TORIC VARIETIES IN ARITHMETIC GEOMETRY

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Aim: propose a notion of toric variety in Arakelov geometry

I. GEOMETRY OF TORIC VARIETIES

$\mathbb{T} := \mathbb{G}_{m,\mathbb{Q}}^n$ (split) torus / $\mathbb{Q}$ of dim n

<u>Def:</u> A toric variety is a (normal) algebraic variety X st

$$\mathbb{T} \subset X \quad \& \quad \mathbb{T} \curvearrowright X$$

CONSTRUCTION:

Σ fan in $\mathbb{R}^n$ (= polyhedral complex of strictly convex cones / $\mathbb{Q}$)

$\sigma \in \Sigma \rightsquigarrow X_\sigma$ affine tv

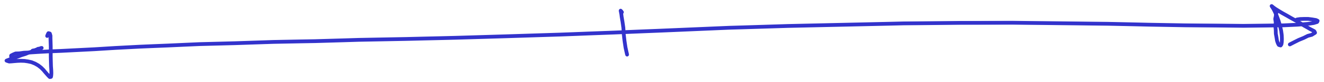
$\tau < \sigma \Rightarrow X_\tau$ open subset of X_σ

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} X_\sigma$$

$$\cup$$

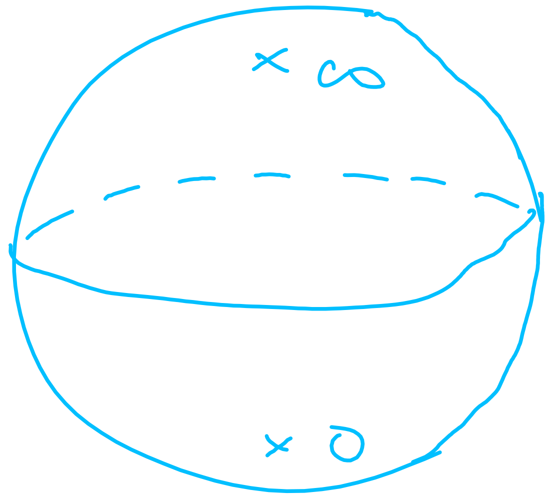
$$X_0 = \mathbb{T}$$

EXAMPLE:



$$\mathbb{A}^1_{\mathbb{Q}} \hookleftarrow \mathbb{G}_{m,\mathbb{Q}} \hookrightarrow \mathbb{A}^1_{\mathbb{Q}}$$

$$\rightsquigarrow \quad \mathbb{P}^1_{\mathbb{Q}}$$



I (Demazure) $\Sigma \to X_\Sigma$ is 1-to-1 between fans and tv of dim n

$\rightsquigarrow$ tv "are" combinatorial objects

<u>Ex:</u> X_Σ proper $\iff$ Σ complete $\left(\bigcup_\sigma \sigma = \mathbb{R}^n \right)$

