

EQUIDISTRIBUTION OF GALOIS ORBITS OF SMALL POINTS

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THE SETTING

- * K number field
- * X projective variety / K of dimension $d \geq 1$
- * $\bar{D}$ adelic divisor on X

$$\leadsto h_{\bar{D}}: X(K) \longrightarrow \mathbb{R} \quad \text{height function}$$

Pl: Determine the asymptotic distribution
of the G -orbits of points of small height

SOME EXPERIMENTS

Take $P_d \in \mathbb{Z}[x]$ irreducible with

$$\deg P_d = d \gg 0 \quad \& \quad \text{coeffs}(P_d) \subset \{0, \pm 1\}$$

Plot the roots of P_d and see what happens...

BILU'S EQUIDISTRIBUTION THM

Thm (Bilu, 1997) Let $(x_l)_l$ in $\mathbb{P}^1(\overline{\mathbb{Q}})$ with

$$\deg(x_l) \xrightarrow{l \rightarrow \infty} \infty \quad h^{\text{Weil}}(x_l) \xrightarrow{l \rightarrow \infty} 0$$

Then

$$\lim_{l \rightarrow \infty} \delta_{G(x_l)_\infty} = \mu_{S^1}$$

Galois orbit
of x_l in $\mathbb{P}^1(\mathbb{C})$



Haar proba measure
on $S^1 \subset \mathbb{P}^1(\mathbb{C})$

In our situation

$$h^{\text{Weil}}(1:x_n) = \frac{m(P_n)}{\deg(P_n)} \leq \frac{\log(n+1)}{n} \xrightarrow{n \rightarrow \infty} 0$$

with $x_n \in \overline{\mathbb{Q}}$ s.t. $P_n(x_n) = 0$

THE EQUIDISTRIBUTION PROPERTY

$$\mathbb{D} = (\mathbb{D}, (g_v)_{v \in M_K}) \in \widehat{\text{Div}}(X)$$

Cartier divisor on X v -adic Green function

$$g_v : X_v^{\text{an}} \setminus |D| \longrightarrow \mathbb{R}$$

$$\text{For } x \in (X \setminus |D|)(\mathbb{K})$$

$$h_{\mathbb{D}}(x) = \sum_{v \in M_K} \frac{1}{\# G(x)_v} \sum_{y \in G(x)_v} g_v(y)$$

$\uparrow$
 p -adic Galois orbit

Two basic quantities:

$$\mu^{\text{abs}}(\overline{D}) = \inf_{x \in X(\mathbb{R})} h_{\overline{D}}(x)$$

absolute minimum

$$\mu^{\text{ess}}(\overline{D}) = \sup_{Y \subsetneq X} \inf_{x \in (X \setminus Y)(\mathbb{R})} h_{\overline{D}}(x)$$

essential minimum

EXAMPLE : $X = (x+y=1) \subset \mathbb{A}^2 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

$$\mu^{\text{abs}} = 0$$

$$0,2481 \leq \mu^{\text{ess}} \leq 0,2539$$

Zagier 1995

Doche 1998

Let $(x_l)_l$ **generic** sequence in $X(\mathbb{K})$

$$\rightarrow \liminf_l h_D(x_l) \geq \mu^{\text{eq}}(\overline{D})$$

Def: $(x_l)_l$ **$\overline{D}$ -small** if $\lim_l h_D(x_l) = \mu^{\text{eq}}(\overline{D})$

Def: $\overline{D}$ satisfies the **equidistribution property** at $v \in M_K$

if $\exists$ probability measure ν_v on X_v^{an} st.

$\forall (x_l)_l$ $\overline{D}$ -small generic sequence in $X(\mathbb{K})$

$$\lim_l \delta_{G(x_l)_v} = \nu_v$$

AET (Yuan 2008) Assume D big, $\bar{D}$ semipositive and

$$\mu^{\text{ess}}(\bar{D}) = \mu^{\text{sh}}(\bar{D})$$

Then $\bar{D}$ satisfies EP for all $v \in M_K$ with $\nu_v = \frac{c_1(\bar{D}_v)^{\text{rd}}}{(D^d)}$

normalized v -adic
Monge-Ampère measure of $\bar{D}$

Application:

$\varphi: X \rightarrow X$ dynamical system

$D \in \text{Div}(X)$ ample st. $\varphi^* D \equiv qD$ ($q > 1$)

