Leuven 10/6/2011

Metrics, measures & heights

joint with Burgos (Madrid) & Philippon (Paris)

Contents

§1 The BKK theorem

Let K a field

$f_1, \dots, f_n \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ generic Laurent polynomials

$$\#\{\xi \in (\overline{K}^\times)^n \mid f_i(\xi) = 0 \ \forall i\} = MV(N(f_1), \dots, N(f_n))$$

Minding 1841 (for $n=2$)

Kushnirenko, Bernstein, Khovanskii mid '70

If $f_1, \dots, f_n$ are "sparse" in the sense that the MV of their Newton polytopes is small, they have a small # of common roots.

Better understood in toric context:

$\Delta \subset \mathbb{R}^n$ lattice polytope

$\rightsquigarrow$ X, L polarized toric variety

$\deg_L(X) = \#\{z \in X / s_i = 0 \ \forall i\} = n! \, \mathrm{Vol}(\Delta)$ (Teissier 1979)

$s_1, \dots, s_n$ generic section of L

If $N(f_i) \subset \Delta \rightsquigarrow f_i$ section of L

Gives entries to the toric dictionary in algebraic geom

polarized tv (X, L) with torus $(\mathbb{C}^\times)^n$ $\longleftrightarrow$ lattice polytope $\Delta \subset \mathbb{R}^n$

degree of X $\longleftrightarrow$ volume of Δ

§2 Height of tv

$q = a/b \quad a, b \in \mathbb{N}^\times$ coprime

$(1:q) \in \mathbb{P}^1(\mathbb{Q}) \qquad h(1:q) = \log(\max a, b)$ height of $(1:q)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$
bit complexity of q

<u>Statement of main result</u>

$K = \mathbb{Q}$

$M_{\mathbb{Q}} = \{\infty\} \cup \{\text{primes}\}$ set of places of $\mathbb{Q}$

For $v \in M_{\mathbb{Q}}$

$\vartheta_v : \Delta \to \mathbb{R}$ concave function st $\vartheta_v \equiv 0 \ \forall v$

$\rightsquigarrow \overline{L}$ "arithmetic structure" on X, L

$h_{\overline{L}}(X)$ height of X wr to $\overline{L}$
$=$ arithmetic degree of X

Thm (BPS) $\quad h_{\overline{L}}(X) = (n+1)! \displaystyle\sum_{v \in M_{\mathbb{Q}}} \int_\Delta \vartheta_v \, dx$

To explain & prove this formula, we have to understand the Arakelov geometry of toric varieties

Basic objects:
- integral models
- metrics
- measures
- heights

The news are that this objects can be translated in terms of convex analysis

Related to:
- polyhedral complexes
- concave functions
- Monge-Ampère measures
- integrals of Legendre-Fenchel duality

We add new words to the arithmetic geometry part of the toric dictionary together with their corresponding translations.

Gives a wealth of examples in Arakelov geometry

§3 Heights in Arakelov geometry

A geometric situation: $k(t)$ a function field

X proper var $/ k(t)$, L lb on X

$\mathcal{X}$ model of $X / \mathbb{P}^1$ = scheme $/ \mathbb{P}^1_k$ with generic fiber X

$$\mathcal{X} \to \mathbb{P}^1_k$$

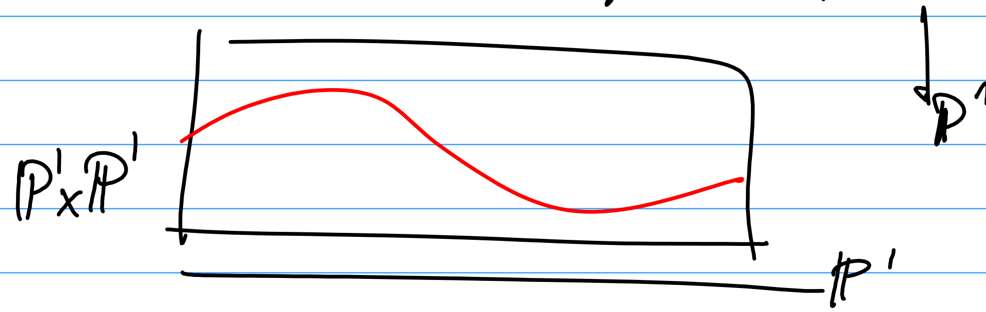
$\mathcal{L}$ model of L = lb on $\mathcal{X}$ with gen fiber L

