Polynomials with multiples factors, unlikely intersections, & osculating spaces of toric curves

Martín Sombra

joint with F. Amoroso (Caen) & U. Zannier (Pisa)

POLYNOMIALS WITH MULTIPLES FACTORS

Q: Let $f \in \mathbb{Z}[t^{\pm 1}]$ sparse Laurent polynomial
When does f have a multiple root?

Here "sparse":
restriction of a fixed regular function on $\mathbb{G}_m^N = (\overline{\mathbb{Q}}^\times)^N$
to a varying 1-parameter subgroup

E.g. $N \geq 1$ $\quad \gamma_0, \gamma_1, \dots, \gamma_N \in \mathbb{Z}$ fixed
For $\underline{a} \in \mathbb{Z}^N$ set

$$f_{\underline{a}} = \gamma_0 + \gamma_1 t^{a_1} + \dots + \gamma_N t^{a_N}$$

restriction of

$$L = \gamma_0 + \gamma_1 x_1 + \dots + \gamma_N x_N$$

to $\quad \mathbb{G}^m \longrightarrow \mathbb{G}_m^N \quad t \mapsto (t^{a_1}, \dots, t^{a_N})$

Multiple roots in f_λ do appear when:

- $F = \gamma_0 + \gamma_1 y^{b_1} + \dots + \gamma_N y^{b_N} \in \mathbb{Z}[y_1, \dots, y_{N-k}]$

- $\underline{\theta} \in \mathbb{Z}^{N-k}$

st $\quad \langle b_i, \theta \rangle = \lambda_i$

$\exists\, P \in \mathbb{Z}[y]\,, \quad P^2 \mid F,$

$p = P(t^{\theta_1}, \dots, t^{\theta_{N-k}})$ not a monomial

$\Rightarrow \quad f_\lambda = F(t^{\theta_1}, \dots, t^{\theta_{N-k}})$

$p^2 \mid f_\lambda$

T1 Let $N \geq 1$ $\quad \gamma_0, \gamma_1, \ldots, \gamma_N \in \mathbb{Z}$. $\exists\, c = c(N, \underline{\gamma})$ st:

Let $\underline{a} \in \mathbb{Z}^N$ st

$$f_{\underline{a}} = \gamma_0 + \gamma_1 t^{a_1} + \cdots + \gamma_N t^{a_N}$$

has a multiple root $\xi \in \overline{\mathbb{Q}}^\times \setminus \mu_\infty$ roots of 1

$\Rightarrow \exists\ b_1, \ldots, b_N, \theta \in \mathbb{Z}^{N-k}$ st

(1) $\|b_i\| \leq c$, $\quad \|\theta\| \leq c\|a\|$ $(= \max_i |a_i|)$

(2) $\langle b_i, \theta \rangle = a_i \quad \forall i$

(3) $F = \gamma_0 + \gamma_1 y^{b_1} + \cdots + \gamma_N y^{b_N}$ has a multiple factor P st ξ root of $P(t^{\theta_1}, \ldots, t^{\theta_{N-k}})$

Cor: The set

$$\{\underline{a} \in \mathbb{Z}^N \mid f_{\underline{a}} \text{ has a multiple root in } \overline{\mathbb{Q}}^\times \setminus \mu_\infty\}$$

is contained in a finite union of proper linear subspaces

UNLIKELY INTERSECTIONS ON TORI

CONJECTURE (ZILBER 2002) $W \subset \mathbb{G}_m^N$

$\exists$ Ξ finite collection of proper algebraic subgroups st

$\forall G$ subgp $\quad Y$ irred component of $G \cap W$ st

$$\dim(Y) > \dim G - \operatorname{codim} W$$

$\Rightarrow \exists H \in \Xi$ st $\quad Y \subset H$

$\dim G = 0$: Manin - Mumford for tori

$\dim G = 1$: Bombieri - Zannier 2000

T (Bombieri-Masser-Zannier 2007) $W \subset \mathbb{G}_m^N$ codim $W \geq 2$

$\exists\, c = c(W)$ st:

