Solving Polynomial Equations

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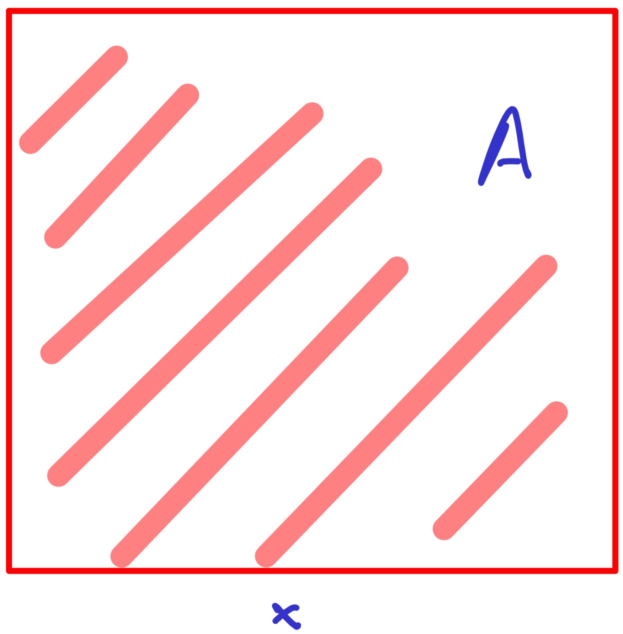
Torino, 30/5/2013

From a Babylonian clay tablet (~1800 BC)

Pb: Add the area and two-third of a square to obtain 0:35.
Which is the side of my square?




MS 3052
Eight mathematical problems with drawings of subdivided trapezoids and triangles.
Babylonia, 1763-1739 BC

$$x^2 + \frac{2}{3}x = \frac{35}{60}$$

Sol: Take 1. Two-thirds of 1 is 0:40. Half of this, 0:20, you multiply by 0:20 and it 0:6:40, you add to 0:35 and the result 0:41:40, has 0:50 as it square root. The 0:20 which you have multiplied by itself, you substract from 0:50, and 0:30 is the side of the square.

D. Burton. The history of mathematics, 1997.

In other words:

$$x = \sqrt{\left(\frac{0:40}{2}\right)^2 + 0:35} - \frac{0:40}{2} = \dots = 0:30 = \frac{30}{60} = \frac{1}{2}$$

Cubic and Quartic Equations

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

CAN BE SOLVED USING SQUARE & CUBIC ROOTS

(Ferrari, Cardano, Tartaglia 1540-1545)

$$r_1 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - \cdots\right.}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3\cdots\right.}\right.} \cdots}$$

$$r_2 = \frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - \cdots\right.}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3\cdots\right.}\right.} \cdots}$$

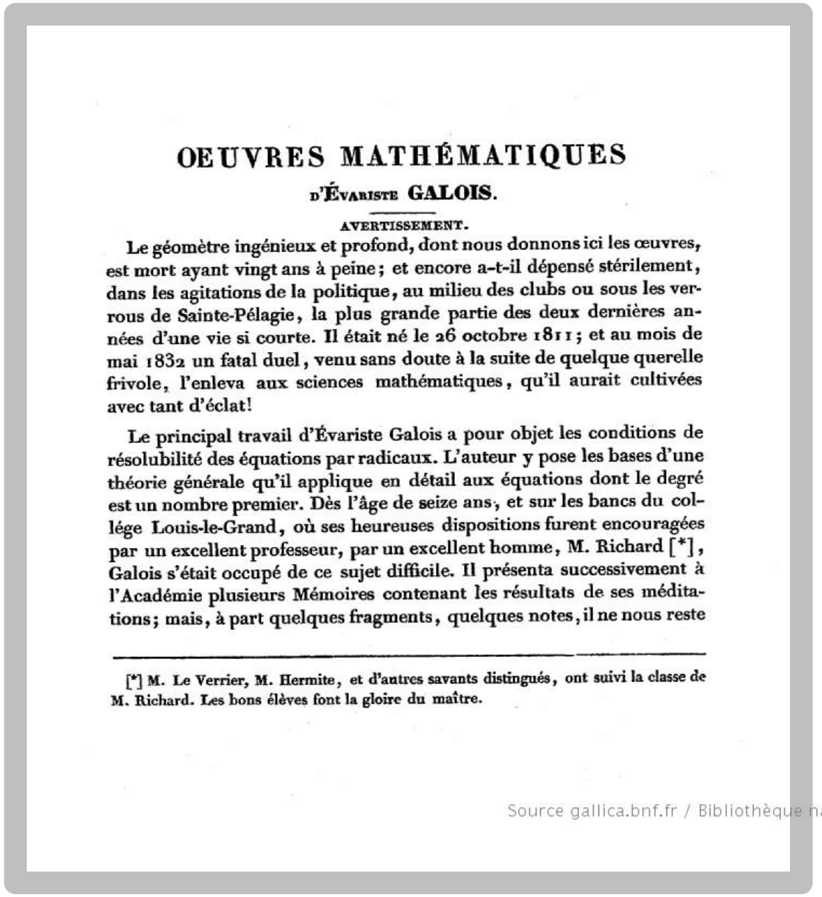
$$r_3 = \frac{-a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - \cdots\right.}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3\cdots\right.}\right.} \cdots}$$

$$r_4 = \frac{-a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - \cdots\right.}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3\cdots\right.}\right.} \cdots}$$

Higher Degree

$d \geq 5, \quad f = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \cdots + \alpha_0$

Thm (Abel–Ruffini 1824)
The equation $f(x) = 0$ cannot always be solved with radicals and arithmetic operations only



OEUVRES MATHÉMATIQUES

D'ÉVARISTE GALOIS.

AVERTISSEMENT.