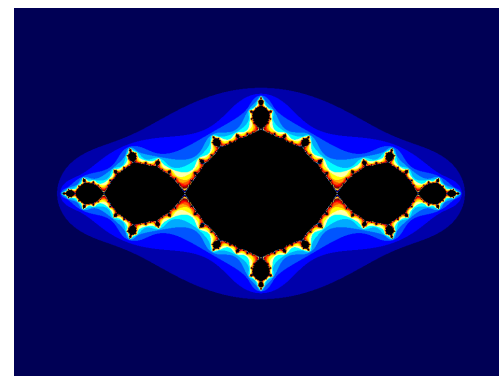
$\leadsto \bar{D}^{\text{can}}$ canonical adelic divisor on D

$h_{\bar{D}^{\text{can}}}(x) \geq 0 \quad \forall x \in X(\mathbb{F})$

$= 0 \iff x$ preperiodic

$$\leadsto \mu^{\text{ess}}(\bar{D}^{\text{can}}) = \mu^{\text{sh}}(\bar{D}^{\text{can}}) = 0$$

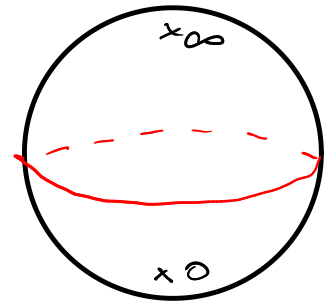
ν_v v -adic equilibrium measure of φ



EXAMPLES $X = \mathbb{P}^1$ $D = (x_0=0)$

(1) $g_\infty(z) = \log^+ |z|_\infty \quad \forall z \rightarrow h(x) = \log \max(|x_0|_\infty, |x_1|_\infty)$
 $\mu^{\text{abs}} = \mu^{\text{ess}} = 0$ for $x = (x_0 : x_1) \in \mathbb{P}^1(\mathbb{Q})$ with $x_0, x_1 \in \mathbb{Z}$ coprime

ν_∞ Haar proba measure on S^1



(2) $g_\infty(z) = \log (1 + |z|_\infty)^{1/2} \rightarrow h(x) = \log (|x_0|_\infty^2 + |x_1|_\infty^2)^{1/2}$
 $\mu^{\text{abs}} = 0 \quad \mu^{\text{ess}} = \frac{\log 2}{2}$

EP with ν_∞ as before

(Burgos, Philippon, Rivera-Letelier, S 2019)

(3) $g_\infty(z) = \log^+ (2|z|_\infty) \rightarrow h(x) = \log \max(|x_0|_\infty, 2|x_1|_\infty)$

$\mu^{\text{abs}} = 0 \quad \mu^{\text{ess}} = \log 2$

No EP

$x_\ell = (1 : \omega_\ell)$ $y_\ell = (1 : \omega_\ell/2)$ with ω_ℓ root of 1
 $\rightarrow h(x_\ell) = h(y_\ell) = \log 2 \quad (\forall \ell)$

THE MAIN RESULT

From now on: X **normal** projective variety / K

D adelic **$\mathbb{R}$ -divisor** on X with **D big**

DEF: A **semipositive approximation** of D is $(\phi, \overline{Q})$ with

* $\phi: X' \rightarrow X$ **normal modification**

* $\overline{Q} \in \widehat{\text{Div}}(X')_{\mathbb{R}}$ **semipositive** with Q big s.t.

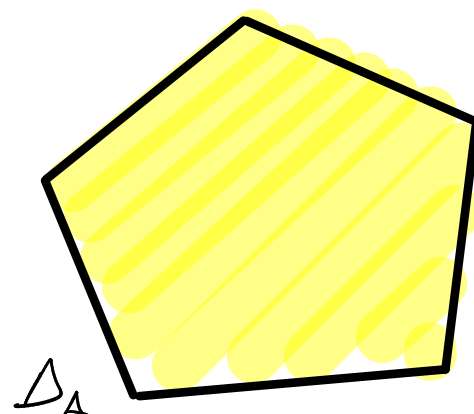
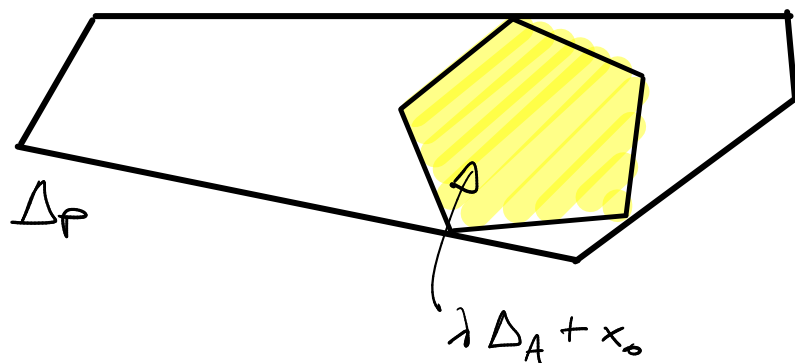
$$\phi^* D - \overline{Q} \quad \text{pscf}$$