$h_{\mathcal{L}}(X) := \deg_{\mathcal{L}}(\mathcal{X})$ = complexity of the fibration

<u>Ex</u> $X = (1 : \frac{g}{f}) \in \mathbb{P}^1(k(t))$ $f, g \in k[t]\setminus\{0\}$ coprime

$\mathcal{X}$ = closure of $\{((1:t),(f(t):g(t))) \mid t \in \overline{k}\} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$

$\mathcal{L} = \pi_2^* \mathcal{O}(1)$

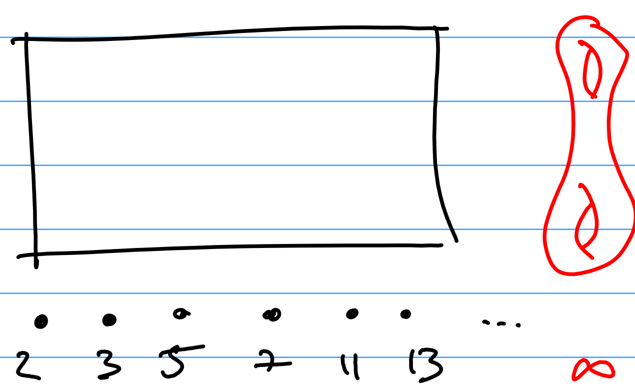


$h(X) = \max\{\deg f, \deg g\}$

How can we do for varieties / $\mathbb{Q}$?

Put $\mathbb{Q}$ instead of $k(t)$ above.
Can consider a model of X over $\mathbb{Z}$

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \operatorname{Spec} \mathbb{Z} \end{array}$$



Major pb: $\operatorname{Spec} \mathbb{Z}$ is not compact
<u>Partial solution</u>: compactify adding a "point at infinity" ∞

But $\operatorname{Spec} \mathbb{Z} \cup \{\infty\}$ is <u>not</u> a scheme.

<u>Second idea</u> $\deg_{\mathcal{L}}(X)$ does not use all the information:
for a rational section s of $\mathcal{L}$, we only need its order of vanishing at points of X.
Leads to the notion of metrised line bundle

§4 Metrised line bundles and measures

Consider absolute values $/\mathbb{Q}$

$|\cdot|_\infty$ standard: completion $\mathbb{R}$

$|\cdot|_p$ p-adic: completion $\mathbb{Q}_p$

Let X algebraic var $/\mathbb{Q}$ (proper) $v \in M_{\mathbb{Q}}$

= object of desire of arith geom.

Consider analytic varieties:

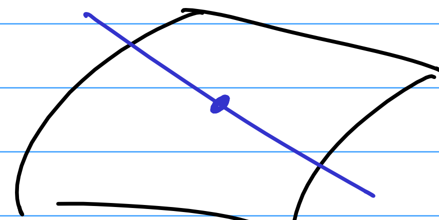
$$X_v \begin{cases} X(\mathbb{C}) \ (v=\infty): \text{complex analytic space} \\ X_{Ber,p} \ (v = p \text{ prime}): \text{Berkovich space} \end{cases}$$

(as in Kedlaya, Tuillier & Gubler talks)

L lb on X $\rightsquigarrow$ L_v analytification

A <u>metric</u> on L_v is way to measure the size of a section at a point of X.
Geometrically: a continuous family of norms

$$s\,,\quad \underset{\substack{\uparrow \\ X_v}}{\xi} \mapsto \|s(\xi)\|_v$$



Recall: a compact analytic space has no nonconstant global functions.
global sections play the role of nonconstant global functions.
Metrics allow to measure their size at a point.

Write $\bar{L}_v = (L, \|\cdot\|_v)$ for short. We will assume $\bar{L}$ <u>semipositive</u> $\rightsquigarrow$ induces a measure μ_v on X_v!

