# THUE 150

# Essential Minimum & Distribution of Small Points in Toric Varieties

http://atlas.mat.ub.es/personals/sombra

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Bordeaux, 3/10/2013

# Weil height

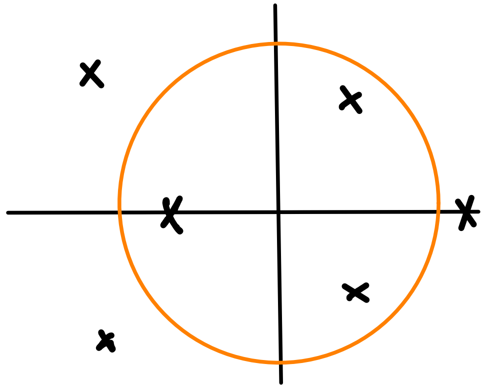
Let $\xi \in \overline{\mathbb{Q}}^\times$ algebraic number of degree $d \geq 1$

$$P_\xi = \alpha_d x^d + \dots + \alpha_0 = \alpha_d \prod_{\eta \in G\xi} (x - \eta)$$

minimal poly of $\xi$ — Galois orbit of $\xi$

The **height** of $\xi$ is

$$h(\xi) = \frac{1}{d} \left( \sum_{\eta \in G\xi} \log \max(1, |\eta|) + \log |\alpha_d| \right)$$



- If $\xi = \frac{a}{b} \in \mathbb{Q}^\times$ then $h(\xi) = \log \max(|a|, |b|)$
- $h(\xi) = 0 \iff \xi$ root of 1 (Kronecker)

# BILU'S EQUIDISTRIBUTION THM

**THM** (Bilu 1997) Let $p_k \in \overline{\mathbb{Q}}^\times$, $k \geq 1$, st

- $\forall p \in \overline{\mathbb{Q}}^\times$, $\#\{k \mid p_k = p\} < \infty$
- $h(p_k) \xrightarrow[k\to\infty]{} 0$

Then $G p_k \to S^1$ uniformly

I.e. $\forall f \in \mathcal{C}^0(\mathbb{C}^\times)$

$$\frac{1}{\# G p_k} \sum_{g \in G p_k} f(g) \xrightarrow[k\to\infty]{} \int f \, d\mu_{S^1}$$

Toric version of Szpiro, Ullmo & Zhang equidistribution (1996)

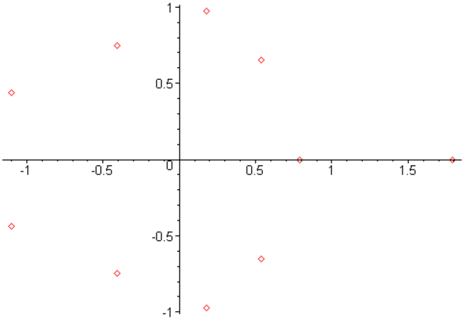
# SOME EXPERIMENTS

Take $P_d \in \mathbb{Z}[x]$ irreducible with

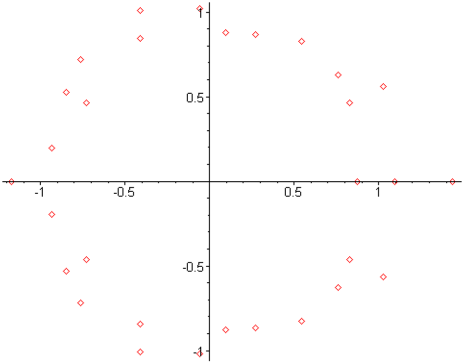
$\deg P_d = d \gg 0$ & $\mathrm{coeffs}(P_d) \subset \{0, \pm 1\}$

Plot the roots of $P_d$ and see what happens...