Assume Σ complete

Def: A virtual support function on Σ is

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$$

st $\psi|_\sigma$ linear $\forall \sigma \in \Sigma$

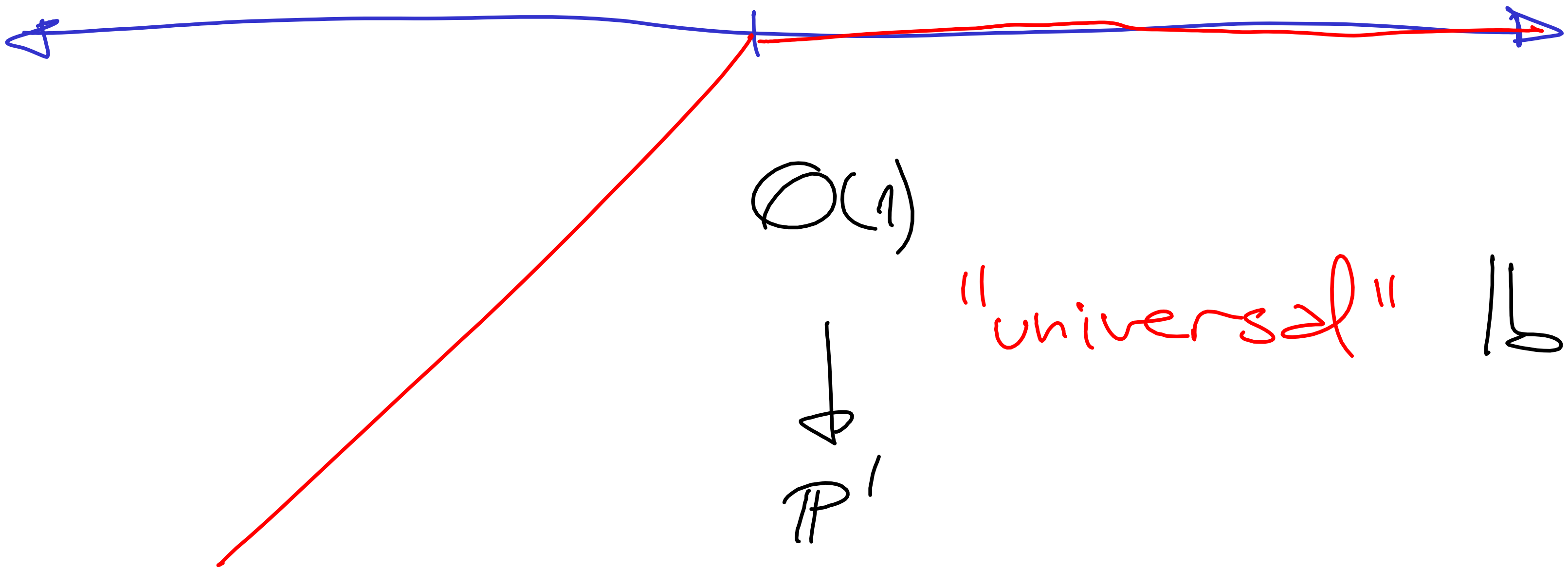
$$\psi \rightsquigarrow \begin{tikzcd} L_\psi \arrow[d] \\ X_\Sigma \arrow[u, dashed, bend right, "s_\psi"'] \end{tikzcd}$$

L_ψ equivariant line bundle

s_ψ invariant rational section

I. $\varphi \to (L_\varphi, s_\varphi)$ is 1-to-1 between vsf & equivariant lb + invariant rtl section

EXAMPLE:



$$\begin{array}{c} \mathcal{O}(1) \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

"universal" lb

Ex: • L_ψ generated by global sections $\iff$ ψ concave

• L_ψ ample $\iff$ ψ strictly concave $/ \Sigma$

X proper, $f : X \longrightarrow \mathbb{A}^1_{\mathbb{Q}}$ $\rightsquigarrow$ f constant

Sections of lb allow global functions $\neq$ ct on proper varieties

II Arithmetic of Toric Varieties

"Would be" examples

$\mathbb{P}^n$ + Weil height of points

$\mathbb{P}^n$ + Euclidean heigh of points

The fan + vsf do not suffice to classify arithmetic structures

II.1 ABSOLUTE VALUES & HEIGHTS

Consider absolute values / $\mathbb{Q}$

$|\cdot|_\infty$ standard : completion $\mathbb{R}$

$|\cdot|_p$ p-adic : completion $\mathbb{Q}_p$

$$\rho = a/b \in \mathbb{Q} \qquad \gcd(a,b) = 1$$

The height (= bitsize) of ρ is

$$h(\rho) = \log \max(|a|,|b|) = \sum \log \max(1, |\rho|_v)$$

Weil's formula

II.2 Metrized Line Bundles

$X/\mathbb{Q}$ $\quad v \quad \mathcal{M}/\mathbb{Q}$

$\rightsquigarrow \quad X_v$ analytic variety
- $X(\mathbb{C}) \quad v = \infty$
- $X_{\mathrm{Ber},p} \quad v = p$ (Berkovich space)

$$\begin{array}{ccc} L & & L_v \\ \downarrow & \rightsquigarrow & \downarrow \\ X & & X_v \end{array}$$

<u>Def.</u> A metric $\|\cdot\|_v$ on L_v: continuous family of norms $x \mapsto \|\cdot\|_{x,v}$. Assume semipositivity. Induces a Borel measure μ_v on X_v

EXAMPLES:

- $v=\infty$, $\|\cdot\|_\infty$ smooth, $c_1(L_\infty, \|\cdot\|_\infty)$ Chern form
  - $\|\ \|_\infty \geq 0$ iff $\int_C c_1 \geq 0 \quad \forall\, C \subset X_\infty$ curve
  - $\mu_\infty(E) = \int_E c_1^{\wedge n}$
- $v=p$ $\quad \|\cdot\|_p$ induced by an integral model $/\mathbb{Z}_p$ $\quad$ ($\|\cdot\|_p$ "algebraic")

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ & & \downarrow \pi \\ & & \operatorname{Spec}(\mathbb{Z}_p) = \{o, \eta\} \end{array}$$

$\|\cdot\|_p \geq 0$ iff $\deg_{\mathcal{L}}(C) \geq 0 \quad \forall\, C \subset \pi^{-1}(o)$ curve

μ_p discrete measure supported on

$\operatorname{red}(V) \qquad V$ component of $\pi^{-1}(o)$

Def: A metrized lb is $\overline{L} = \left(L, (\|\cdot\|_v)_v\right)$

$+\ h$

(= arithmetic structure on $\begin{array}{c} L \\ \downarrow \\ X \end{array}$)

Allows to define height of subvarieties

Arithmetic Bézout thm:

$$\begin{array}{ccc} & & L_{\mathbb{R}} \\ & & \downarrow \uparrow s \\ Y & \subset & X \end{array}$$

$$h(Y\cdot \mathrm{div}(s)) = h(Y) + \sum_v \int_{Y_v} \log \|s\|_v \, \mu_v$$

The μ_v's also control the distribution of points of small height (Bilu, Szpiro-Ullmo-Zhang, Yuan)

Other results: Hilbert-Samuel

Lefschetz fixed pt formula

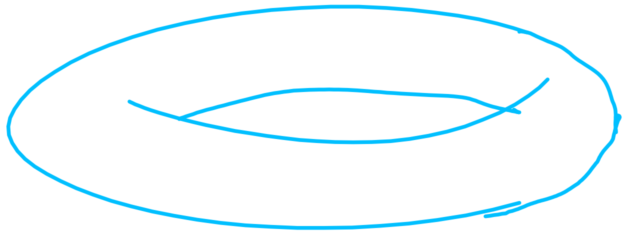
Riemann-Roch

...

II.3 Arithmetic Toric Varieties

$\mathbb{S}_v = \{ x \in \mathbb{T} : \ |x|_v = 1 \}$ compact torus

$\mathbb{S}_\infty = (S^1)^n$



Def (BPS) An arithmetic toric variety is
$(X_\Sigma, L_\psi, (\|\cdot\|_v)_v)$

- X_Σ proper ($\iff \Sigma$ complet)
- L_ψ generated by sections ($\iff \psi$ concave)
- $\|\cdot\|_v$ $\mathbb{S}_v$-invariant ($\forall v$)

CONSTRUCTION ($v = \infty$)

$$
\begin{tikzcd}
x \arrow[d, mapsto] & X_0(\mathbb{C}) \arrow[d] \arrow[dr, dashed, "-\log \| s_\varphi \|_f"] & \\
-\log|x| = \mathrm{val}(x) & \mathbb{R}^n \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}
$$