$\forall\ \zeta \in \mu_\infty^N,\ a \in \mathbb{Z}^N,\quad \alpha \in \overline{\mathbb{Q}}^\times$ st

$$(\zeta_1 \alpha^{a_1}, \dots, \zeta_N \alpha^{a_N}) \in W$$

$\exists\ b \in \mathbb{Z}^N \setminus \{0\}$ st $\|b\| \leq c$ & $\prod_{i=1}^{N} (\zeta_i \alpha^{a_i})^{b_i} = 1$

Set

$$W^{\circ} = W \setminus \bigcup_{T} T$$

with T torus coset of $\dim \geq 1$

<u>T2</u>: $W \subset \mathbb{G}_m^N$ $\operatorname{codim} W \geq 2$ defined by equations of degree $\leq d_0$ and height $\leq h_0$

$\exists\, c = c(N, d_0)$ st

$\forall\, \underline{\zeta} \in \mu_\infty^N$, $\underline{a} \in \mathbb{Z}^N$, $\alpha \in \overline{\mathbb{Q}}^\times$ st

$$(\zeta_1 \alpha^{a_1}, \dots, \zeta_N \alpha^{a_N}) \in W^{\circ}$$

either $\|\underline{a}\| \leq c\,(1+h_0)^{2(N-1)}$

or $\exists\, \underline{b} \in \mathbb{Z}^N \setminus \{0\}$ st $\|\underline{b}\| \leq c$ & $\prod_{i=1}^{N} (\zeta_i \alpha^{a_i})^{b_i} = 1$

Ex : $W = \{(2, 2^a)\} \times \mathbb{G}_m \subset \mathbb{G}_m^3$

$W^\circ = \emptyset$ $\qquad d_0 = 1 \quad h_0 \approx a$

$(2, 2^a, 2^{a'}) \in W$ $\qquad (a' \gg a)$

A rough version of T2 $\Rightarrow$ T1

Let $a \in \mathbb{Z}^N$ st

$$f_a = \gamma_0 + \gamma_1 t^{a_1} + \cdots + \gamma_N t^{a_N}$$

has a multiple root $\xi \in \overline{\mathbb{Q}}^\times \setminus \mu_\infty$. Set $D = \|a\|$

Set $W = (\gamma_0 + \gamma_1 x_1 + \cdots + \gamma_N x_N,\ a_1\gamma_1 x_1 + \cdots + a_N \gamma_N x_N)$

codim $W = 2$

$d_0 = 1$ $\qquad h_0 \leq \max \log|\gamma_i| + \log D$

$$(\xi^{a_1}, \ldots, \xi^{a_N}) \in W$$

$\overset{T2}{\Longrightarrow}$ either $D \ll c'$

or $\exists\ \underline{b} \in \mathbb{Z}^N \setminus \{0\}$ st $\|b\| \ll c$ & $b \perp a$

$$\Longrightarrow \quad a \in \bigcup_{\|b\| \leq c''} b^\perp \qquad \boxtimes$$

OSCULATING SPACES OF TORIC CURVES

Let $0 < a_1 < \cdots < a_N$ coprime

$C_a = \{ (t^{a_1}, \ldots, t^{a_N}) \mid t \in \mathbb{C}^\times \} \subset (\mathbb{C}^\times)^N$

"toric" curve

$L_a \subset \mathbb{C}^N$

$$\operatorname{rank} \begin{pmatrix} a_1 & a_1^2 & \cdots & a_1^{N-2} & x_1 - 1 \\ a_2 & a_2^2 & \cdots & a_2^{N-2} & x_2 - 1 \\ \vdots & & & \vdots & \vdots \\ a_N & a_N^2 & \cdots & a_N^{N-2} & x_N - 1 \end{pmatrix} \leqslant N-2$$

osculating $(N-2)$-space of C_a at $\xi = \underline{1}$

Conj (Bolognesi - Pirola 2011) $C_{\underline{a}} \cap L_{\underline{a}} = \{\underline{1}\}$

ok for $N=3$ (B-P)

I3: The set

$$\{\underline{a} \in \mathbb{Q}^N \mid \exists\, \zeta \in \overline{\mathbb{Q}}^\times \setminus \mu_\infty \;\; (\zeta^{a_1}, \dots, \zeta^{a_N}) \in L_{\underline{a}}\}$$

is contained in a finite union of proper linear subspaces