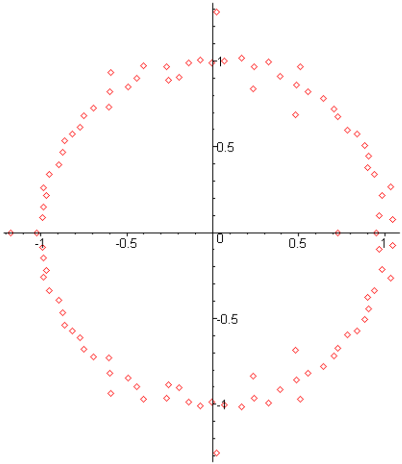
For instance, let $d = 10$ and $f = -x^{10}+x^9+x^8+x^6+x^5-x^4+x^3-x^2+x-1$



$d = 30$ and $f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$



$d = 100$ and $f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \cdots$



# THE ADELIC POINT OF VIEW

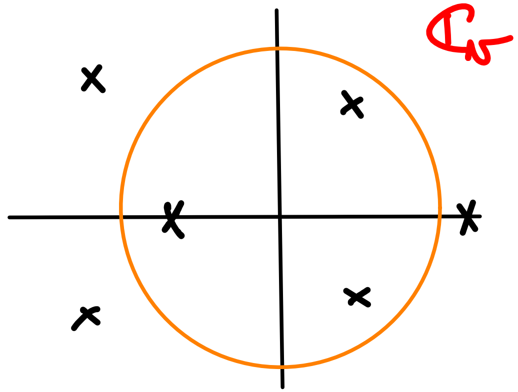
$M_{\mathbb{Q}} = \{\infty\} \cup \{\text{primes of } \mathbb{Z}\}$ places of $\mathbb{Q}$

For each $v \in M_{\mathbb{Q}}$ choose an embedding

$$\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_v$$

The height of $\xi \in \overline{\mathbb{Q}}^{\times}$ can also be written as

$$h(\xi) = \frac{1}{\# G\xi} \sum_{v \in M_{\mathbb{Q}}} \sum_{\eta \in G\xi} \log \max(1, |\eta|_v)$$



# $v$-ADIC EQUIDISTRIBUTION

There is a $v$-adic version of Bilu's thm:

THM (Chambert-Loir, Favre & Rivera-Letelier, Baker & Rumely 2005)

Let $p_k \in \overline{\mathbb{Q}}^\times$ st $\#\{k \mid p_k = p\} < \infty$ $(\forall p \in \overline{\mathbb{Q}}^\times)$ and $h(p_k) \xrightarrow[k\to\infty]{} 0$. Let $v \in M_{\mathbb{Q}}$.

Then $\forall P \in \mathbb{C}_v[x^{\pm 1}]$,

$$\frac{1}{\# G_{p_k}} \sum_{g \in G_{p_k}} P(g) \xrightarrow[k\to\infty]{} \sup_{|x| \le 1} |P(x)|_v$$

Better: $\forall f \in \mathcal{C}^0(\overline{\mathbb{Q}}_v)$ — $v$-adic analytification of $\overline{\mathbb{Q}}^\times$

$$\frac{1}{\# G_{p_k}} \sum_{g \in G_{p_k}} P(g) \longrightarrow \int f \, d\delta_{\mathrm{Gauss}_v}$$

Dirac delta measure on Gauss point

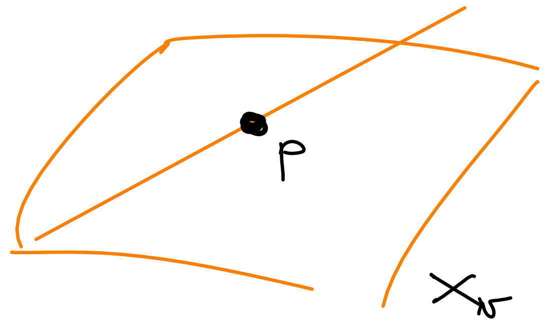
$$\left.\begin{array}{c}\text{Zariski density}\\ + \\ \text{small height}\end{array}\right\} \Longrightarrow \begin{array}{c}\text{analytic}\\ \text{density}\end{array}$$

