Approximation of adelic divisors and equidistribution of small points

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Arithmetic geometry at Cabourg

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* K number field
* X projective variety $/K$ of dimension $d \geq 1$
* $\overline{D}$ adelic divisor on X

$\rightsquigarrow \quad h_{\overline{D}} : X(\overline{K}) \longrightarrow \mathbb{R}$ height function

I. The equidistribution property

$$\overline{D} = \left(D\,,\,(g_v)_{v\in M_K}\right) \in \widehat{\mathrm{Div}}(X)$$

Cartier divisor on X

v-adic Green function

$$g_v : X_v^{an} \setminus |D| \longrightarrow \mathbb{R}$$

For $x \in (X \setminus |D|)(\overline{K})$

$$h_{\overline{D}}(x) = \sum_{v \in M_K} \frac{1}{\# G(x)_v} \sum_{y \in G(x)_v} g_v(y)$$

v-adic Galois orbit

Two basic quantities:

$$\mu^{abs}(\bar{D}) = \inf_{x \in X(\bar{K})} h_{\bar{D}}(x)$$ absolute minimum

$$\mu^{ess}(\bar{D}) = \sup_{Y \subsetneq X} \inf_{x \in (X \setminus Y)(\bar{K})} h_{\bar{D}}(x)$$ essential minimum

EXAMPLE: $X = (x+y=1) \subset \mathbb{A}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

$\mu^{abs} = 0$ $\qquad 0{,}2481 \leq \mu^{ess} \leq 0{,}2539$

Zagier 1995 $\qquad$ Doche 1998

Let $(x_\ell)_\ell$ generic sequence in $X(\bar{K})$

$$\rightarrow \liminf_\ell h_{\bar{D}}(x_\ell) \geq \mu^{ess}(\bar{D})$$

DEF: $(x_\ell)_\ell$ $\bar{D}$-small if $\lim_\ell h_{\bar{D}}(x_\ell) = \mu^{ess}(\bar{D})$

DEF: $\bar{D}$ satisfies the equidistribution property at $v \in M_K$ if $\exists$ probability measure ν_v on X_v^{an} s.t. $\forall (x_\ell)_\ell$ $\bar{D}$-small generic sequence in $X(\bar{K})$

$$\lim_\ell \delta_{O(x_\ell)_v} = \nu_v$$

uniform probability measure on $O(x_\ell)_v$

AET (Yuan 2008) Assume D big, $\overline{D}$ semipositive and

$$\mu^{ess}(\overline{D}) = \mu^{abs}(\overline{D})$$

Then $\overline{D}$ satisfies EP for all $v \in M_K$ with $\nu_v = \dfrac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}$

normalized v-adic Monge-Ampère measure of $\overline{D}$

Application:

$\varphi : X \to X$ dynamical system

$D \in \mathrm{Div}(X)$ ample s.t. $\varphi^* D \equiv q D$ $(q > 1)$

$\rightsquigarrow$ $\overline{D}^{can}$ canonical adelic divisor on D

$h_{\overline{D}^{can}}(x) \geq 0$ $\forall x \in X(\overline{K})$

$= 0 \iff x$ preperiodic

$\rightsquigarrow$ $\mu^{ess}(\overline{D}^{can}) = \mu^{abs}(\overline{D}^{can}) = 0$

ν_v v-adic equilibrium measure of φ

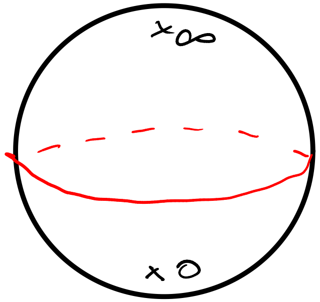
EXAMPLES $\quad X=\mathbb{P}^1 \quad D=(x_0=0)$

(1) $g_v(z)=\log^+|z|_v \quad \forall v \quad \rightsquigarrow \quad h(x)=\log\max(|x_0|_\infty,|x_1|_\infty)$

for $x=(x_0:x_1)\in\mathbb{P}^1(\mathbb{Q})$ with $x_0,x_1\in\mathbb{Z}$ coprime

$\mu^{abs}=\mu^{ess}=0$

ν_∞ Haar proba measure on S^1



(2) $g_\infty(z)=\log(1+|z|_\infty)^{1/2} \quad \rightsquigarrow \quad h(x)=\log(|x_0|_\infty^2+|x_1|_\infty^2)^{1/2}$

$\mu^{abs}=0 \qquad \mu^{ess}=\frac{\log 2}{2}$

EP with ν_∞ as before

(Burgos, Philippon, Rivera-Letelier, S 2019)

(3) $g_\infty(z)=\log^+(2|z|_\infty) \quad \rightsquigarrow \quad h(x)=\log\max(|x_0|_\infty,2|x_1|_\infty)$

