# Kähler-Einstein toric submanifolds of the projective space

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# Embeddings into complex space forms

Let $(X, \omega)$ be a Kähler manifold of dimension $n \geqslant 1$.

The basic examples *(complex space forms)*:

- $(\mathbb{H}^n, \omega_{\mathsf{hyp}})$ hyperbolic space
- $(\mathbb{E}^n, \omega_{\mathsf{eucl}})$ Euclidean space
- $(\mathbb{P}^n, \omega_{\mathsf{FS}})$ projective space

**Question**

*Can $(X, \omega)$ be embedded into a given complex space form?*

The same question for

- differentiable manifolds: yes (Withney 1936)
- Riemannian manifolds: yes (Nash 1956)

# Curves

For instance let $E = \mathbb{C}/\Lambda$ an elliptic curve with its flat metric. Can $E$ be isometrically embedded into a $\mathbb{P}^N$?

## Definition

The Kähler manifold $(X, \omega)$ is *projectively induced* if there exists $\varphi\colon X \to \mathbb{P}^N$ s.t. $\omega = \varphi^*\omega_{\mathsf{FS}}$.

## Theorem (Calabi 1950)

*Let $(X, \omega)$ be a curve with constant scalar curvature that is projectively induced. Then $(X, \omega) \simeq (\mathbb{P}^1, k\,\omega_{\mathsf{FS}})$.*

Indeed $(\mathbb{P}^1, k\,\omega_{\mathsf{FS}})$ is induced by the Veronese map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^k, \quad (z_0 : z_1) \longmapsto \left(\binom{k}{j}^{1/2} z_0^{k-j} z_1^j\right)_j$$

# Kähler-Einstein manifolds

We say that $(X, \omega)$ is *Kähler-Einstein* if there exists $\lambda \in \mathbb{R}$ s.t.

$$\mathrm{Ric}(\omega) = \lambda\,\omega \qquad \text{(Ricci form)}$$

**Example**

$(\mathbb{P}^n, \omega_{\mathsf{FS}})$ is KE with $\lambda = n + 1$.

**Conjecture**

*Let $(X, \omega)$ be a projectively induced KE (compact) manifold. Then $(X, \omega)$ is a homogeneous space.*

Known cases:

- $N = n + 1$ (Chern 1967)
- $N = n + 2$ (Tsukada 1986)

# Toric manifolds

Let $(X, \omega)$ be a *toric* Kähler manifold:

- $(\mathbb{C}^\times)^n$ acts on $X$ with a dense orbit $X_0 \simeq (\mathbb{C}^\times)^n$
- $\omega$ is invariant under the action of $(S^1)^n$

## Conjecture

*Let $(X, \omega)$ be a projectively induced KE toric Fano manifold. Then*

$$(X, \omega) \simeq \left( \prod_{i=1}^{r} \mathbb{P}^{n_i}, \bigoplus_{i=1}^{r} k_i \operatorname{pr}_i^* \omega_{FS} \right)$$

Known (under hypothesis) for $n \leqslant 4$ using the classification of Fano polytopes (Arezzo, Loi and Zuddas 2012).

# KE metrics on toric manifolds

Let $X$ be a toric Fano manifold. Its *anticanonical polytope*

$$\Delta_{-K_X} \subset \mathbb{R}^n$$

is a unimodular reflexive polytope. Up to a translation

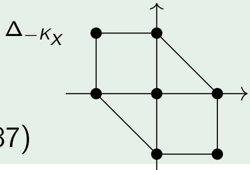
$$\Delta^{\circ}_{-K_X} \cap \mathbb{Z}^n = \{0\}.$$

**Theorem (Wang and Zhu 2004)**

*$X$ admits a KE metric iff the baycenter of $\Delta_{-K_X}$ vanishes.*

**Example**

Let $X$ be the blow up of $\mathbb{P}^2$ at $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.




It admits a KE metric (Tian and Yau 1987)

# Toric potentials

Let $(X, \omega)$ be a toric Kähler manifold and $\varphi \colon X \to \mathbb{P}^N$ a toric map s.t. $\omega = \varphi^* \omega_{\mathsf{FS}}$. It writes down as

$$\varphi(x) = (\alpha_0 x^{m_0} : \cdots : \alpha_N x^{m_N})$$

with $\alpha_j \in \mathbb{C}^\times$ and $m_j \in \mathbb{Z}^n$.