# Heights from Arakelov geometry

- $X^n/\mathbb{Q}$ (proper) algebraic variety
- $D$ Cartier divisor on $X$

For each $v \in M_\mathbb{Q}$

- $X_v$ $v$-adic analytic space
  - $X(\mathbb{C})$ $\quad v=\infty$
  - Berkovich space $\quad v\neq\infty$
- $\|\ \|_v$ $v$-adic metric on $\mathcal{O}(D)$
- $\overline{D} = \left(D, (\|\cdot\|_v)_{v\in M_\mathbb{Q}}\right)$ adelic metrized Cartier divisor




The height of $p \in X(\mathbb{Q})$ wr to $\overline{D}$ is

$$h_{\overline{D}}(p) = -\sum_{v\in M_\mathbb{Q}} \log\|s(p)\|_v$$

for any rational section $s$ regular and $\neq 0$ at $p$

# SOME EXAMPLES

$X = \mathbb{P}^1_{\mathbb{Q}}$, $D = (0:1)$

$$\ell \in \mathbb{Q}[x_0, x_1]_1 \longleftrightarrow s_\ell \in H^0(X, \mathcal{O}(D))$$

1) WEIL HEIGHT

For each $v \in \mathcal{M}_{\mathbb{Q}}$ set

$$\| s_\ell(p) \|_{v,\mathrm{can}} = \frac{|\ell(p)|_v}{\max(|p_0|_v, |p_1|_v)}$$

"canonical" metric

for $p \in \mathbb{P}^1(\mathbb{C}_v)$ and $\ell \in \mathbb{C}_v[x_0, x_1]_1$

$\Longrightarrow$ $h_{\overline{D}}$ = Weil height

2) FUBINI-STUDY HEIGHT

$$\begin{cases} \| s_\ell(p) \|_\infty = \dfrac{|\ell(p)|_\infty}{\sqrt{|p_0|_\infty^2 + |p_1|_\infty^2}} \\ \| \cdot \|_v = \| \cdot \|_{v,\mathrm{can}} \quad (v \neq \infty) \end{cases}$$

$$h_{FS}(p) = \log \sqrt{|p_0|_\infty^2 + |p_1|_\infty^2} + \sum_{v \neq \infty} \log \max(|p_0|_v, |p_1|_v)$$

3) TWISTED WEIL HEIGHT

$$\begin{cases} \| s_\ell(p) \|_\infty = \dfrac{|\ell(p)|_\infty}{\max(|p_0|_\infty, 2|p_1|_\infty)} \\ \| \cdot \|_v = \| \cdot \|_{v,\mathrm{can}} \quad (v \neq \infty) \end{cases}$$

$$h_{\widetilde{\mathrm{Weil}}}(p) = \log \max(|p_0|_\infty, 2|p_1|_\infty) + \sum_{v \neq \infty} \log \max(|p_0|_v, |p_1|_v)$$

# ESSENTIAL MINIMUM

$X, \bar{D}$ as before

$$\mu^{ess}_{\bar{D}}(X) = \inf\{\theta \in \mathbb{R} \mid \{p \in X(\bar{\mathbb{Q}}) \mid h_{\bar{D}}(p) \leq \theta\} \text{ Zariski dense}\}$$

<u>Fact:</u> $(p_k)_{k \geq 1}$ generic sequence in $X(\bar{\mathbb{Q}})$

i.e. $\forall Y \subsetneq X \qquad \#\{k \mid p_k \in Y\} < \infty$

Then
$$\underline{\lim}\, h_{\bar{D}}(p_k) \geq \mu^{ess}_{\bar{D}}(X)$$

<u>Pb:</u> For $(p_k)_{k \geq 1}$ generic st $\lim_{k \to \infty} h_{\bar{D}}(p_k) = \mu^{ess}_{\bar{D}}(X)$

study the Limit distribution of $G p_k$

# An abridged toric dictionary

joint with Burgos (Madrid) & Philippon (Paris)