$\mu^{abs}=0 \qquad \mu^{ess}=\log 2$

No EP

$x_\ell=(1:\omega_\ell) \quad y_\ell=(1:\omega_\ell/2)$ with ω_ℓ root of 1

$\rightsquigarrow h(x_\ell)=h(y_\ell)=\log 2 \quad (\forall \ell)$

II. THE MAIN RESULT

From now on: X normal projective variety $/K$

$\overline{D}$ adelic $\mathbb{R}$-divisor on X with D big

DEF: A semipositive approximation of $\overline{D}$ is $(\phi, \overline{Q})$ with

* $\phi : X' \longrightarrow X$ normal modification
* $\overline{Q} \in \widehat{\mathrm{Div}}(X')_{/\mathbb{R}}$ semipositive with Q big s.t.

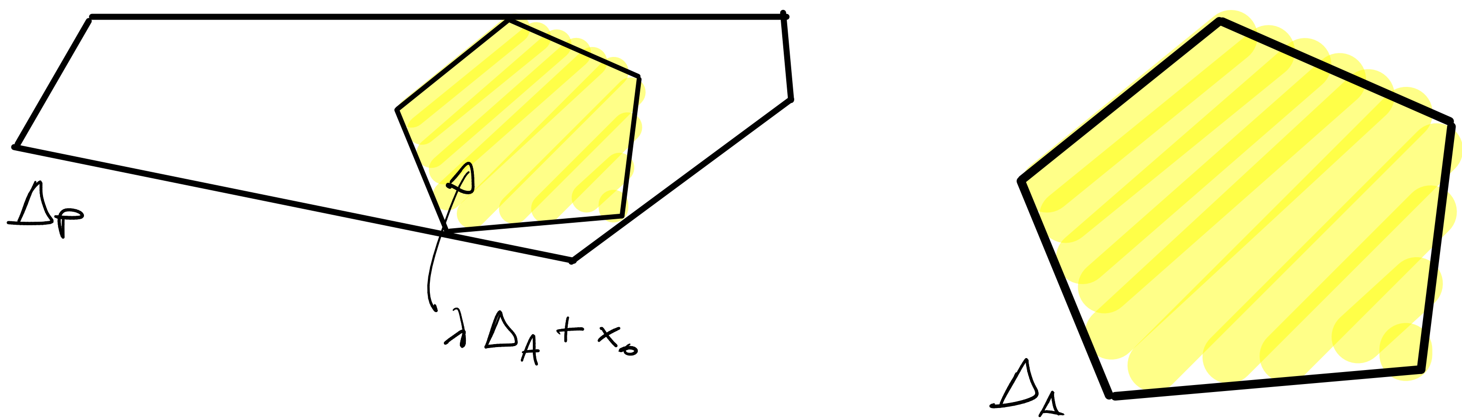
$$\phi^*\overline{D} - \overline{Q} \quad \text{psef}$$

Def: For $P, A \in \mathrm{Div}(X)_{\mathbb{R}}$ big

$$r(P;A) = \sup\{\lambda \in \mathbb{R} \mid P - \lambda A \text{ big}\}$$ inradius

Teissier 1982

If P, A toric $\rightsquigarrow$ $r(P;A) =$ inradius of Δ_P w.r.t. Δ_A



THM (Ballaÿ–S)

Assume $\exists$ $(\phi_n, \overline{Q}_n)$ semipositive approximations of $\overline{D}$ st

$$\lim_n \frac{\mu^{ess}(\overline{D}) - \mu^{abs}(\overline{Q}_n)}{r(Q_n; D)} = 0$$

For $v \in M_K$ set $\nu_{v,n} = \dfrac{c_1(\overline{Q}_{n,v})^{\wedge d}}{(Q_n^d)}$ probability measure on X_v^{an}

Then $\overline{D}$ satisfies EP with $\nu_v = \lim_n \nu_{v,n}$

REM.1 if $\overline{D}$ semipositive and $\mu^{ess}(\overline{D}) = \mu^{abs}(\overline{D})$ can take

$\overline{Q}_n = \overline{D}$ $\forall n$ $\leadsto$ AET

III. Sums of canonical divisors

Let $\varphi : X \to X$ dynamical system

$D_1, \dots, D_s \in \mathrm{Div}(X)_{\mathbb{R}}$ st

* $\varphi^* D_i \equiv q_i D_i \quad (q_i > 1)$
* D_i effective and $\overline{D}_i^{can}$ semipositive
* $D = D_1 + \dots + D_s$ ample

EXAMPLE:

$$0 \to \mathbb{G}_m^r \to G \xrightarrow{\pi} A \to 0$$

semiabelian variety

$\overline{D} = \overline{M}^{can} + \pi^* \overline{N}^{can}$ on $\overline{G}$

$$[2]^* M \equiv 2M \qquad [2]^* \pi^* N \equiv 4 \pi^* N$$

We have $\quad \mu^{abs}(\overline{D}) < \mu^{ess}(\overline{D}) = 0$ unless G isotrivial