EXAMPLES:

- $v=\infty$, $\|\cdot\|_\infty$ smooth, $c_1(L_\infty, \|\cdot\|_\infty)$ Chern form of

  - $\|\ \|_\infty \geq 0$ iff $\int_D c_1 \geq 0$ $\forall D \hookrightarrow X_\infty$ disk

  - $\mu_\infty(E) = \int_E c_1^{\wedge n}$

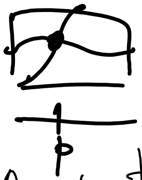
- $v=p$ $\|\cdot\|_p$ induced by an integral model $/\mathbb{Z}_p$

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} \\ & & \downarrow \pi \\ & & \mathrm{Spec}(\mathbb{Z}_p) = \{0, \eta\} \end{array}$$

($\|\cdot\|_p$ "algebraic")

$\|\cdot\|_p \geq 0$ iff $\deg_{\mathcal{L}}(C) \geq 0$ $\forall C \subset \pi^{-1}(0)$ curve

μ_p discrete measure supported on $\mathrm{red}(V)$ $\quad V$ component of $\pi^{-1}(0)$

ξ  $\xi \in X$ $\quad p^{-\mathrm{ord}_{\xi_0}(s)}$) val

Def: A metrized lb is $\overline{L} = (L, (\|\cdot\|_v)_v)$
+ global condition

= arithmetic structure on X, L

Allows to define height of subvarieties.

L, s ut. prop Y

Arithmetic Bézout thm: $Y \subset X$

$$h(Y \cdot \mathrm{div}(s)) = h(Y) + \sum_v \int_{Y_v} \log \|s\|_v \, \mu_v$$

The μ_v's also control the distribution of points of small height
(Bilu, Szpiro–Ullmo–Zhang, Yuan)

Other results: Hilbert–Samuel
Lefschetz fixed pt formula
Riemann–Roch
...

The "adelic" point of view of heights due to Zhang, Gubler, Chambert-Loir

Ex $X = (a:b) \in \mathbb{P}^1(\mathbb{Q})$ $a, b \in \mathbb{Z}$ coprime

$\ell \in \mathbb{Q}[x_0, x_1]$ linear form $\Leftrightarrow$ section s_ℓ of $\mathcal{O}(1)$

$\xi \in \mathbb{P}^1_v$

$$\| s_\ell(\xi) \|_{v,\mathrm{can}} = \frac{|\ell(\xi)|_v}{\max |\xi_0|_v, |\xi_1|_v}$$

$$h_{\overline{\mathcal{O}(1)}}(\xi) = \max \log|a|, \log|b|$$

§5 Integral models of tv

$T \simeq \mathbb{G}^n_{m,\mathbb{Q}}$ split torus$/\mathbb{Q}$, $N = \mathrm{Hom}(T, \mathbb{Q}^\times)$, $M = N^\vee$

$\mathbb{Z}_p$ p-adic integers: discrete valuation ring

$\mathbb{A}^1$ tv$/\mathbb{Z}_p$ is a normal scheme

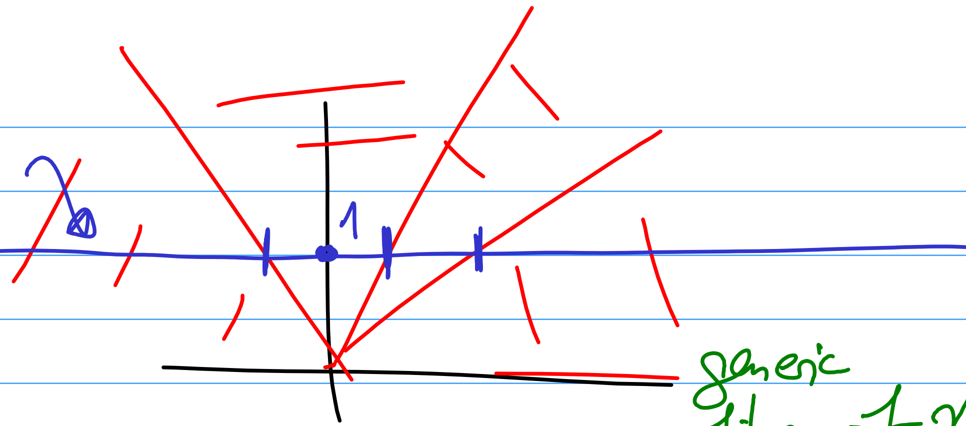
$$\begin{array}{ccc} \mathcal{X} & \curvearrowleft \mathbb{T} & \text{torus}/\mathbb{Z}_p \quad \text{as in Gubler} \\ \downarrow & & \\ \mathrm{Spec}\,\mathbb{Z}_p & & \end{array}$$

