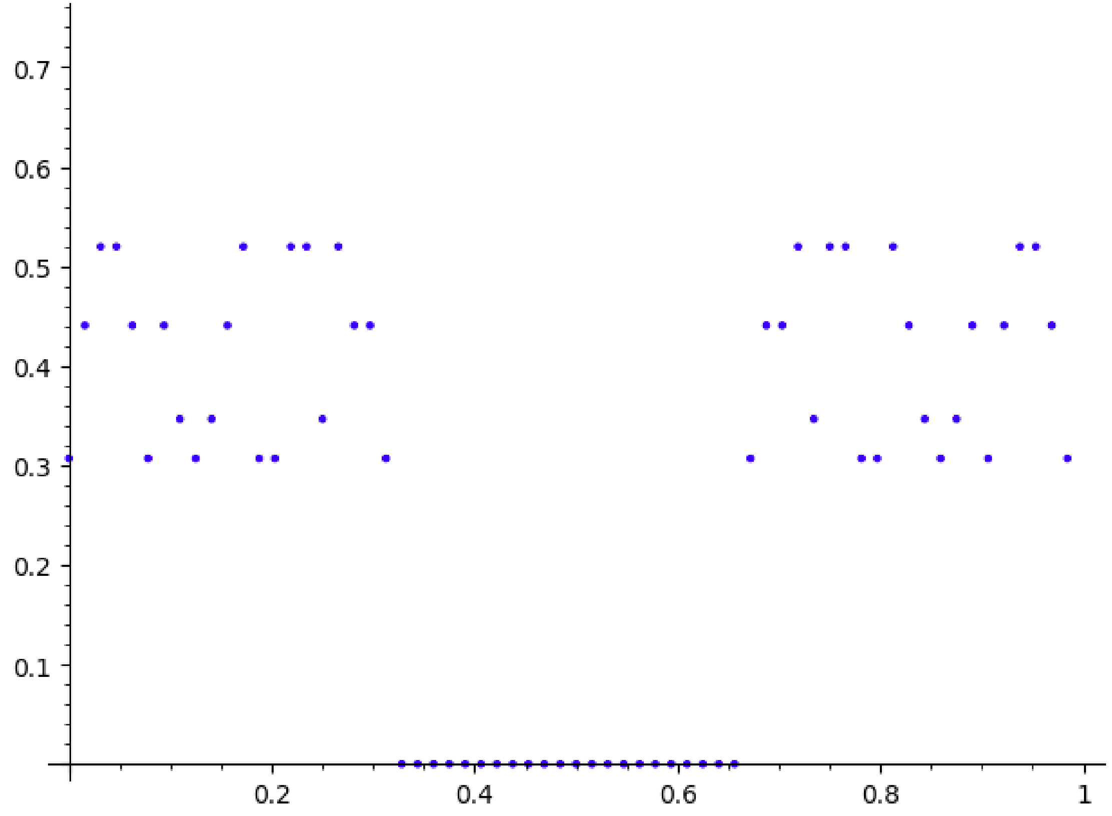
$$\zeta(s) = \sum_{n \geq 1} n^{-s} \quad (\mathrm{Re}(s) > 1)$$