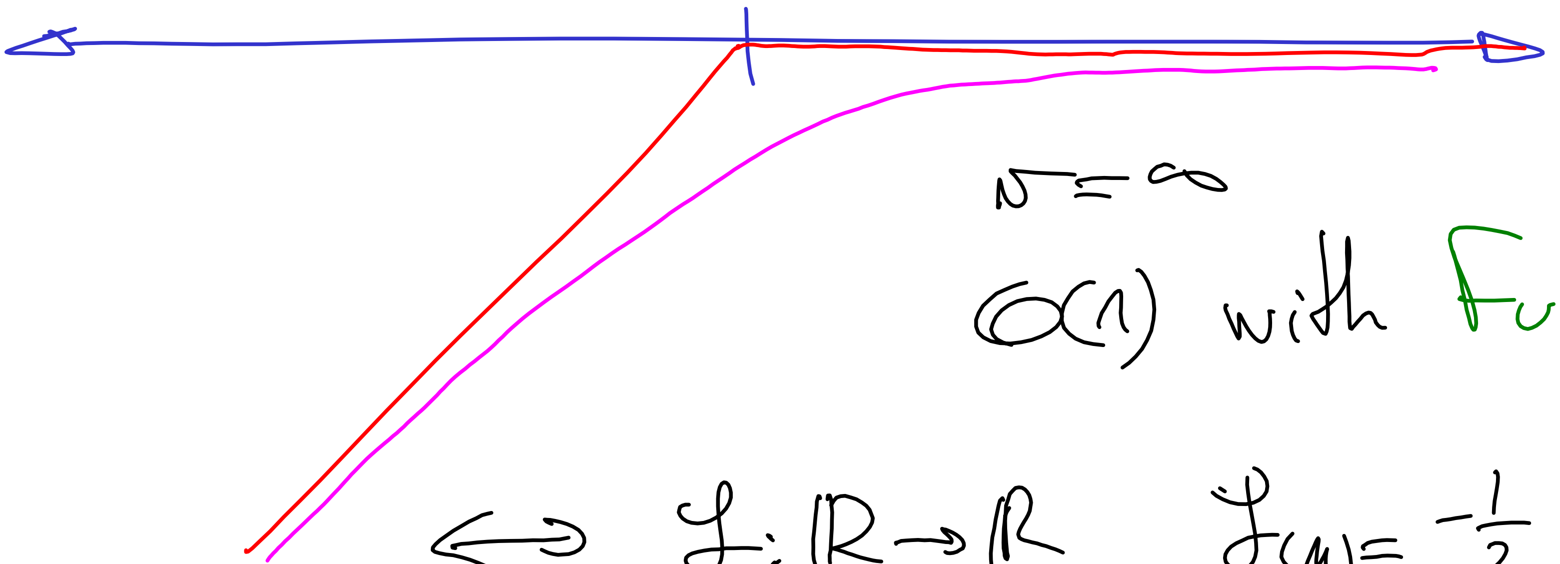
$$\left\{ \begin{array}{c} \text{continuous} \\ \text{functions} \end{array} \right\} \longleftrightarrow \left\{ \text{continuous metrics} \,/\, X_0(\mathbb{C}) \right\}$$

Th (BPS) (1) $\|\cdot\|_f$ extends to a metric ≥ 0 iff f concave & $f = \psi + O(1)$

(2) $(f_v)_v \to (\|\cdot\|_{f_v})_v$ 1-to-1 between

- concave functions st $\begin{cases} f_v = \psi + O(1) & (\forall v) \\ f_v = \psi & (\mathring{\forall} v) \end{cases}$
- $\mathbb{S}$-invariant arithmetic structures $/ (X_\Sigma, L_\psi)$

EXAMPLES

- 

$\nu = \infty$

$\mathcal{O}(1)$ with Fubini Study

$\Longleftrightarrow \quad \varphi : \mathbb{R} \to \mathbb{R} \qquad \varphi(u) = -\frac{1}{2}\log(1+e^{-2u})$

$\nu = p$:

- φ piecewise affine concave ("tropical" polynomial)

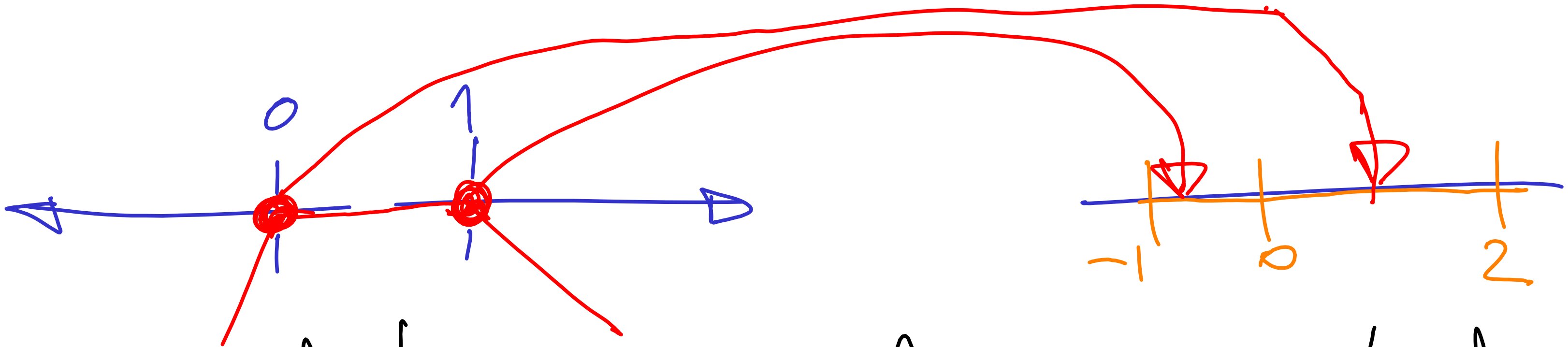
$\Longleftrightarrow \quad \| \; \|_{\varphi}$ algebraic