The *toric potential* of $\omega$ is the $n$-variate Laurent polynomial

$$p_\varphi = \sum_{j=0}^{N} |\alpha_j|^2 x^{m_j} \in \mathbb{R}_{>0}[x^{\pm 1}]$$

Setting $f_\varphi(u + iv) = \log(p_\varphi)(e^{2u})$ we have

$$\omega = i\partial\overline{\partial} f_\varphi \qquad \text{on } \mathbb{C}^n / 2\pi\mathbb{Z}^n \simeq (\mathbb{C}^\times)^n$$

# The algebraic Einstein condition

Let $D_i = x_i \dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, and for $p \in \mathbb{R}[x^{\pm 1}]$ set

$$\mu(p) = p^{n+1} \det(D^2 \log(p)) \in \mathbb{R}[x^{\pm 1}]$$

with $D^2 \log(p) = (D_i D_j \log(p))_{i,j}$ the $D$-Hessian matrix of $\log(p)$.

**Proposition (DS 25)**

*A Fano toric manifold $X$ admits a projectively induced KE metric iff there exists $p \in \mathbb{R}_{>0}[x^{\pm 1}]$ s.t.*

$$\mathrm{NP}(p) = \Delta_{-K_X} \quad \textit{and} \quad \mu(p) = p^n$$

The equation $\mu(p) = p^n$ boils down to a system over $\mathbb{R}_{>0}$ with

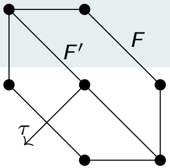
$$\#(n\Delta_{-K_X} \cap \mathbb{Z}^n) \text{ equations in } \#(\Delta_{-K_X} \cap \mathbb{Z}^n) \text{ variables}$$

# The toric adjuntion formula

**Theorem (DS 25)**

*Let $p \in \mathbb{R}[x^{\pm 1}]\backslash\{0\}$ and $\tau$ the inner normal direction to a facet of $\mathrm{NP}(p)$. Then*

$$\mathrm{init}_\tau(\mu(p)) = \mu(\mathrm{init}_\tau(p)) \cdot p|_{F'_\tau}$$




**Definition**

A Laurent polynomial $p$ satisfies the *generalized Einstein condition* if

$$\mu(p) \mid p^\kappa \quad \text{for some } \kappa \geqslant 0$$

GEC is hereditary on initial parts: if $p$ satisfies GEC so does $\mathrm{init}_\tau(p)$.

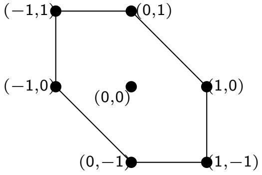
# Low dimensions

## Proposition (DS 25)

*Let*

$$p = \alpha_0 x_2^{-1} + \alpha_1 x_1 x_2^{-1} + \alpha_2 x_1^{-1} + \alpha_3 + \alpha_4 x_1 + \alpha_5 x_1^{-1} x_2 + \alpha_6 x_2$$

*with* $\alpha_j \neq 0$ *for all* $j \neq 3$*. Then* $p$ *does not satisfy GEC.*



# Large dimensions

A Fano toric manifold $X$ is *centrally symmetric* if $\Delta_{-K_X}$ is symmetric with respect to $0 \in \mathbb{R}^n$.

**Theorem (DS 2025)**

*Let $X$ be a centrally symmetric toric Fano manifold admitting a projectively induced KE metric. Then $X \simeq (\mathbb{P}^1)^n$.*

**Proof.**

A toric Fano manifold $X$ is centrally symmetric iff

$$X \simeq \prod_{i=1}^{r} X_i \quad \text{with } X_i = \mathbb{P}^1 \text{ or } V_k \text{ (\textit{del Pezzo} toric of dim } 2k)$$

(Klyachko and Voskrenskij 1987)

Now if $X$ admits a projectively induced KE metric then no $V_k$ can appear, since otherwise the hexagon would be a face of $\Delta_{-K_X}$. □

# Other cases

The conjecture is verified for all *known* toric Fanos, including

- Batyrev-Selivanova's families of symmetric toric Fanos
- Nill and Paffenholz non-symmetric examples

**Question**

*Let $X$ be a toric Fano that is not a product of projective spaces. Does $\Delta_{-K_X}$ have a 2-dimensional face that is not a triangle or a rectangle?*

Thanks!