$$= 0{,}487175\ldots$$

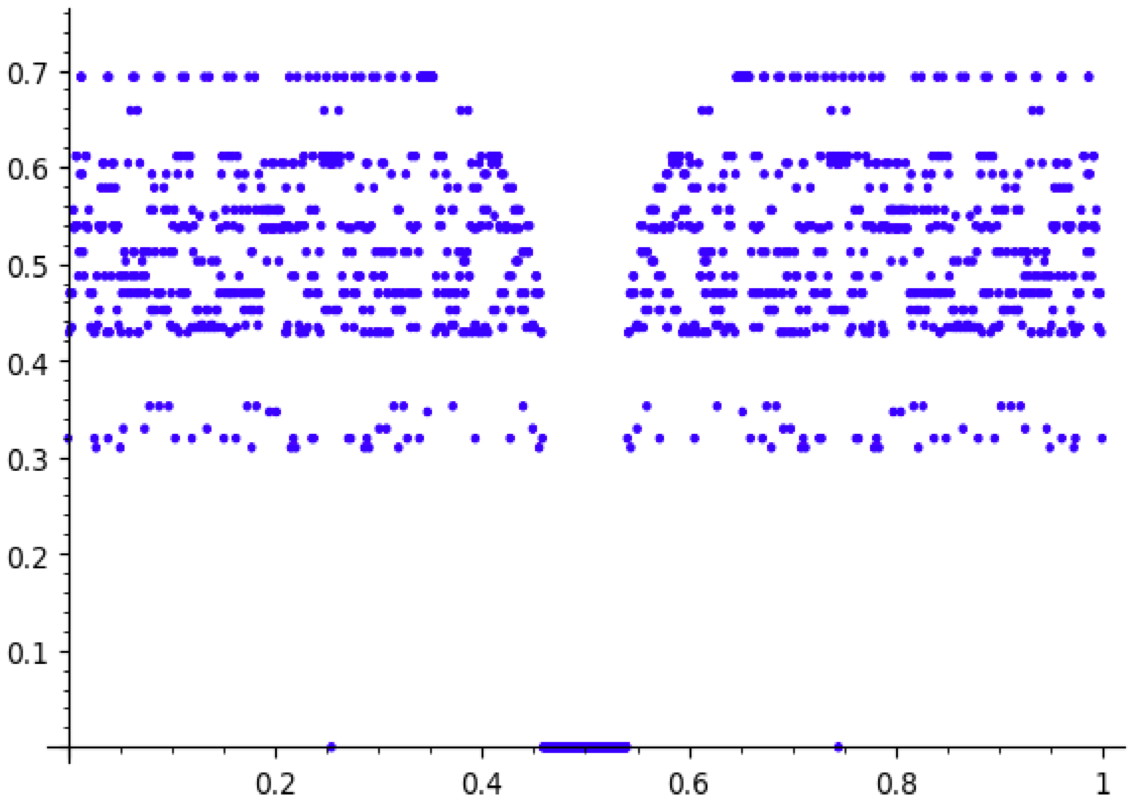
Moreover, the heights concentrate around the expected value

