Kähler-Einstein toric submanifolds of the projective space

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Based on joint work with Antonio Di Scala (Torino)

Embeddings into complex space forms

Let (X, ω) be a Kähler manifold of dimension $n \geqslant 1$.

The basic examples *(complex space forms)*:

- $(\mathbb{H}^n, \omega_{\text{hyp}})$ hyperbolic space
- $(\mathbb{E}^n, \omega_{\text{eucl}})$ Euclidean space
- $(\mathbb{P}^n, \omega_{\text{FS}})$ projective space

Question

Can (X, ω) be embedded into a given complex space form?

The same question for

- differentiable manifolds: yes (Withney 1936)
- Riemannian manifolds: yes (Nash 1956)

Curves

For instance let $E = \mathbb{C}/\Lambda$ an elliptic curve with its flat metric. Can E be isometrically embedded into a $\mathbb{P}^N$?

Definition

The Kähler manifold (X, ω) is *projectively induced* if there exists $\varphi\colon X \to \mathbb{P}^N$ s.t. $\omega = \varphi^*\omega_{\mathrm{FS}}$.

Theorem (Calabi 1950)

Let (X, ω) be a curve with constant scalar curvature that is projectively induced. Then $(X, \omega) \simeq (\mathbb{P}^1, k\,\omega_{\mathrm{FS}})$.

Indeed $(\mathbb{P}^1, k\,\omega_{\mathrm{FS}})$ is induced by the Veronese map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^k, \quad (z_0 : z_1) \longmapsto \left(\binom{k}{j}^{1/2} z_0^{k-j} z_1^j\right)_j$$

Kähler-Einstein manifolds

We say that (X, ω) is *Kähler-Einstein* if there exists $\lambda \in \mathbb{R}$ s.t.

$$\mathrm{Ric}(\omega) = \lambda\,\omega \qquad \text{(Ricci form)}$$

Example

$(\mathbb{P}^n, \omega_{\mathrm{FS}})$ is KE with $\lambda = n + 1$.

Assume X compact and let K_X be its *canonical line bundle*.

- If $\lambda < 0$ then K_X is ample *(canonically polarized)*
- If $\lambda = 0$ then K_X is trivial *(Calabi-Yau)*
- It $\lambda > 0$ then $-K_X$ is ample *(Fano)*

Projectively induced KE manifolds

Theorem (Umehara 1988)

Let (X, ω) be a KE manifold with $\varphi \colon X \to \mathbb{H}^N$ (resp. $\varphi \colon X \to \mathbb{E}^N$) s.t. $\omega = \varphi^*\omega_{\mathrm{hyp}}$ (resp. $\omega = \varphi^*\omega_{\mathrm{eucl}}$). Then $\varphi(X)$ is totally geodesic.

Conjecture

Let (X, ω) be a projectively induced KE manifold. Then (X, ω) is a homogeneous space.

Known cases:

- $N = n + 1$ (Chern 1967)
- $N = n + 2$ (Tsukada 1986)

The homogeneous projectively induced KE manifolds are classified (Takeuchi 1978)

Toric manifolds

Let (X, ω) be a *toric* Kähler manifold:

- $(\mathbb{C}^\times)^n$ acts on X with a dense orbit $X_0 \simeq (\mathbb{C}^\times)^n$
- ω is invariant under the action of $(S^1)^n$

Conjecture

Let (X, ω) be a projectively induced KE toric Fano manifold. Then

$$(X, \omega) \simeq \Big(\prod_{i=1}^{r} \mathbb{P}^{n_i}, \bigoplus_{i=1}^{r} k_i \, \mathrm{pr}_i^* \omega_{FS} \Big)$$

Known (under hypothesis) for $n \leqslant 4$ using the classification of Fano polytopes (Arezzo, Loi and Zuddas 2012).

KE metrics on toric manifolds

Let X be a toric Fano manifold. Its *anticanonical polytope*

$$\Delta_{-K_X} \subset \mathbb{R}^n$$

is a unimodular reflexive polytope. Up to a translation

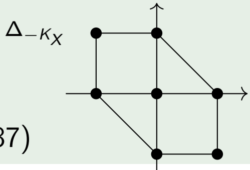
$$\Delta^{\circ}_{-K_X} \cap \mathbb{Z}^n = \{0\}.$$

Theorem (Wang and Zhu 2004)

X admits a KE metric iff the baycenter of Δ_{-K_X} vanishes.

Example

Let X be the blow up of $\mathbb{P}^2$ at $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$.



It admits a KE metric (Tian and Yau 1987)

Centrally symmetric toric manifolds

A Fano toric manifold X is *centrally symmetric* if the inversion

$$(\mathbb{C}^\times)^n \longrightarrow (\mathbb{C}^\times)^n, \quad x \longmapsto x^{-1}$$

extends to a map $X \to X$, or equivalently if Δ_{-K_X} is symmetric with respect to $0 \in \mathbb{R}^n$.

Theorem (DS 2025)

Let X be a centrally symmetric toric Fano manifold admitting a projectively induced KE metric. Then $X \simeq (\mathbb{P}^1)^n$.