II.4 Measures

Thm (BPS) $E \subset \mathbb{R}^n$

$$\mathrm{val}_* \mu_v (E) = n! \, \overbrace{\mathrm{vol}\, \mathrm{grad}(f)(E)}^{\text{Monge-Ampère measure of } f}$$

Ex:



$v = p$ $\|\ \|$ algebraic $\Rightarrow$ $\mathrm{val}_* \mu_v$ supported in the vertices of the tropical hyp of f

$$\sum_{i=0}^{3} \delta_i$$

II.5 Heights & Legendre-Fenchel Duality

Def: The Legendre-Fenchel dual a concave $f : \mathbb{R}^n \to \mathbb{R}$;

$$f^\vee(x) = \inf_{\mu \in \mathbb{R}^n} \langle x, \mu \rangle - f(\mu)$$

$f^\vee$ concave with domain

$$\mathrm{dom}(f^\vee) = \{ x \mid f^\vee(x) > -\infty \}$$

Prop $f = \psi + O(1) \iff \begin{cases} \mathrm{dom}(f^\vee) = \Delta_\psi = \mathrm{grad}(\psi)(0) \\ f^\vee : \Delta \to \mathbb{R} \text{ continuous} \end{cases}$

I(BPS) $h_{\overline{L_\psi}}(X_\Sigma) = (n+1)! \sum_{v} \int_\Delta \vartheta_v^\vee \, d\underline{x}$

Arithmetic analogue of

$$\deg_{L_\psi}(X_\Sigma) = n! \operatorname{vol}(\Delta)$$

Ex:

$\mathcal{O}(1)$ with Fubini-Stud,

$\downarrow$

$\mathbb{P}^n$ $\quad \vartheta^\vee : \Delta^n \longrightarrow \mathbb{R}$

$$\vartheta(x) = \frac{1}{2} \underbrace{\left(-\sum_{i=0}^{n} x_i \log x_i \right)}_{\text{entropy of } (x_0, \dots, x_n)} \qquad \left(x_0 = 1 - \sum_{i=1}^{n} x_i \right)$$

$$h(\mathbb{P}^n) = \frac{(n+1)!}{2} \int_{\Delta^n} \mathcal{E}(x) \, dx = \frac{(n+1)}{2} \sum_{j=2}^{n+1} \frac{1}{j} \quad \left(= \frac{1}{2}, \frac{5}{4}, \dots \right)$$

II.6 Toric Bundles

$a_r \geq a_{r-1} \geq \cdots \geq a_0 \geq 1$

$$E = \mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_r)$$
$$\downarrow$$
$$\mathbb{P}^n$$

$\mathbb{P}(E)$ bundle of hyperplanes of E

Has a natural metrized lb $\overline{\mathcal{O}_{\mathbb{P}(E)}(1)}$

We can compute f, $f^\vee$

- $\deg \mathbb{P}(E) = \sum_{|\underline{i}|=n} \underline{a}^{\underline{i}}$

- $h(\mathbb{P}(E)) = \left(\sum_{|\underline{i}|=n+1} \underline{a}^{\underline{i}} \right) h(\mathbb{P}^n) + \sum_{|\underline{i}|=n} \underline{a}^{\underline{i}} \, A_{n,r}(\underline{i})$

$$A_{n,r}(\underline{i}) = \sum_{m=0}^{r} (i_m+1) \sum_{j=2}^{n+r+1} \frac{1}{2j}$$

En particular $\mathcal{H}_b = \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(b+1))$ Hirzebruch surf.

$$h(\mathcal{H}_b) = \frac{b^2}{2} + \frac{9}{4} b + 3$$

Happy Birthday RSME!!!