ARITHMETIC GEOMETRY OF TORIC VARIETIES (I)

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RENCONTRES ARITHMÉTIQUES DE CAEN

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The Weil height

Let $\xi \in \overline{\mathbb{Q}}^\times$ of degree $d \geq 1$

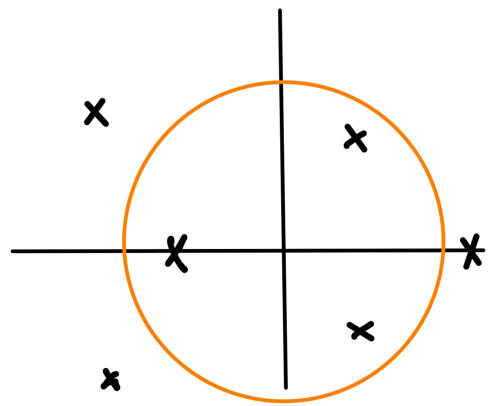
$$P_\xi = \alpha_d x^d + \cdots + \alpha_0 = \alpha_d \prod_{\eta \in G\xi} (x-\eta) \in \mathbb{Z}[x]$$

minimal poly of ξ

Galois orbit of ξ

The height of ξ is

$$h(\xi) = \frac{1}{d}\left(\sum_{\eta \in G\xi} \log \max(1, |\eta|) + \log|\alpha_d| \right)$$



- If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h(\xi) = \log \max(|a|, |b|)$
- $h(\xi) = 0 \iff \xi$ root of 1 (Kronecker)

BILU'S EQUIDISTRIBUTION THM (1997)

THM Let $p_k \in \overline{\mathbb{Q}}^\times$, $k \geq 1$, st

- $\forall p \in \overline{\mathbb{Q}}^\times$, $\#\{k \mid p_k = p\} < \infty$
- $h(p_k) \xrightarrow[k\to\infty]{} 0$

Then $G p_k \to S^1$

I.e. $\forall f \in \mathcal{C}^0(\mathbb{C}^\times)$ with compact support

$$\frac{1}{\# G p_k} \sum_{g \in G p_k} f(g) \xrightarrow[k\to\infty]{} \int f \, d\mu_{S^1}$$

Toric version of Szpiro, Ullmo & Zhang equidistribution (1996)