Is a sort of "T-variety of complexity 1" over a germ of a curve $\mathrm{Spec}(R)$

cf. talk of Jürgen Hausen

ts/dvr are in 1-to-1 correspondence with fans in $N \times \mathbb{R}_{\geq 0}$ (Mumford) polyhedral complex $\mathcal{X}$

special fiber of $\mathcal{X}$



generic fiber of $\mathcal{X}$

In the complete case $|\tilde{\Sigma}| = N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ all polyhedral complexes of $N_{\mathbb{R}}$ arise in this way (Burgos – S.)

Σ complete fan in $N_{\mathbb{R}}$ $\rightsquigarrow$ X_Σ tv

Ψ virtual support function $/\Sigma$

$\rightsquigarrow$ L_ψ, s_ψ equivariant lb on Σ + invariant section

Entry 1 (1) proper models $/\mathbb{Z}_p$ of X_Σ

$\Updownarrow$

complete polyhedral complex Π in $N_{\mathbb{R}}$

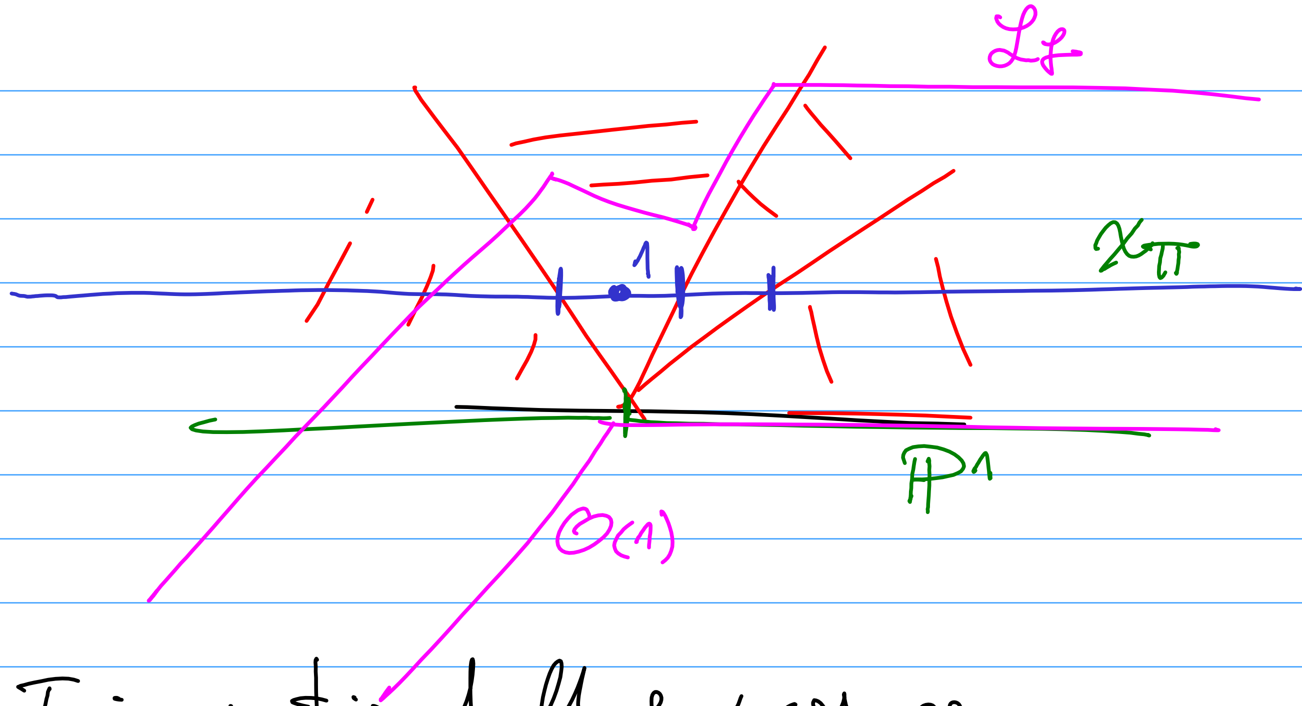
(2) $\mathcal{X}_\Pi$ model of X_Σ

models of L_ψ $\longleftrightarrow$ piecewise affine function f on Π st $\mathrm{rec}(f) = \psi$

recession function = asymptotic behavior

(3) $\mathcal{L}_f$ generated by sections

$\Leftrightarrow$ f concave (toric Nakai-Moishezon criterion)