DEF: For $P, A \in \text{Div}(X)_{\mathbb{R}}$ big

$$r(P, A) = \sup \{ \lambda \in \mathbb{R} \mid P - \lambda A \text{ big} \} \quad \text{inradius}$$

Teissier 1982

If P, A toric $\leadsto r(P, A) = \text{inradius of } \Delta_P \text{ w.r. to } \Delta_A$



Thm (Bellaï-S)

Assume $\exists (\phi_n, \overline{Q}_n)$ semipositive approximations of $\overline{D}$ st

$$\lim_n \frac{\mu^{\text{ess}}(\overline{D}) - \mu^{\text{sls}}(\overline{Q}_n)}{r(\overline{Q}_n, \overline{D})} = 0$$

For $\nu \in M_K$ set $\nu_{\nu, n} = \frac{e_1(\overline{Q}_{n, \nu})^{nd}}{(Q_n^d)}$ probability measure on X_n

Then $\overline{D}$ satisfies EP with $\nu_\nu = \lim_n \nu_{\nu, n}$

REM.: if $\overline{D}$ semipositive and $\mu^{\text{ess}}(\overline{D}) = \mu^{\text{sls}}(\overline{D})$ can take

$$\overline{Q}_n = \overline{D} \quad \forall n \quad \leadsto \text{AET}$$

SUMS OF CANONICAL DIVISORS

Let $\varphi: X \rightarrow X$ dynamical system

$D_1, \dots, D_s \in \text{Div}(X)_{\mathbb{R}}$ st

* $\varphi^* D_i \equiv q_i D_i \quad (q_i > 1)$

* D_i *effective* and $\overline{D}_i^{\text{can}}$ *semipositive*

* $D = D_1 + \dots + D_s$ *ample*

EXAMPLE:

$$0 \rightarrow G_m^r \rightarrow G \xrightarrow{\pi} A \rightarrow 0$$

$\nwarrow$ *semiabelian variety*

$$\overline{D} = \overline{M}^{\text{can}} + \pi^* \overline{N}^{\text{can}} \text{ on } \overline{G}$$

$$[2]^* M \equiv 2M$$

$$[2]^* \pi^* N \equiv 4 \pi^* N$$

We have $\mu^{\text{abs}}(\overline{D}) < \mu^{\text{ess}}(\overline{D}) = 0$ unless G is trivial

Thm (Bellaï-S)

$\bar{D}$ satisfies EP for all $n \in \mathbb{N}_k$ with $\nu_n = \frac{c_1(\bar{D}_n)^{1/d}}{(\bar{D})}$

Pf: Suppose $1 < q_1 < \dots < q_s$ and set

$$\bar{Q}_n = \frac{1}{q_s^n} (\phi^n)^* \bar{D} = \left(\frac{q_1}{q_s}\right)^n \bar{D}_1 + \dots + \left(\frac{q_s}{q_s}\right)^n \bar{D}_s$$

We have (i) $\bar{Q}_n$ semipositive approximation of $\bar{D}$

$$(ii) \quad r(\bar{Q}_n, \bar{D}) \geq \left(\frac{q_1}{q_s}\right)^n$$

$$(iii) \quad \mu^{abs}(\bar{Q}_n) = \frac{\mu^{abs}(\bar{D})}{q_s^n}$$

Hence

$$0 \leq \frac{\mu^{ess}(\bar{D}) - \mu^{abs}(\bar{Q}_n)}{r(\bar{Q}_n, \bar{D})} \leq \frac{-\mu^{abs}(\bar{D})}{q_1^n} \xrightarrow{n \rightarrow \infty} 0$$

□

Toric varieties

$\times$ projective toric variety with torus G_m^d

$\overline{D}$ s.f. toric adelic $\mathbb{R}$ -divisor with D big

$D \rightsquigarrow \Delta_D \subset \mathbb{R}^d$ polytope

$\overline{D} \rightsquigarrow \vartheta_{\overline{D}} : \Delta_D \rightarrow \mathbb{R}$ roof function

$$\mu^{\text{ess}}(\overline{D}) = \max_{x \in \Delta_D} \vartheta_{\overline{D}}(x)$$

$$\mu^{\text{log}}(\overline{D}) = \min_{x \in \Delta_D} \vartheta_{\overline{D}}(x)$$