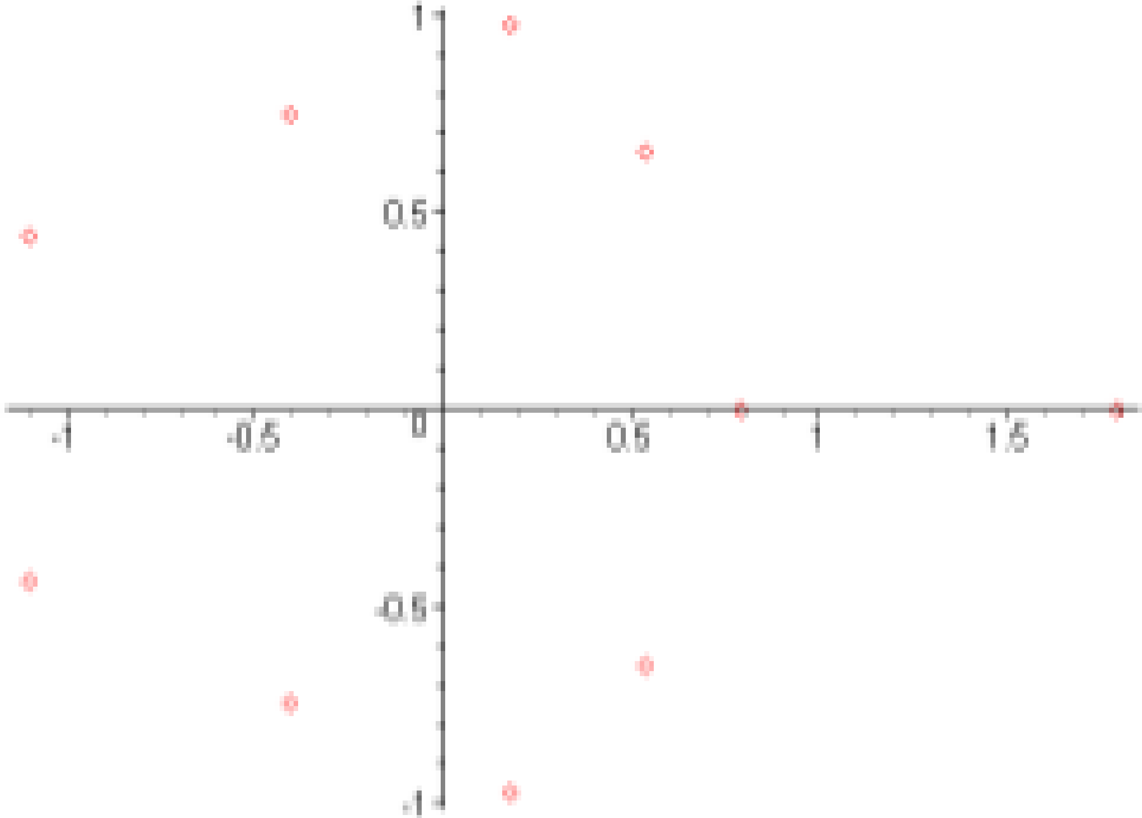
Some experiments

Take $P_d \in \mathbb{Z}[x]$ irreducible with

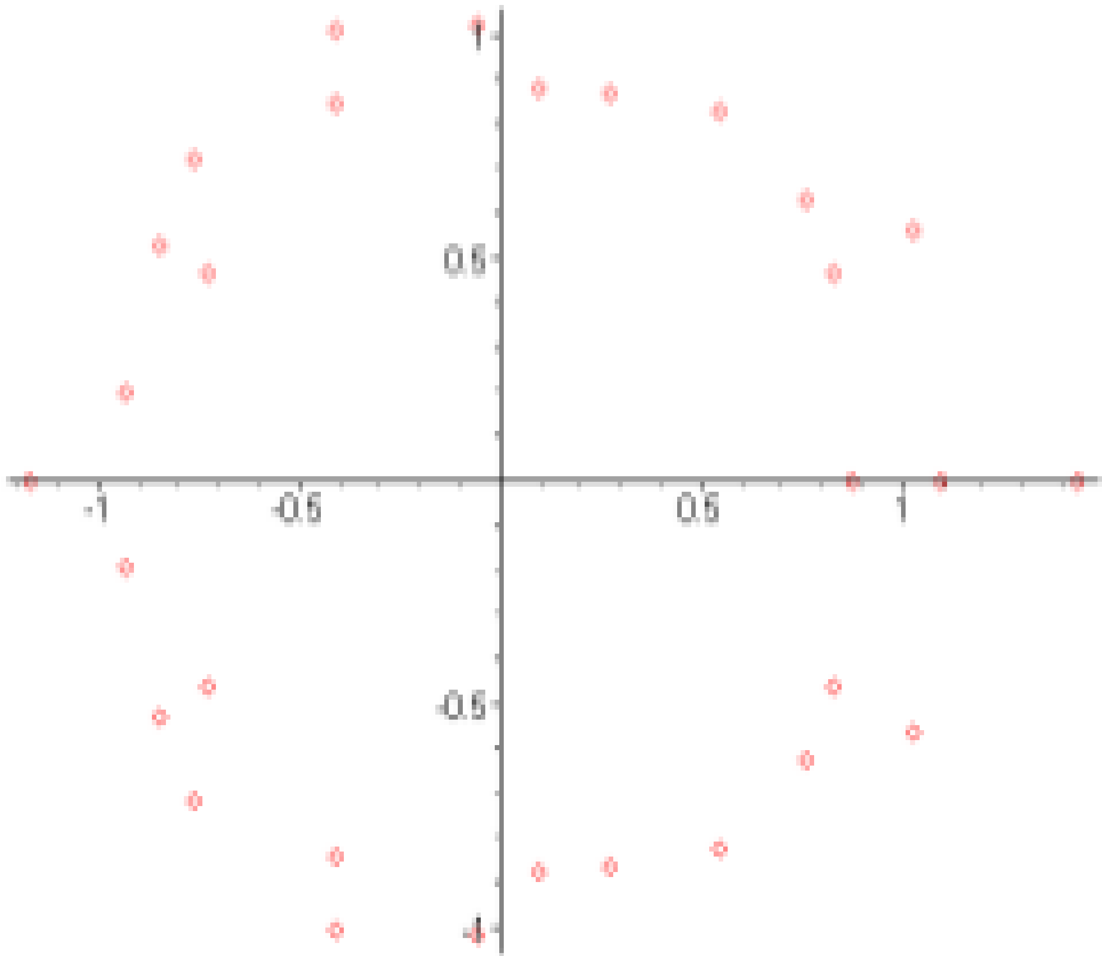
$$\deg P_d = d \gg 0 \quad \& \quad \mathrm{coeffs}(P_d) \subset \{0, \pm 1\}$$

Plot the roots of P_d and see what happens...

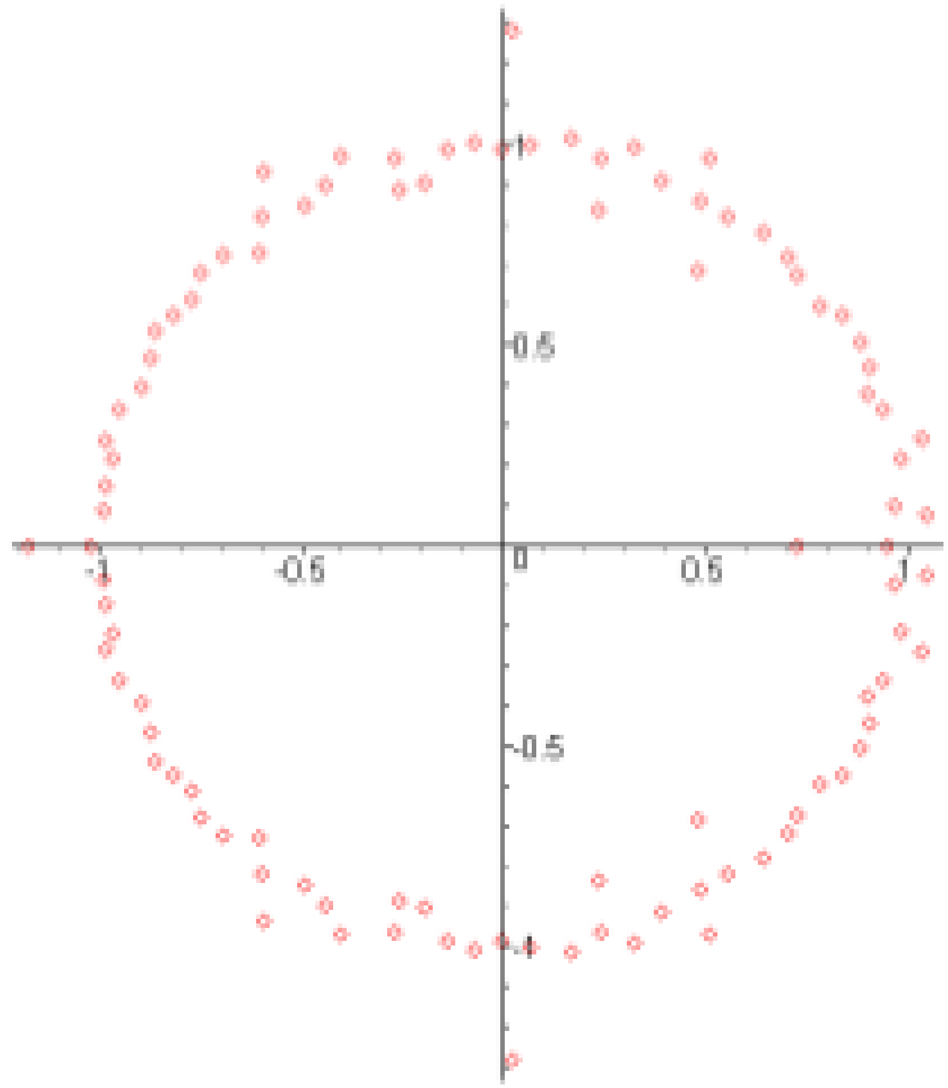
For instance, let $d = 10$ and $f = -x^{10} + x^9 + x^8 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$