- $\mathbb{T} = (\overline{\mathbb{Q}}^\times)^n$ split algebraic torus
- $\mathbb{S}_v$ compact subtorus ($v \in \mathfrak{M}_{\mathbb{Q}}$)

| | |
|---|---|
| $X$ toric variety with torus $\mathbb{T}$ | $\Sigma$ fan on $\mathbb{R}^n$ |
| $D$ toric Cartier divisor on $X$ | $\Psi : \mathbb{R}^n \to \mathbb{R}$ piecewise linear<br>$\Delta \subset \mathbb{R}^n$ lattice polytope |
| $\lVert\cdot\rVert_v$ toric metric on $\mathcal{O}(D)_v$ ($\mathbb{S}_v$-invariant) | $\psi_v : \mathbb{R}^n \to \mathbb{R}$ st $\lvert\psi_v - \Psi\rvert$ bounded<br>$\vartheta_v : \Delta \to \mathbb{R}$ concave |
| $\overline{D}$ adelic metrized divisor | $\vartheta = \sum_v \vartheta_v$ |

# SOME SAMPLE CONSEQUENCES

- If $D$ nef $\Longrightarrow$ $\deg_D(X) = n!\,\mathrm{vol}(\Delta)$ (Teissier 1975?)
- If $\overline{D}$ semipositive $\Longrightarrow$ $h_{\overline{D}}(X) = (n+1)! \int_\Delta \vartheta(x)\, d\mathrm{vol}$

(BPS 2008)

- $\overline{D}$ nef $\Longleftrightarrow$ $\psi_v$ concave $(\forall v)$ & $\vartheta \geq 0$

(BPS + Moriwaki 2011)

<u>THM</u> (BPS 2013)

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x \in \Delta} \vartheta(x)$$

# THE THEOREM ON SUCCESSIVE ALGEBRAIC MINIMA

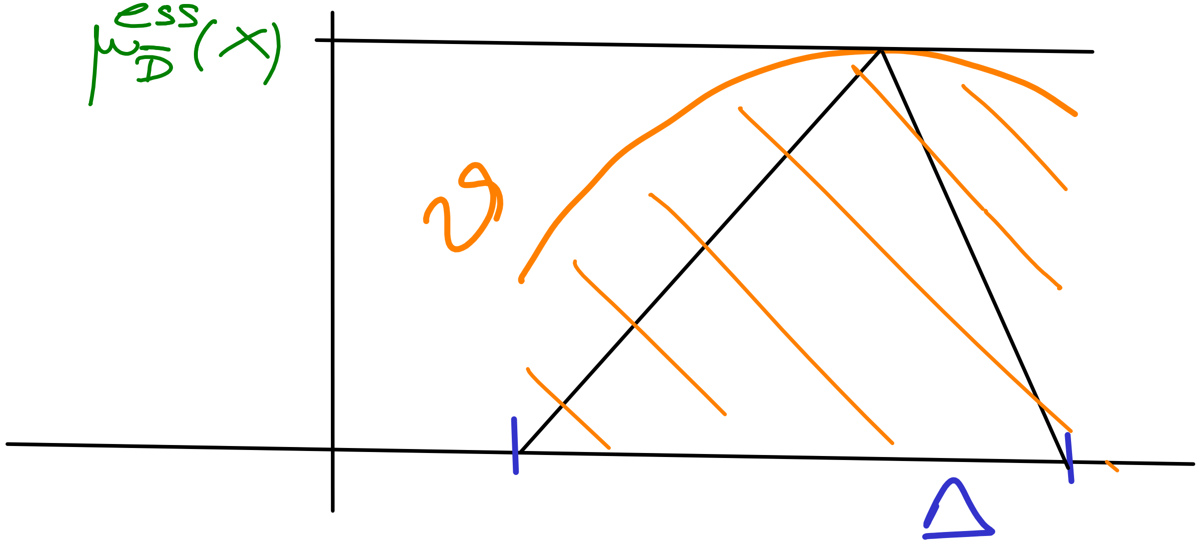
THM (ZHANG 1995) $X^n/\mathbb{Q}$ proper variety