§6 Toric metrized lb & measures

$v \in M_{\mathbb{Q}}$

$\mathbb{S}_v = \{ x \in \mathbb{T}_v : |x|_v = 1 \}$ compact torus

$\mathbb{S}_\infty = (S^1)^n$

X_Σ, L_ψ, s_ψ t.v/$\mathbb{Q}$ + lb + section

$\|\cdot\|_v$ <u>toric</u> if $\mathbb{S}_v$-invariant

<u>Entry 2</u> X_Σ, L_ψ as before

(toric) algebraic metrics on L_ψ
(≥ 0)

$\Updownarrow$

Concave piecewise affine functions f on $N_{\mathbb{R}}$
st $\mathrm{rec}(f) = \psi$ (= "tropical poly")

L, $\|\cdot\|_v$ toric metric = combinatorial

CONSTRUCTION $(v = \infty)$

$v = p$: trop
@ Gubler

$$\begin{tikzcd} x \arrow[d, mapsto] & X(\mathbb{C}) \arrow[d] \arrow[dr, dashed, "-\log\|s_\psi\|_f"] & \\ -\log|x| = \mathrm{val}(x) & N_\mathbb{R} \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

"combinatorial Green function"

$$\left\{\begin{array}{c}\text{continuous}\\ \text{functions}\end{array}\right\} \longleftrightarrow \{\text{continuous metrics} / X_0(\mathbb{C})\}$$

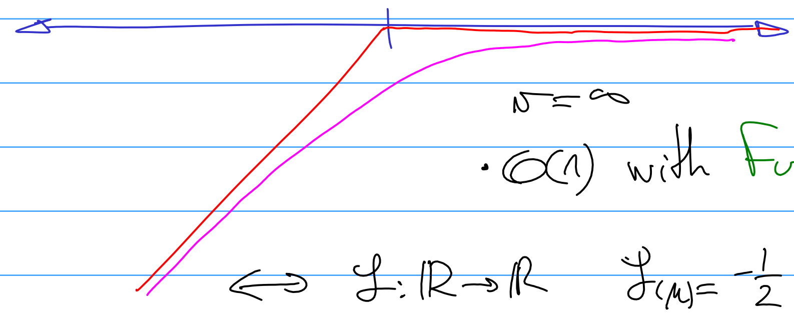
X_Σ, L_ψ, s_ψ as before

Entry 3 (1) $\|\cdot\|_f$ extends to a metric ≥ 0 iff f concave & $f - \psi$ bounded

(2) $(f_v)_v \longrightarrow (\|\cdot\|_{f_v})_v$ 1-to-1 between

- concave functions st $\begin{cases} f_v - \psi \text{ bounded} & (\forall v) \\ f_v = \psi & (\overset{\circ}{\forall} v) \end{cases}$
- toric adelic metrics on L_ψ

EXAMPLE

• 

$v=\infty$

· $\mathcal{O}(1)$ with Fubini Study

$\iff f:\mathbb{R}\to\mathbb{R}$ $\quad f(u) = -\frac{1}{2}\log(1+e^{-2u})$

$v = p$ we have a triangle

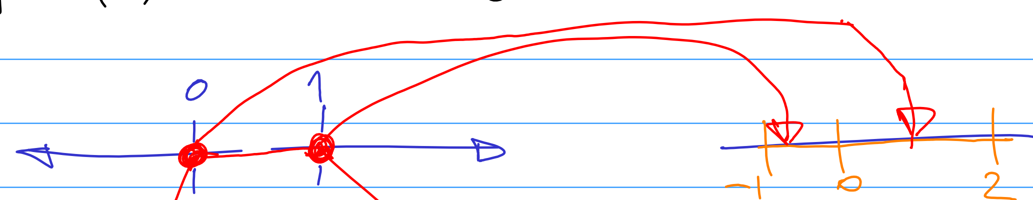
toric model of X_Σ, L_ψ

toric algebraic metrics (≥ 0) on L_ψ $\iff$ concave piecewise affine functions f on $N_\mathbb{R}$ st $\mathrm{rec}(f)=\psi$