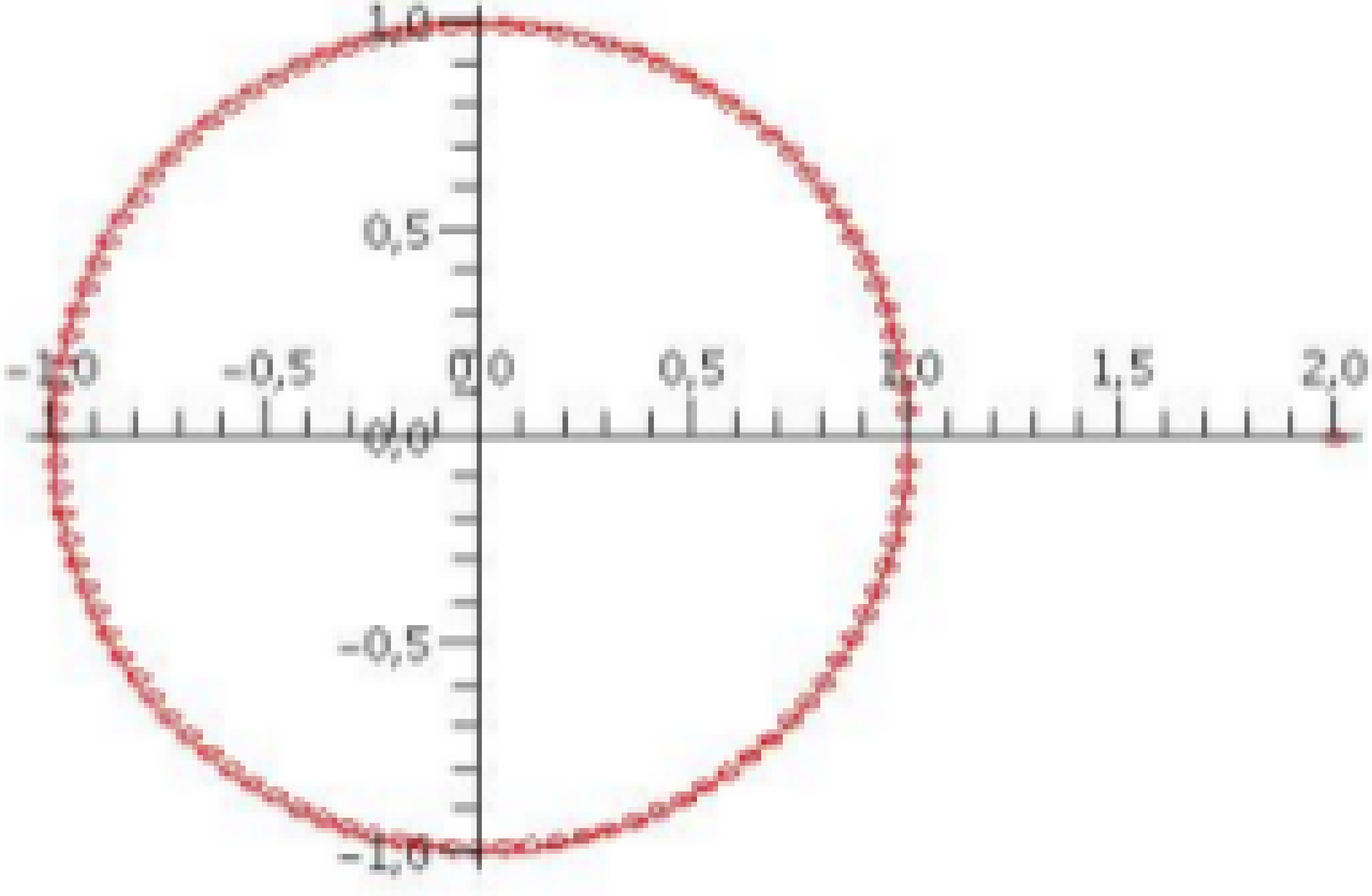
$d = 30$ and $f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$



$d = 100$ and $f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$



$$f = x^{100} - x^{99} - x^{98} - x^{97} - \cdots - x - 1$$



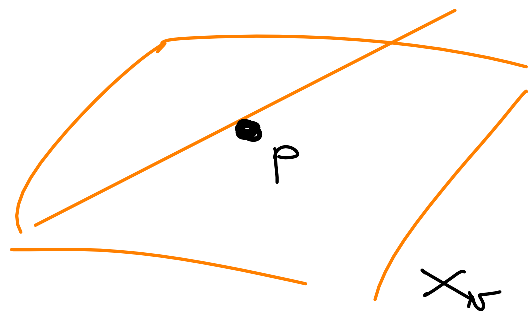
$$\left.\begin{array}{c}\text{Zariski density}\\ + \\ \text{small height}\end{array}\right\} \longrightarrow \begin{array}{c}\text{analytic}\\ \text{density}\end{array}$$