$\overline{D}$ adelic metrized divisor on $X$

Sup. $\overline{D}$ nef. Then

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) \leq \frac{h_{\overline{D}}(X)}{\deg_D(X)} \leq (n+1)\,\mu^{\mathrm{ess}}_{\overline{D}}(X)$$

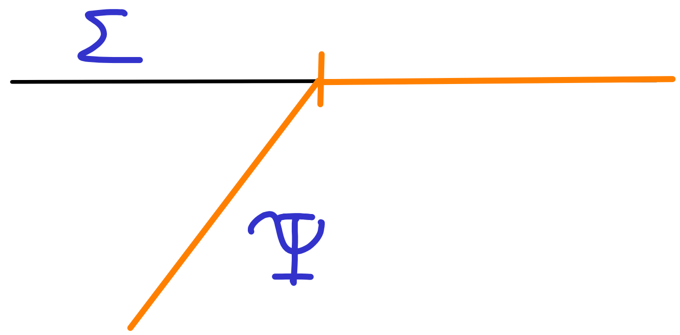
Sup. $X, \overline{D}$ toric. Then this means

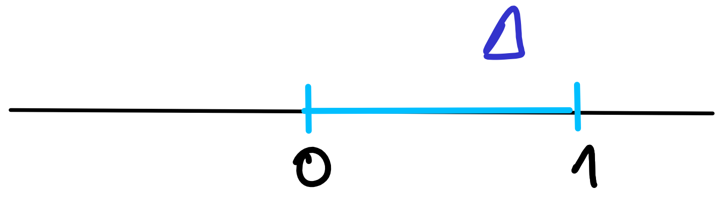


# Examples revisited

$X = \mathbb{P}^1$

$D = (0:1)$




$\Rightarrow \deg_D(\mathbb{P}^1) = 1$

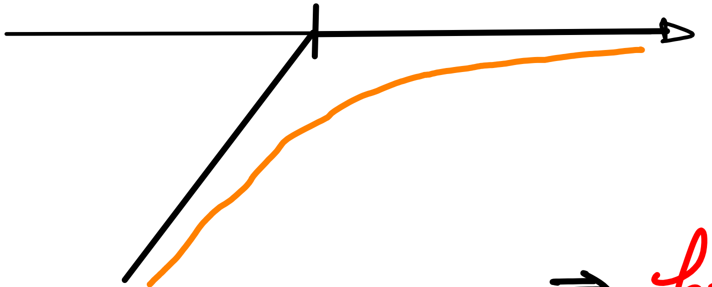
## 1) Weil height

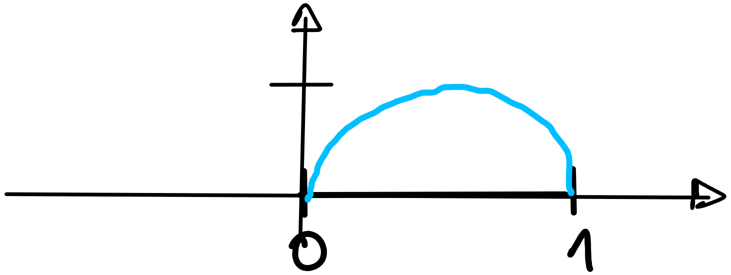
$\psi_v = \Psi$ & $\vartheta_v \equiv 0$ $(\forall v)$, $\vartheta \equiv 0$

$$\Rightarrow h_D(\mathbb{P}^1) = \mu_D^{ess}(\mathbb{P}^1) = 0$$

## 2) Fubini-Study height

$$\psi_\infty(u) = \frac{1}{2}\log(1+e^{-2u}) \qquad \vartheta(x) = \vartheta_\infty(x) = -\frac{1}{2}\left(x\log x + (1-x)\log(1-x)\right)$$