Some experiments

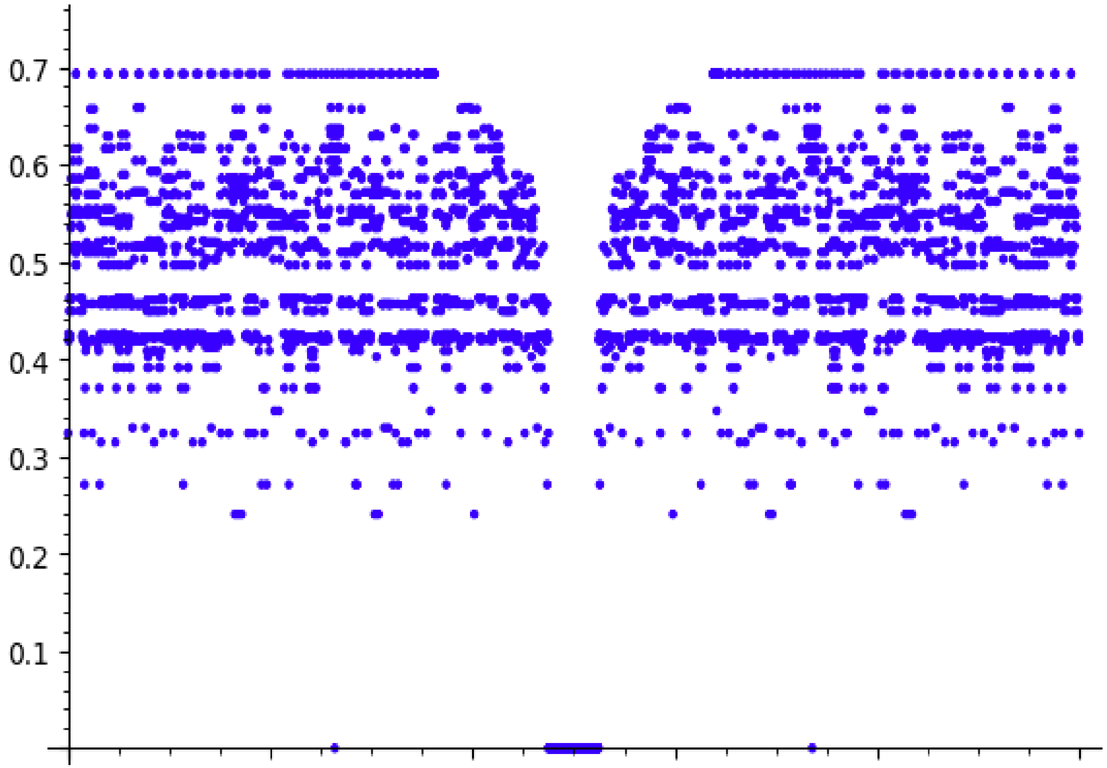
```
d =  8
average:  0.27015389
```



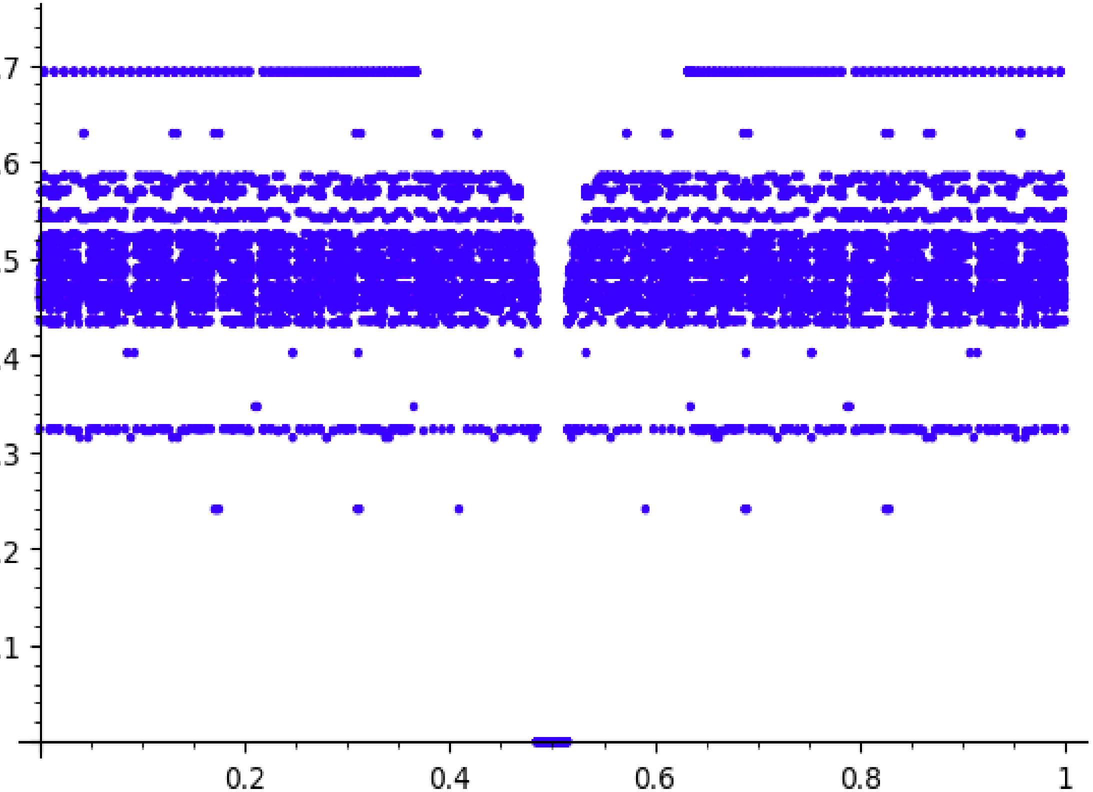
```
d =  36
average:  0.46991504
```



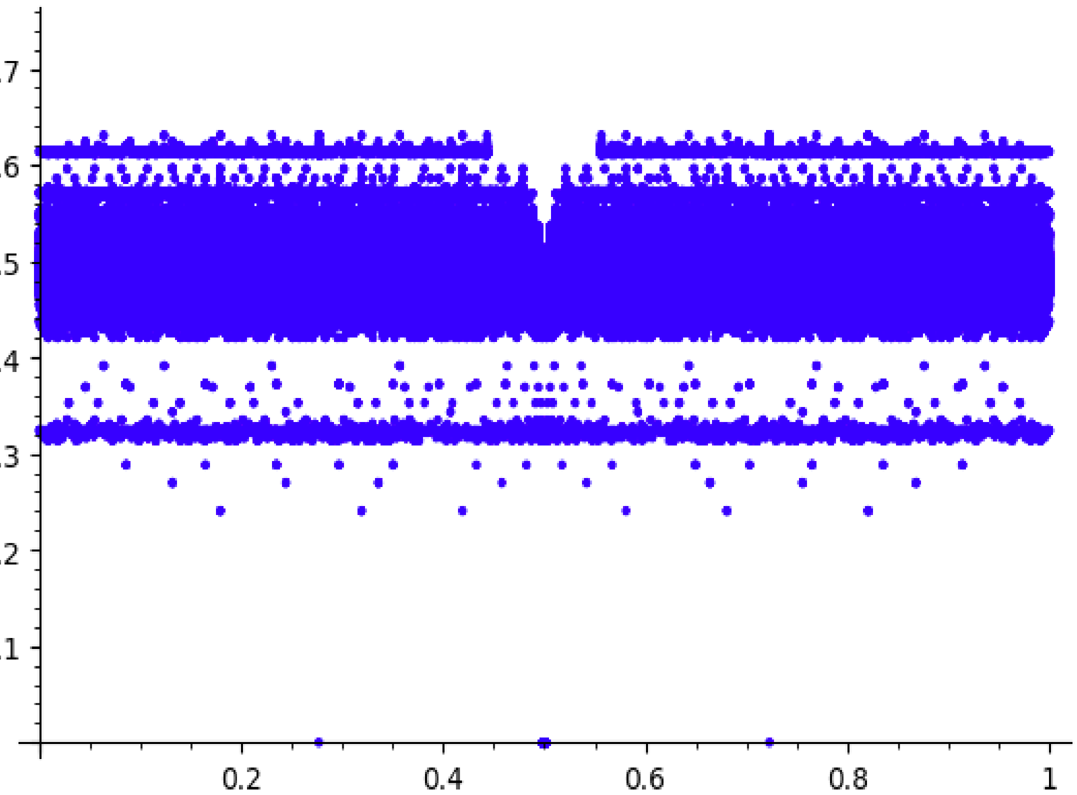
```
d =  60
average:  0.48201223
```



d = 100
average: 0.48095365



```
d =  1155
average:  0.48713620
```



TOWARDS AN ARITHMETIC BKK THEOREM

$f_i \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}, \quad i = 1, \dots, n$

$\overline{D}$ toric SP metrized divisor on X

For $\omega \in \mathrm{Tors}(\mathbb{T})^n$ set

$$\eta(\omega) := \begin{cases} h_{\overline{D}}(Z(\omega^* f)) & \text{if } Z(\omega^* f) \text{ finite} \\ 0 & \text{else} \end{cases}$$

Conj. $\lim_{d \to \infty} \mathbb{E}(\eta|_{T_d}) = \sum_v \mathrm{MI}(\vartheta_v, \rho_{f_1,v}^\vee, \dots, \rho_{f_n,v}^\vee)$

and $\eta|_{T_d}$ should concentrate around the expected value

Status. Work in progress with Gualdi, verified in several situations ✓

THANKS !

Measuring the Solution set of a system of polynomial equations

Martín Sombra (ICREA & UB)

BGSMath MdM closing workshop

Barcelona, June 6th 2019