HEIGHTS FROM ARAKELOV GEOMETRY

- $X^n/\mathbb{Q}$ (proper) algebraic variety
- D Cartier divisor on X

For each $v \in M_{\mathbb{Q}} = \{\infty\} \cup \{\text{primes of } \mathbb{Z}\}$ places of $\mathbb{Q}$

- X_v v-adic analytic space $\begin{cases} X(\mathbb{C}) & (v=\infty) \\ \text{Berkovich space} & (v \neq \infty) \end{cases}$
- $\|\ \|_v$ v-adic metric on $\mathcal{O}(D)_v$
- $\overline{D} = (D, (\|\cdot\|_v)_{v \in M_{\mathbb{Q}}})$ metrized Cartier divisor



The **height** of $p \in X(\mathbb{Q})$ wr to $\overline{D}$ is

$$h_{\overline{D}}(p) = -\sum_{v \in M_{\mathbb{Q}}} \log \|s(p)\|_v$$

for any rational section s regular and $\neq 0$ at p

Some examples

$X = \mathbb{P}^1_{\mathbb{Q}}$, $\quad D = (0:1)$

$$\ell \in \mathbb{Q}[x_0, x_1]_1 \longleftrightarrow s_\ell \in H^0(X, \mathcal{O}(D))$$

1) Weil height

For each $v \in M_{\mathbb{Q}}$ set

$$\| s_\ell(p) \|_{v,\mathrm{can}} = \frac{|\ell(p)|_v}{\max(|p_0|_v, |p_1|_v)} \qquad \text{"canonical" metric}$$

for $p \in \mathbb{P}^1(\mathbb{C}_v)$ and $\ell \in \mathbb{C}_v[x_0, x_1]_1$

$$\Longrightarrow \quad h_{\overline{D}} = \text{Weil height}$$

2) Fubini-Study height

$$\begin{cases} \|s_\ell(p)\|_\infty = \dfrac{|\ell(p)|_\infty}{\sqrt{|p_0|_\infty^2 + |p_1|_\infty^2}} \\ \|\cdot\|_v = \|\cdot\|_{v,\mathrm{can}} \quad (v \neq \infty) \end{cases}$$

If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h_{FS}(\xi) = \log\sqrt{a^2+b^2}$

3) Twisted Weil height

$$\begin{cases} \|s_\ell(p)\|_\infty = \dfrac{|\ell(p)|_\infty}{\max(|p_0|_\infty, 2|p_1|_\infty)} \\ \|\cdot\|_v = \|\cdot\|_{v,\mathrm{can}} \quad (v \neq \infty) \end{cases}$$

$h_{\widetilde{\mathrm{Weil}}}(\xi) = \log\max(|a|, 2|b|)$

ESSENTIAL MINIMUM

X, $\overline{D}$ as before

$$\mu^{ess}_{\overline{D}}(X) = \inf\{\theta \in \mathbb{R} \mid \{p \in X(\overline{\mathbb{Q}}) \mid h_{\overline{D}}(p) \leq \theta\} \text{ Zariski dense}\}$$

Fact: $(p_k)_{k\geq 1}$ generic sequence in $X(\overline{\mathbb{Q}})$

i.e. $\forall\, Y \subsetneq X$ $\quad \#\{k \mid p_k \in Y\} < \infty$

Then

$$\underline{\lim}\, h_{\overline{D}}(p_k) \geq \mu^{ess}_{\overline{D}}(X)$$

Pb: For $(p_k)_{k\geq 1}$ generic st $\lim_{k\to\infty} h_{\overline{D}}(p_k) = \mu^{ess}_{\overline{D}}(X)$

study the limit distribution of $G p_k$

EQUIDISTRIBUTION OF GALOIS ORBITS OF SMALL POINTS

THM (Yuan 2008 after Szpiro-Ullmo-Zhang, Bilu, Chambert-Loir, Favre-Rivera, Baker-Rumely, ...)

$X^n/\mathbb{Q}$ proper, $\overline{D}$ metrized divisor.
Sup D ample & $\overline{D}$ semipositive.

Let $(p_k)_{k\geq 1}$ generic st

$$h_{\overline{D}}(p_k) \xrightarrow[k\to\infty]{} \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)}$$

Let $v \in \mathfrak{M}_{\mathbb{Q}}$. Then

$$G\,p_k \xrightarrow[k\to\infty]{} c_1(\|\cdot\|_v)^{\wedge n}$$

proba measure on X_v

By Zhang's theorem on successive algebraic minima

$$\mu^{ess}_{\overline{D}}(X) \leq \frac{h_{\overline{D}}(X)}{\deg_D(X)} \leq (n+1)\ \mu^{ess}_{\overline{D}}(X)$$

$\Rightarrow$ the equidistribution thm can only be applied when

$$\mu^{ess}_{\overline{D}}(X) = \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)}$$

Toric varieties

$\mathbb{T} = \mathbb{G}_m^n$ algebraic torus $/\mathbb{Q}$

A toric variety (with torus $\mathbb{T}$) is a normal variety X
st $\mathbb{T} \subset X$ and $\mathbb{T} \circlearrowright X$