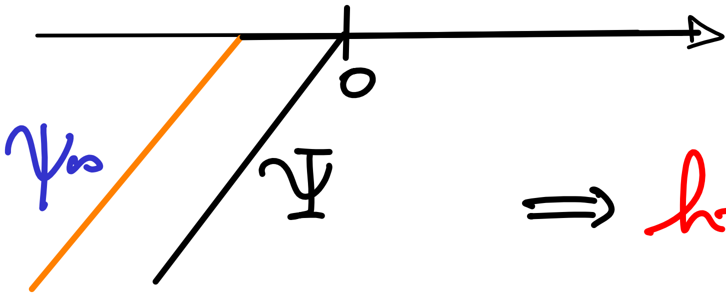


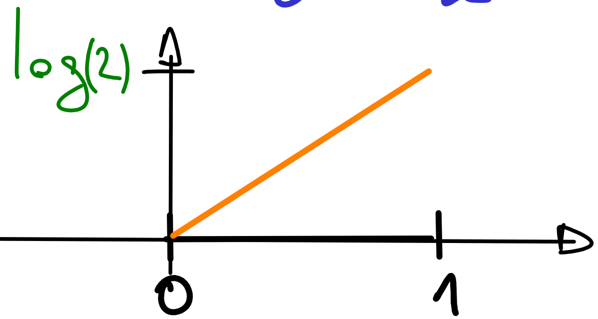



$$\Rightarrow h_{\overline{D}}(\mathbb{P}^1) = \frac{1}{2}, \quad \mu^{ess}_{\overline{D}}(\mathbb{P}^1) = \frac{\log 2}{2}$$

## 3) Twisted Weil height

$$\vartheta = \vartheta_\infty$$




$$\Rightarrow h_{\overline{D}}(\mathbb{P}^1) = \mu^{ess}_{\overline{D}} = \log 2$$

# DISTRIBUTION OF SMALL POINTS IN THE TORIC CASE

**THM** (BPS + Rivera-Letelier 2013)

Let $X$ toric variety & $\overline{D}$ semipositive toric metrized divisor on $X$

Sup $\vartheta$ differentiable at $x_{max}$

Let $(p_k)_{k\geq 1}$ generic st $\lim_{k\to\infty} h_{\overline{D}}(p_k) = \mu^{ess}_{\overline{D}}(X)$

Let $v \in \mathfrak{M}_{\mathbb{Q}}$. Then $\exists\, u_v \in \mathbb{R}^n$ st.

$$\delta_{G p_k} \xrightarrow[k\to\infty]{} \exp_v(u_v)\cdot \mu_v$$

uniform proba measure on $Gp_k$

with $\mu_v = \begin{cases} \text{Haar on } \mathbb{S}_v & v=\infty \\ \delta_{\mathrm{Gauss},v} & v\neq\infty \end{cases}$

If $\vartheta = \vartheta_v + \tilde{\vartheta}$ then $\{u_v\} = \partial\vartheta_v(x_{max})$

# WHAT DO WE KNOW FOR GENERAL VARIETIES?

**THM** (Yuan 2008 after SUZ, B, C-L, F-RL, BR, ...)

$X^n/\mathbb{Q}$ projective variety, $\overline{D}$ metrized divisor.

Sup $D$ big & $\overline{D}$ semipositive.

Let $(p_k)_{k\geq 1}$ generic st

$$h_{\overline{D}}(p_k) \xrightarrow[k\to\infty]{} \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)}$$

Let $v \in \mathfrak{M}_{\mathbb{Q}}$. Then

$$\delta_{G p_k} \xrightarrow[k\to\infty]{} c_1(\|\cdot\|_v)^{\wedge n}$$

proba measure on $X$

**Obs:** Can only be applied when

$$\mu^{ess}_{\overline{D}}(X) = \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)} \quad !$$

WHAT DOES THIS SAY IN THE TORIC CASE?