THE TORIC EQUIDISTRIBUTION THEOREM

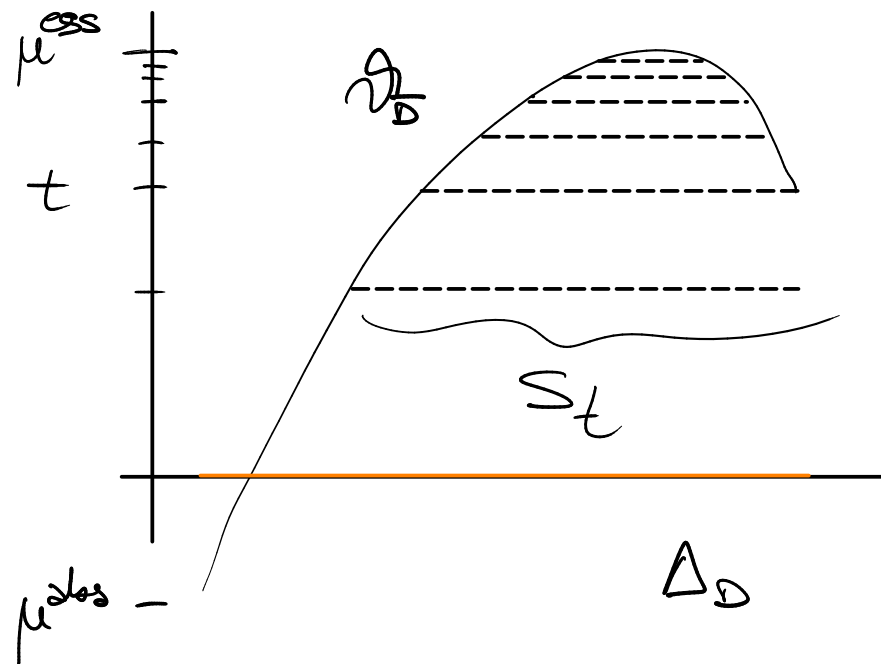
Thm (Burgos, Philippon, Rivera-Letelier, S 2019)

TFAE

$$(1) \lim_{t \rightarrow \mu^{\text{ess}}} \frac{\mu - t}{r(S_t)} = 0$$

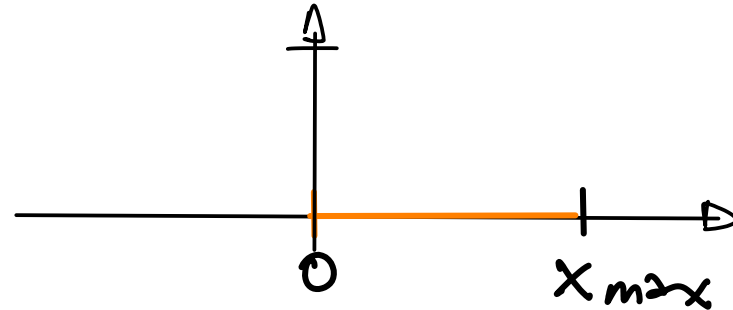
inradius of S_t

(2) $\mathcal{D}$ satisfies EP
for all $N \in M_{\mathbb{R}}$

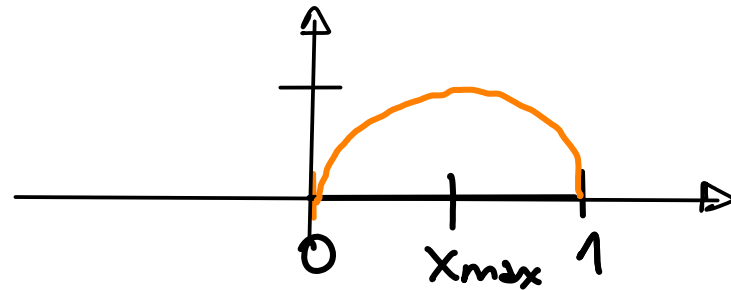


EXAMPLES

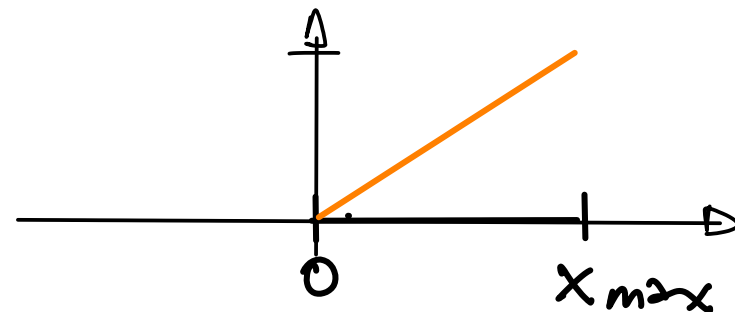
(1) $h(x) = \log \max(|x_0|_\infty, |x_1|_\infty)$



(2) $h(x) = \log (|x_0|_\infty^2 + |x_1|_\infty^2)^{1/2}$



(3) $h(x) = \log \max(|x_0|_\infty, 2|x_1|_\infty)$



Thank you!