Construction

Σ fan on $\mathbb{R}^n$

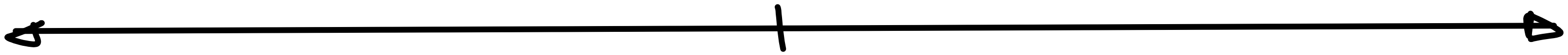
$\sigma \in \Sigma \rightsquigarrow X_\sigma$ affine toric variety

$\tau \subset \sigma \implies X_\tau \hookrightarrow X_\sigma$ open immersion

$$X_\Sigma := \bigcup_{\sigma \in \Sigma} X_\sigma$$

Examples

1)

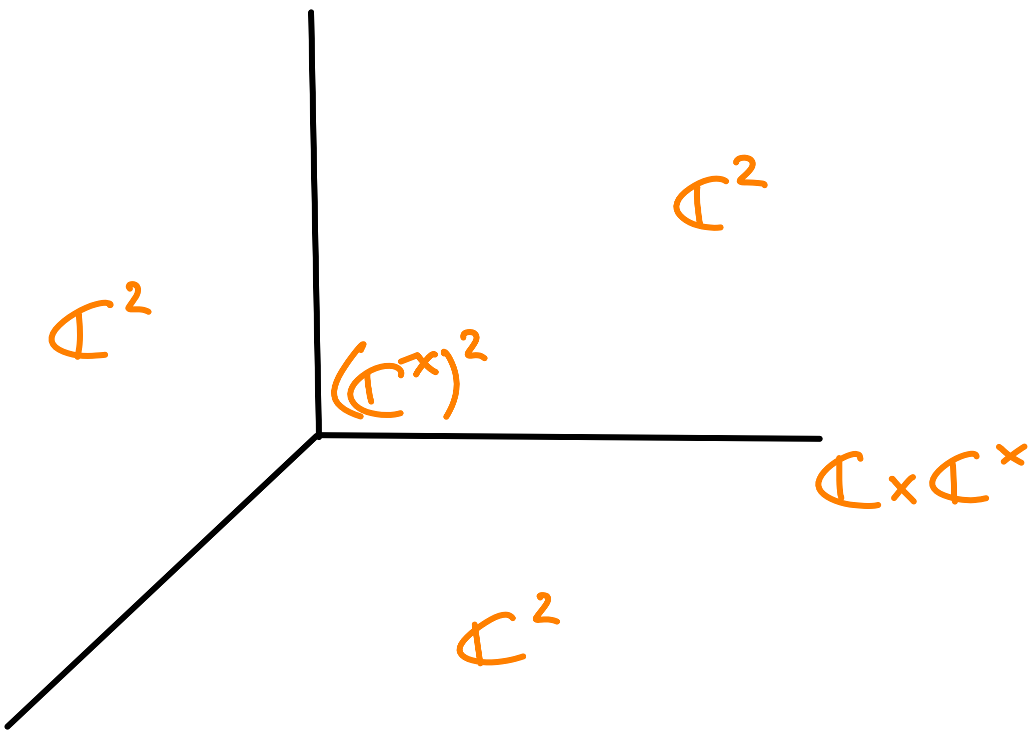


$$\mathbb{C} \hookleftarrow \mathbb{C}^\times \hookrightarrow \mathbb{C}$$

$$x^{-1} \mapsfrom x \mapsto x$$

$X_\Sigma = \mathbb{P}^1$

2)



$X_\Sigma = \mathbb{P}^2$

Toric Cartier divisors

Assume Σ covers $\mathbb{R}^n$ (Σ "complete")

Equivalently X_Σ proper

A virtual support function is $\Psi : \mathbb{R}^n \to \mathbb{R}$

st $\Psi|_\sigma = m_\sigma \in (\mathbb{Z}^n)^\vee \quad \forall \sigma \in \Sigma$

$\Psi \rightsquigarrow D_\Psi = \left(X_\sigma, x^{-m_\sigma} \right)_{\sigma \in \Sigma}$ toric Cartier divisor

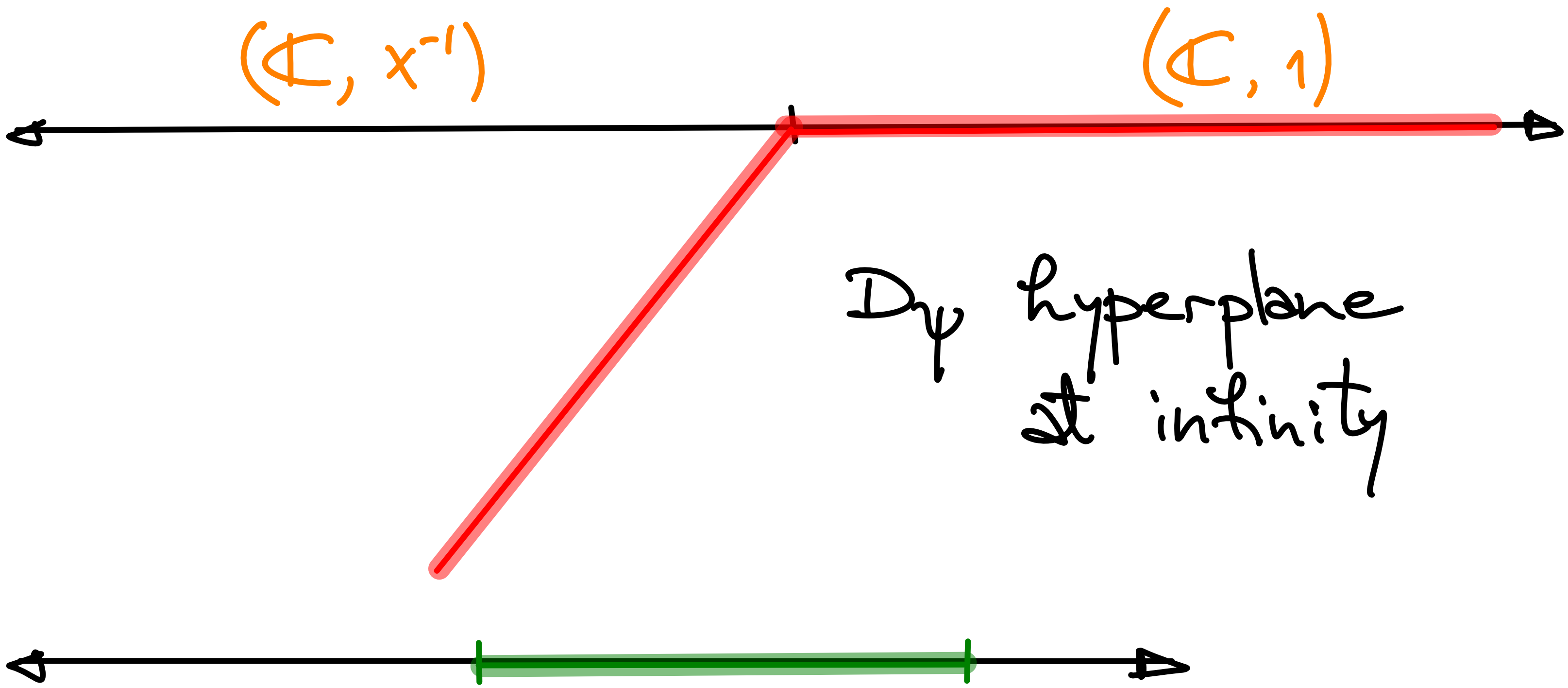
$\Psi \rightsquigarrow \Delta_\Psi = \{ x \in (\mathbb{R}^n)^\vee \mid x \geq \Psi \}$ polytope