Sup. $X, \overline{D}$ toric. Then

$$\mu^{ess}_{\overline{D}}(X) = \frac{h_{\overline{D}}(X)}{(n+1)\deg_D(X)} \iff \vartheta \equiv \text{constant}$$

$$\iff \exists\ \gamma \in \overline{\mathbb{Q}}^{\times}\ \&\ c \in \mathbb{R}\ \text{s.t.}$$

$$h_{\overline{D}}(p) = h_{Weil}(\gamma \cdot p) + c$$

i.e. $\text{Yuan's thm}\Big|_{\text{toric case}} = \text{Bilu's thm}$

# EXAMPLES REVISITED (AGAIN!)

$X = \mathbb{P}^1$, $D = (0:1)$, $(p_k)_{k \geq 1}$ s.t. $\lim_{k\to\infty} h_{\bar{D}}(p_k) = \mu^{ess}_{\bar{D}}(X)$

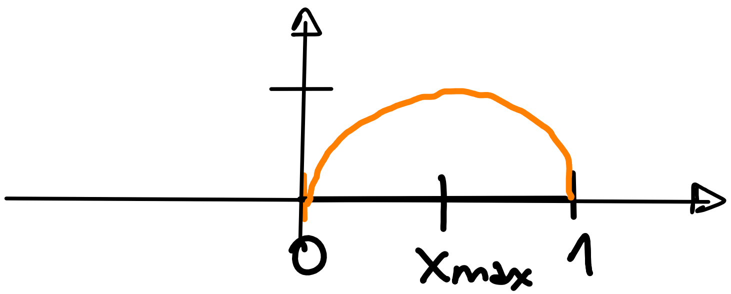
$v = \infty$

## 1) WEIL HEIGHT

$\vartheta \equiv 0$ diff at any $x_{max} \in (0,1)$

$\Longrightarrow$ $G_{p_k} \xrightarrow[k\to\infty]{} S^1$

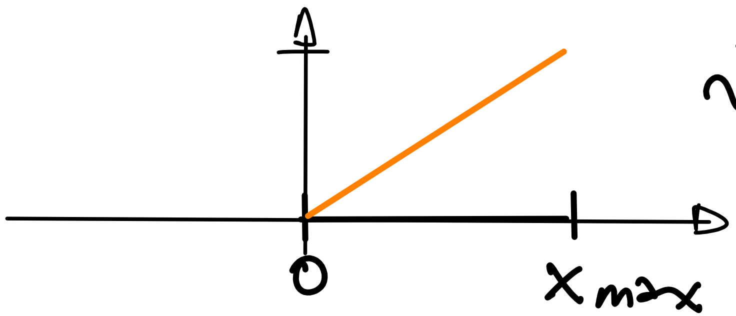
## 2) FUBINI-STUDY HEIGHT




$\vartheta$ diff at $x_{max} = \frac{1}{2}$

$\Longrightarrow$ $G_{p_k} \xrightarrow[k\to\infty]{} S^1$

## 3) TWISTED WEIL HEIGHT



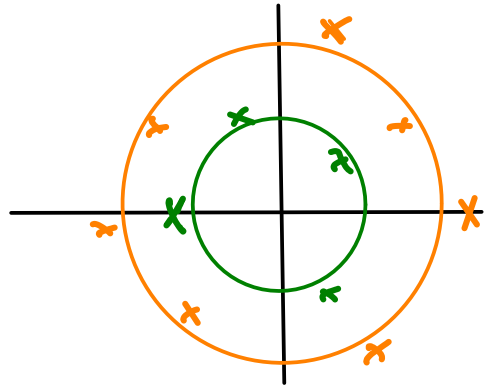

$\rightsquigarrow$ not differentiable at $x_{max}=1$

Recall that

$$h_{Weil}(\varphi) = \log\max(|p_0|_\infty, 2|p_1|_\infty) + \sum_{v\neq\infty} \log\max(|p_0|_v, |p_1|_v)$$

Take $\omega_k$ roots of 1 and set

$$p_k = (1:\omega_k) \qquad q_k = \left(1:\frac{\omega_k}{2}\right)$$



$$h(p_k) = h(q_k) = \log 2 = \mu_{\overline{D}}^{ess}(X)$$

$\Rightarrow$ no equidistribution in this case!

# THUE 150

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