Toric potentials

Let (X, ω) be a toric Kähler manifold and $\varphi \colon X \to \mathbb{P}^N$ a toric map s.t. $\omega = \varphi^* \omega_{\mathsf{FS}}$. It writes down as

$$\varphi(x) = (\alpha_0 x^{m_0} : \cdots : \alpha_N x^{m_N})$$

with $\alpha_j \in \mathbb{C}^\times$ and $m_j \in \mathbb{Z}^n$.

The *toric potential* of ω is the n-variate Laurent polynomial

$$p_\varphi = \sum_{j=0}^{N} |\alpha_j|^2 x^{m_j} \in \mathbb{R}_{>0}[x^{\pm 1}]$$

Setting $f_\varphi(u + iv) = \log(p_\varphi)(e^{2u})$ we have

$$\omega = i\partial\overline{\partial} f_\varphi \qquad \text{on } \mathbb{C}^n / 2\pi\mathbb{Z}^n \simeq (\mathbb{C}^\times)^n$$

The algebraic Einstein condition

Let $D_i = x_i \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, and for $p \in \mathbb{R}[x^{\pm 1}]$ set

$$\mu(p) = p^{n+1} \det(D^2 \log(p)) \in \mathbb{R}[x^{\pm 1}]$$

with $D^2 \log(p) = (D_i D_j \log(p))_{i,j}$ the D-Hessian matrix of $\log(p)$.

Proposition (DS 25)

A Fano toric manifold X admits a projectively induced KE metric iff there exists $p \in \mathbb{R}_{>0}[x^{\pm 1}]$ s.t.

$$\mathrm{NP}(p) = \Delta_{-K_X} \quad \textit{and} \quad \mu(p) = p^n$$

The equation $\mu(p) = p^n$ boils down to a system over $\mathbb{R}_{>0}$ with

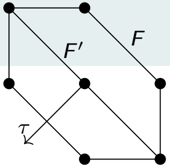
$$\#(n\Delta_{-K_X} \cap \mathbb{Z}^n) \text{ equations in } \#(\Delta_{-K_X} \cap \mathbb{Z}^n) \text{ variables}$$

The toric adjuntion formula

Theorem (DS 25)

Let $p \in \mathbb{R}[x^{\pm 1}] \backslash \{0\}$ *and* τ *the inner normal direction to a facet of* $\mathrm{NP}(p)$*. Then*

$$\mathrm{init}_\tau(\mu(p)) = \mu(\mathrm{init}_\tau(p)) \cdot p|_{F'_\tau}$$



Definition

A Laurent polynomial p satisfies the *generalized Einstein condition* if

$$\mu(p) \mid p^\kappa \quad \text{for some } \kappa \geqslant 0$$

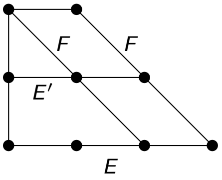
GEC is hereditary on initial parts: if p satisfies GEC so does $\mathrm{init}_\tau(p)$.

Low dimensions

Proposition (DS 25)

Let $p \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ satisfying GEC. Then for every pair of edges E, F of $\mathrm{NP}(p)$ and their adjacents segments E', F' we have

$$\frac{\ell(F')}{\ell(F)} = \frac{\ell(E')}{\ell(E)} \qquad \text{(lattice length)}$$



$$\frac{\ell(E')}{\ell(E)} = \frac{2}{3} \neq 1 = \frac{\ell(F')}{\ell(F)}$$

A three-dimensional toric Fano

Set

$$\Delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -1 \leqslant x_1, x_2, \; -1 \leqslant x_3 \leqslant 1, \; x_2 - x_3 \leqslant 1\}$$

It is the only "new" smooth reflexive polytope in dimension 3 whose associated toric Fano manifold admits a KE metric.

The trapezoid is a face of Δ and so this KE metric is not projectively induced.

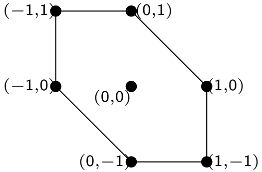
The hexagon

Proposition (DS 25)

Let

$$p = \alpha_0 x_2^{-1} + \alpha_1 x_1 x_2^{-1} + \alpha_2 x_1^{-1} + \alpha_3 + \alpha_4 x_1 + \alpha_5 x_1^{-1} x_2 + \alpha_6 x_2$$

with $\alpha_j \neq 0$ *for all* $j \neq 3$. *Then* p *does not satisfy GEC.*



Large dimensions

A toric Fano manifold X is centrally symmetric iff

$$X \simeq \prod_{i=1}^{r} X_i$$

with X_i equal to either $\mathbb{P}^1$ or a *del Pezzo* toric manifold V_k
(Klyachko and Voskrenskij 1987)

Now if X admits a projectively induced KE metric then no V_k can appear, because otherwise the hexagon is a face of Δ_{-K_X}.

Hence

$$X \simeq (\mathbb{P}^1)^n$$

Other applications

The conjecture is verified for all *known* toric Fanos, including

- Batyrev-Selivanova's families of symmetric toric Fano manifolds
- Nill and Paffenholz non-symmetric examples

Question

Let X be a toric Fano that is not a product of projective spaces. Does Δ_{-K_X} have a 2-dimensional face that is not a triangle or a rectangle?

Bon anniversaire André!