Prop: D_Ψ nef $\iff$ Ψ concave

If so,

$$\deg_D(X) = n!\,\mathrm{vol}(\Delta)$$

Ex



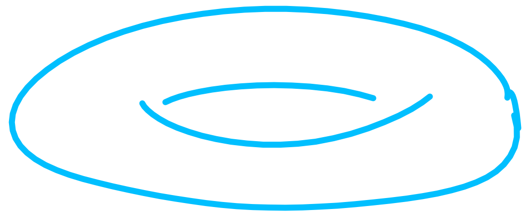
$\Delta = [0,1]$

Toric metrics

$v \in \mathfrak{M}_{\mathbb{Q}}$

$\mathbb{S}_v \subset \mathbb{T}_v$ compact torus

Ex: $\mathbb{S}_\infty = \{ (t_1, \dots, t_n) \in (\mathbb{C}^\times)^n \mid |t_i| = 1 \ \forall i \}$



v-adic toric metric on $\mathcal{O}(D_\Psi)_v :=$ $\mathbb{S}_v$-invariant

Construction

Suppose $v = \infty$

$$\begin{tikzcd} & (\mathbb{C}^\times)^n \arrow[d, "p \mapsto \operatorname{Log}|p|"'] \arrow[dr, "-\log\|s_\Psi\|_v"] & \\ & \mathbb{R}^n \arrow[r, "f_v"'] & \mathbb{R} \end{tikzcd}$$

Thm 1

$$\left\{ \begin{array}{l} f: \mathbb{R}^n \to \mathbb{R} \text{ concave} \\ \text{st } |f - \psi| \text{ bounded} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} v\text{-adic semipositive} \\ \text{metrics on } \mathcal{O}(D_\psi)_v \end{array} \right\}$$

The v-adic roof function is the Legendre-Fenchel dual

$$\vartheta_v : \Delta \to \mathbb{R} \qquad x \mapsto \inf_{\mu \in \mathbb{R}^n} \langle x, \mu \rangle - f_v(\mu)$$

The global roof function is

$$\vartheta := \sum_v \vartheta_v$$

AN ABRIDGED TORIC DICTIONARY

X toric variety with torus $\mathbb{T}$	Σ fan on $\mathbb{R}^n$
D nef toric divisor on X	$\Psi: \mathbb{R}^n \to \mathbb{R}$ concave Σ-linear $\Delta \subset \mathbb{R}^n$ lattice polytope
$\Vert \cdot \Vert_v$ semipositive toric metric on $\mathcal{O}(D)_v$	$\psi_v: \mathbb{R}^n \to \mathbb{R}$ concave st $\lvert \psi_v - \Psi \rvert$ bounded $\vartheta_v : \Delta \to \mathbb{R}$ concave
$\overline{D}$ metrized divisor	$\vartheta = \sum_v \vartheta_v$

Some consequences

Thm 2: If $\overline{D}$ semipositive

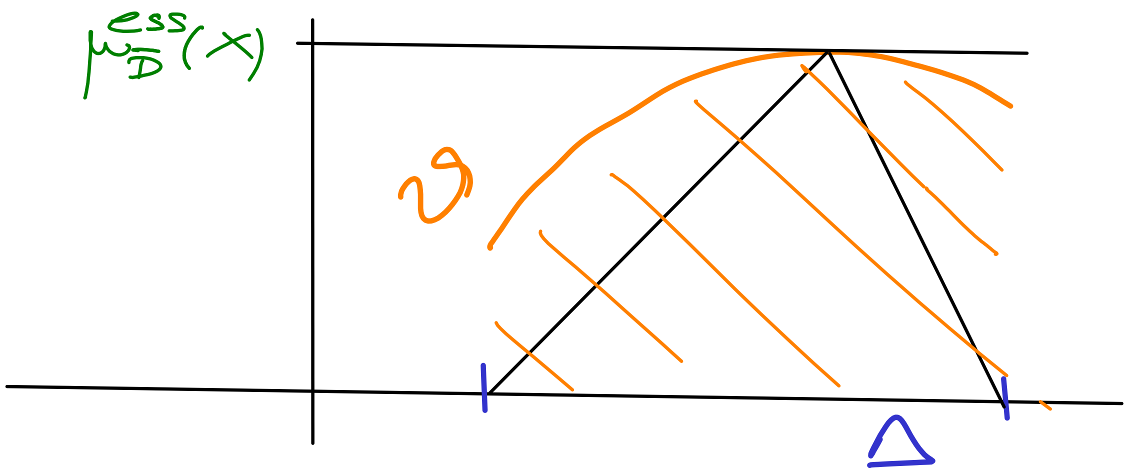
$$h_{\overline{D}}(X) = (n+1)! \int_{\Delta} \vartheta(x)\, d\mathrm{vol}$$