I(BPS) $E \subset \mathbb{R}^n$

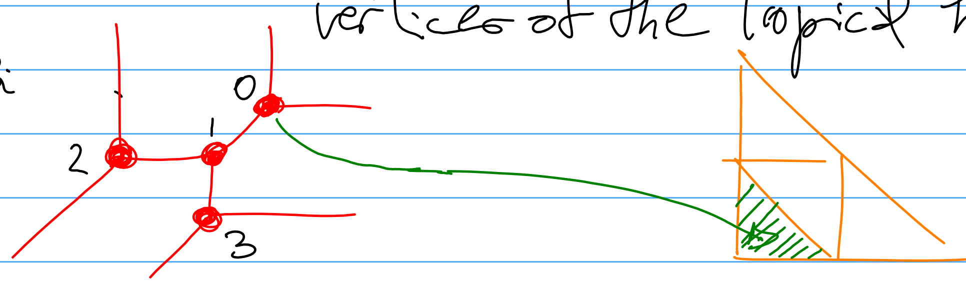
$$\mathrm{val}_* \mu_v(E) = n!\, \overbrace{\mathrm{vol}\,\mathrm{grad}(f)(E)}^{\text{Monge-Ampère measure of } f}$$

Ex:



$v=p$ $\quad \|\ \|$ algebraic $\Rightarrow \mathrm{val}_*\mu_v$ supported in the vertices of the tropical hyp of f

$\sum_{i=0}^{3} \delta_i$



II.5 Heights & Legendre-Fenchel Duality

Def: The Legendre-Fenchel dual a concave $f: N \to \mathbb{R}$:

$$\cancel{f}^{\vee}(x) = \inf_{\mu \in \mathbb{R}^n} \langle x, \mu \rangle - f(\mu)$$

$f^{\vee}$ concave with domain

$$\operatorname{dom}(f^{\vee}) = \{ x \mid f^{\vee}(x) > -\infty \}$$

$$f = \psi + O(1) \iff \begin{cases} \operatorname{dom}(f^{\vee}) = \Delta_\psi = \operatorname{grad}(\psi)(0) \\ f^{\vee} : \Delta \to \mathbb{R} \text{ continuous} \end{cases}$$

$$\vartheta_v = \begin{cases} f^{\vee}_{\infty} & v = \infty \\ \log p \, f^{\vee}_{p} & v = p \end{cases}$$ "roof function"

T(BPS) $\quad h_{\overline{L}_\psi}(X_\Sigma) = (n+1)! \sum_v \int_\Delta \vartheta_v \, d\underline{x}$

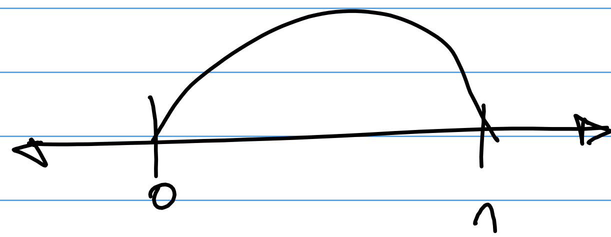
Ex:

$$\begin{array}{ll} \mathcal{O}(1) & \text{with Fubini-Study} \\ \downarrow & \\ \mathbb{P}^n & f^{v} : \Delta^n \longrightarrow \mathbb{R} \end{array}$$

$$f(x) = \frac{1}{2}\underbrace{\left(-\sum_{i=0}^{n} x_i \log x_i \right)}_{\text{entropy of } (x_0, \dots, x_n)} \qquad \left(x_0 = 1 - \sum_{i=1}^{n} x_i \right)$$

$$h(\mathbb{P}^n) = \frac{(n+1)!}{2} \int_{\Delta^n} \mathcal{E}(x)\, dx = \frac{(n+1)}{2} \sum_{j=2}^{n+1} \frac{1}{j} \quad \left(= \frac{1}{2}, \frac{5}{4}, \dots \right)$$

$$\frac{1}{2}\left(\log x + \log 1-x \right)$$



The dictionary is very concrete & allows to compute many examples:

- translated toric varieties with canonical metrics
- toric curves with Fubini Study
- tv with metrics from polytopes (Guillemin)
- toric bundles (eg Hirzebruch surfaces).