THM (Ballay - S)

$\overline{D}$ satisfies EP for all $v \in M_K$ with $\nu_v = \dfrac{c_1(\overline{D}_v)^{\wedge d}}{(D^d)}$

Pf: Suppose $1 < q_1 < \cdots < q_s$ and set

$$\overline{Q}_n = \frac{1}{q_s^n} (\phi^n)^* \overline{D} = \left(\frac{q_1}{q_s}\right)^n \overline{D}_1 + \cdots + \left(\frac{q_s}{q_s}\right)^n \overline{D}_s$$

We have (i) $\overline{Q}_n$ semipositive approximation of $\overline{D}$

(ii) $r(Q_n, D) \geq \left(\frac{q_1}{q_n}\right)^n$

(iii) $\mu^{abs}(\overline{Q}_n) = \dfrac{\mu^{abs}(\overline{D})}{q_s^n}$

Hence

$$0 \leq \frac{\mu^{ess}(\overline{D}) - \mu^{abs}(\overline{Q}_n)}{r(Q_n; D)} \leq \frac{-\mu^{abs}(\overline{D})}{q_1^n} \xrightarrow[n \to \infty]{} 0 \qquad \boxtimes$$

IV. THE PROOF ($d=1$)

* X smooth projective curve $/K$
* $\overline{D} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$ with D big
* $(\overline{Q}_n)_n$ SP approximations of $\overline{D}$ s.t. $\lim_n \dfrac{\mu^{ess}(\overline{D}) - \mu^{abs}(\overline{Q}_n)}{(Q_n)} = 0$

cor $\quad r(Q_n; D) \asymp (Q_n) \qquad$ Siu bigness thm

Fix $v \in M_K$ and set $\nu_{v,n} = \dfrac{c_1(\overline{Q}_{n,v})}{(Q_n)} \qquad n \in \mathbb{N}$

Let $(x_\ell)_\ell$ $\overline{D}$-small sequence in $X(\overline{K})$

Aim: show that $\forall \varphi \in C^0(X_v^{an})$

$$\lim_{\ell} \int_{X_v^{an}} \varphi \, d\delta_{O(x_\ell)_v} = \lim_n \int_{X_v^{an}} \varphi \, d\nu_{v,n}$$

Standard reductions:

* φ $\mathrm{Gal}(\overline{K_v}/K_v)$-invariant

$\rightsquigarrow$ $E := \overline{O}^{\varphi} \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$

* $\overline{O}^{\varphi}$ DSP

$\rightsquigarrow$ LHS: $\lim_{\ell} h_E(x_\ell)$ RHS: $\lim_n \dfrac{(\overline{Q}_n \cdot E)}{(Q_n)}$

Set

$$F(\lambda) = \liminf_{\ell} h_{\overline{D}+\lambda\overline{E}}(x_\ell) = \mu^{ess}(\overline{D}) + \liminf_{\ell} \lambda\, h_{\overline{E}}(x_\ell)$$

$(x_\ell)_\ell$ $\overline{D}$-small

then for $\lambda > 0$

$$F(\lambda) \geq \mu^{ess}(\overline{D} + \lambda\overline{E})$$

$(x_\ell)_\ell$ generic

$$\geq \mu^{ess}(\overline{Q}_n + \lambda\overline{E})$$

$\overline{D} - \overline{Q}_n$ psef

$$\geq \frac{\widehat{\mathrm{vol}}_\chi(\overline{Q}_n + \lambda\overline{E})}{2\,(Q_n)}$$

Zhang's inequality

$$\geq \frac{(\overline{Q}_n^2) + 2\lambda(\overline{Q}_n \cdot \overline{E}) - c\lambda^2}{2(Q_n)}$$

arithmetic Siu bigness (Yuan)

$$\geq \mu^{abs}(\overline{Q}_n) + \lambda\frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)} - \frac{c\lambda^2}{(Q_n)}$$

Zhang's successive minima

Hence

$$\liminf_{\ell} h_{\overline{\mathcal{O}}_\varphi}(x_\ell) \geq \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)} - \frac{\mu^{ess}(\overline{D}) - \mu^{abs}(\overline{Q}_n)}{\lambda} - \frac{c\lambda^2}{(Q_n)}$$

$$\geq \limsup_n \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)}$$

optimize for λ and take $\limsup_n$

Hence

$$\liminf_n \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)} \geq \limsup_\ell h_{\overline{E}}(x_\ell) \geq \liminf_\ell h_{\overline{E}}(x_\ell) \geq \limsup_n \frac{(\overline{Q}_n \cdot \overline{E})}{(Q_n)}$$

$\boxtimes$

Thanks!