Thm 3:

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x\in\Delta} \vartheta(x)$$

Thm 4: $\overline{D}$ nef $\iff$ ψ_v concave $(\forall v)$ & $\vartheta \geq 0$

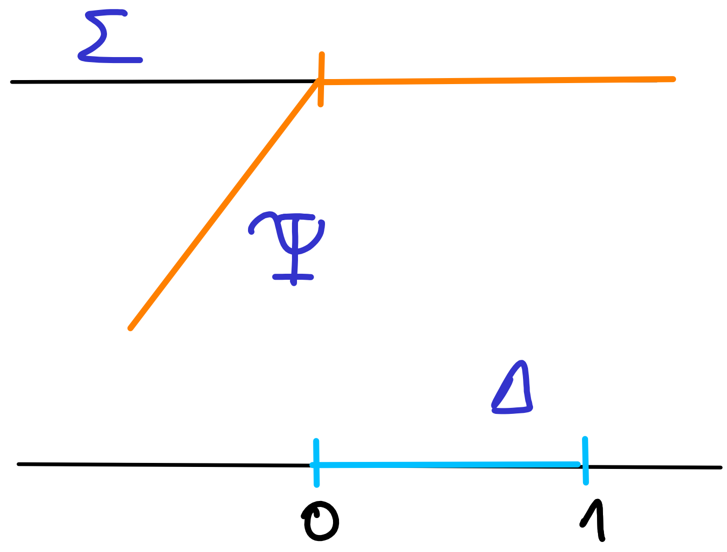
Cor 1 (successive algebraic minima)



Examples revisited

$X = \mathbb{P}^1$

$D = (0:1)$



$\Rightarrow \deg_D(\mathbb{P}^1) = 1$

1) Weil height

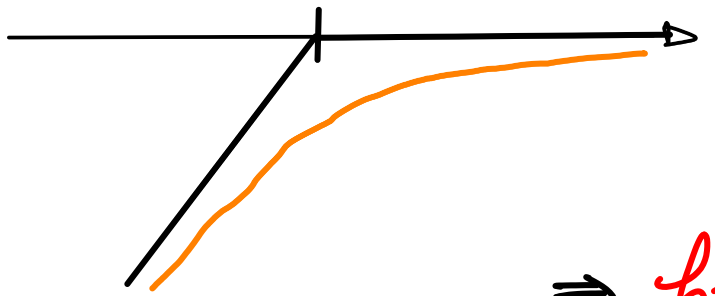
$\psi_v = \Psi$ & $\vartheta_v \equiv 0$ $(\forall v)$, $\vartheta \equiv 0$

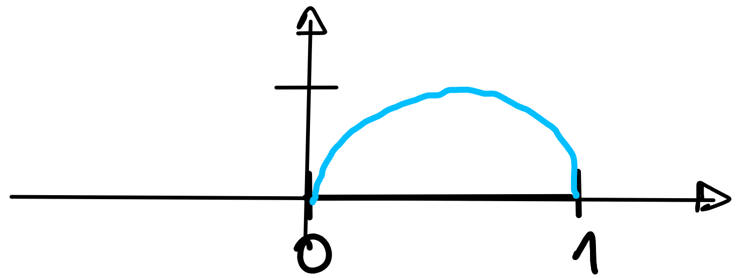
$\Rightarrow h_D(\mathbb{P}^1) = \mu_D^{ess}(\mathbb{P}^1) = 0$

2) Fubini–Study height

$$\psi_\infty(u) = \frac{1}{2}\log(1+e^{-2u})$$

$$\vartheta(x) = \vartheta_\infty(x) = -\frac{1}{2}\big(x\log x + (1-x)\log(1-x)\big)$$

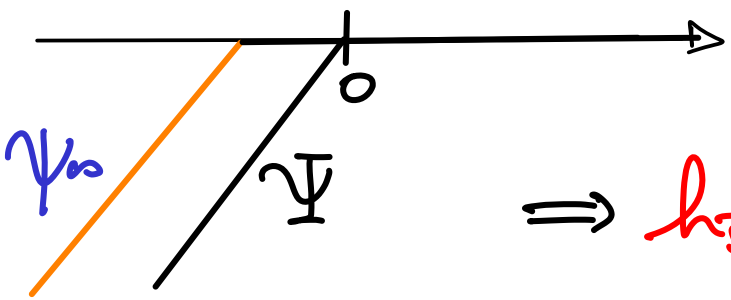


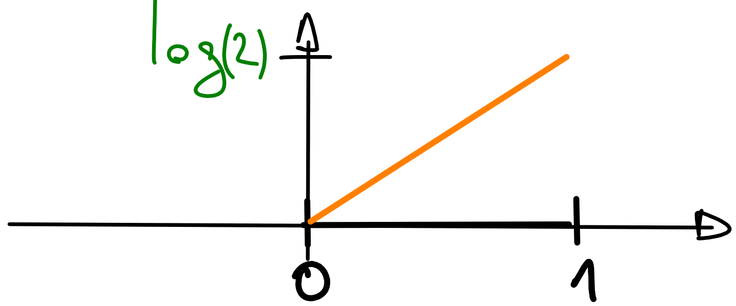


$$\Rightarrow h_{\overline{D}}(\mathbb{P}^1) = \frac{1}{2}, \quad \mu^{ess}_{\overline{D}}(\mathbb{P}^1) = \frac{\log 2}{2}$$

3) Twisted Weil height

$$\vartheta = \vartheta_\infty$$





$$\Rightarrow h_{\overline{D}}(\mathbb{P}^1) = \mu^{ess}_{\overline{D}} = \log 2$$

DISTRIBUTION GALOIS ORBITS OF SMALL POINTS

Cor 2: X toric variety, $\overline{D}$ semipositive with D ample

$$\mu^{ess}_{\overline{D}}(X) = \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)} \iff \vartheta \equiv \text{constant}$$

$\rightsquigarrow$ Yuan's thm$\big|_{\text{toric case}}$ = Bilu's thm

THM 5 X toric variety & $\overline{D}$ semipositive toric metrized divisor on X

Sup ϑ differentiable at x_{max}

Let $(p_k)_{k\geq 1}$ generic st $\lim_{k\to\infty} h_{\overline{D}}(p_k) = \mu^{ess}_{\overline{D}}(X)$

Let $v \in \mathfrak{M}_{\mathbb{Q}}$. Then $\exists\, u_v \in \mathbb{R}^n$ st.

$$G_{p_k} \xrightarrow[k\to\infty]{} \exp_v(u_v)\cdot\mu_v$$

$$\text{with } \mu_v \begin{cases} \text{Haar on } S_v & v=\infty \\ \delta_{\text{Gauss},v} & v\neq\infty \end{cases}$$

If $\vartheta = \vartheta_v + \widetilde{\vartheta}$ then $\{u_v\} = \partial\vartheta_v(x_{max})$

Examples revisited

$X = \mathbb{P}^1$, $D = (0:1)$, $(p_k)_{k \geq 1}$ s.t. $\lim_{k \to \infty} h_D(p_k) = \mu_D^{ess}(X)$

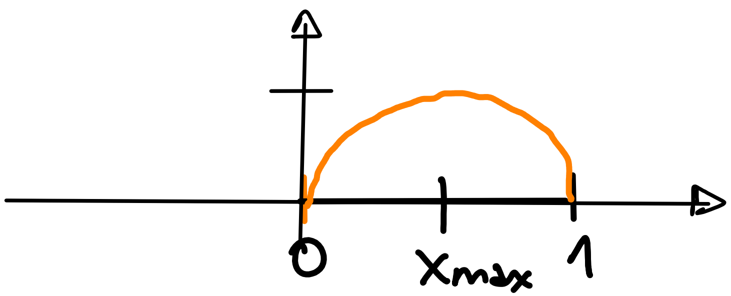
$v = \infty$

1) Weil height

$\vartheta \equiv 0$ diff at any $x_{max} \in (0,1)$

$\Longrightarrow$ $G_{p_k} \xrightarrow[k \to \infty]{} S^1$

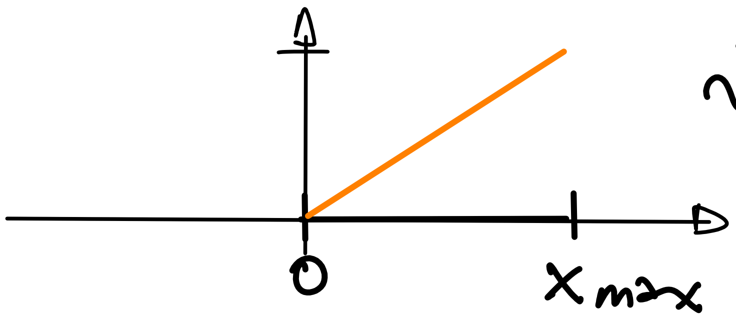
2) Fubini-Study height



ϑ diff at $x_{max} = \frac{1}{2}$

$\Longrightarrow$ $G_{p_k} \xrightarrow[k \to \infty]{} S^1$

3) TWISTED WEIL HEIGHT



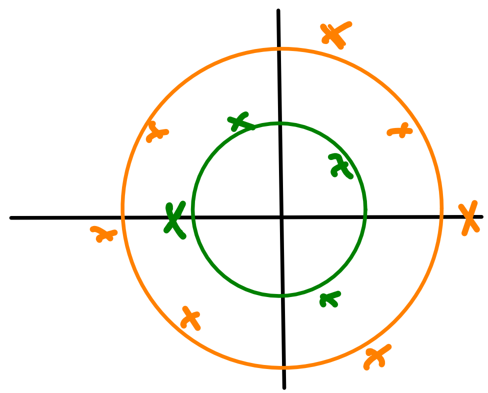
$\rightsquigarrow$ not differentiable at $x_{max} = 1$

Recall that

$$h_{Weil}(\varphi) = \log\max(|p_0|_\infty, 2|p_1|_\infty) + \sum_{v\neq\infty} \log\max(|p_0|_v, |p_1|_v)$$

Take ω_k roots of 1 and set

$$p_k = (1 : \omega_k) \qquad q_k = \left(1 : \frac{\omega_k}{2}\right)$$



$$h(p_k) = h(q_k) = \log 2 = \mu_D^{ess}(X)$$

$\Longrightarrow$ no equidistribution in this case!