Le géomètre ingénieux et profond, dont nous donnons ici les œuvres, est mort ayant vingt ans à peine; et encore a-t-il dépensé stérilement, dans les agitations de la politique, au milieu des clubs ou sous les verrous de Sainte-Pélagie, la plus grande partie des deux dernières années d'une vie si courte. Il était né le 26 octobre 1811; et au mois de mai 1832 un fatal duel, venu sans doute à la suite de quelque querelle frivole, l'enleva aux sciences mathématiques, qu'il aurait cultivées avec tant d'éclat!

Le principal travail d'Évariste Galois a pour objet les conditions de résolubilité des équations par radicaux. L'auteur y pose les bases d'une théorie générale qu'il applique en détail aux équations dont le degré est un nombre premier. Dès l'âge de seize ans, et sur les bancs du collége Louis-le-Grand, où ses heureuses dispositions furent encouragées par un excellent professeur, par un excellent homme, M. Richard [*], Galois s'était occupé de ce sujet difficile. Il présenta successivement à l'Académie plusieurs Mémoires contenant les résultats de ses méditations; mais, à part quelques fragments, quelques notes, il ne nous reste

[*] M. Le Verrier, M. Hermite, et d'autres savants distingués, ont suivi la classe de M. Richard. Les bons élèves font la gloire du maître.

Ex.: $f = x^5 - x - 1$

The **end** of polynomial equation solving?

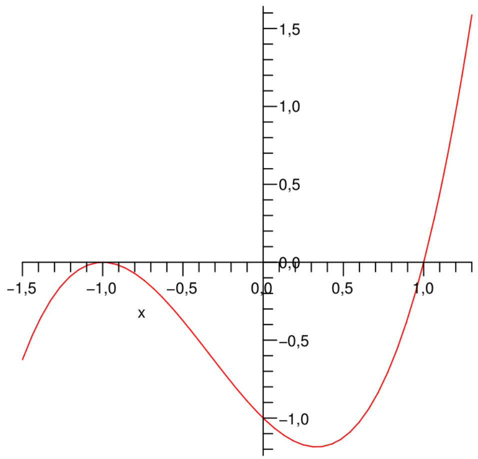
The Fundamental Theorem of Algebra

<u>Thm</u> (D'Alembert 1746 - Gauss 1798)

Let $f \in \mathbb{C}[x]$. Then $f(x) = 0$ has $\deg(f)$ solutions

<u>Ex:</u> $f = x^3 + x^2 - x - 1$

$V(f) = \{ x \in \mathbb{C} \mid f(x) = 0 \} = \{\pm 1\}$



$\mathbb{C}$ is "algebraically closed"

THE DISTRIBUTION OF THE ZEROS

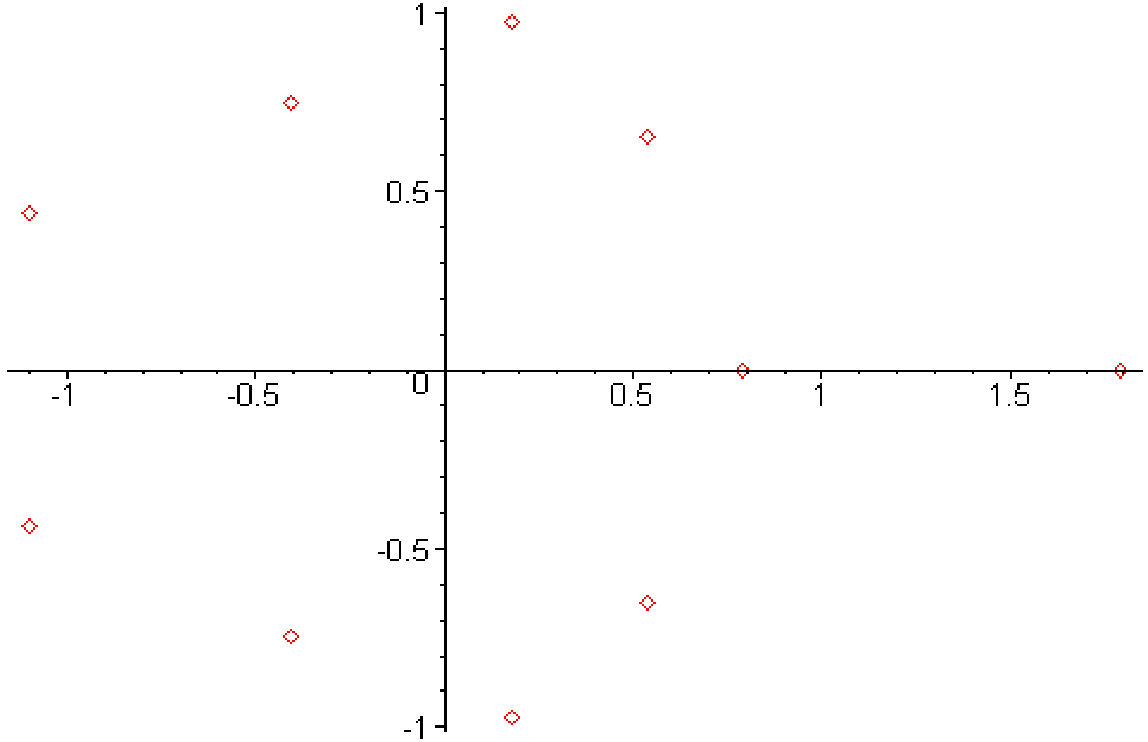
ACTIVITY

1) TAKE $f \in \mathbb{C}[x]$ WITH COEFFICIENTS ± 1 OR 0 AND DEGREE $d \gg 0$.
2) PLOT THE SOLUTIONS OF $f(x) = 0$.

WHAT DO YOU SEE ?

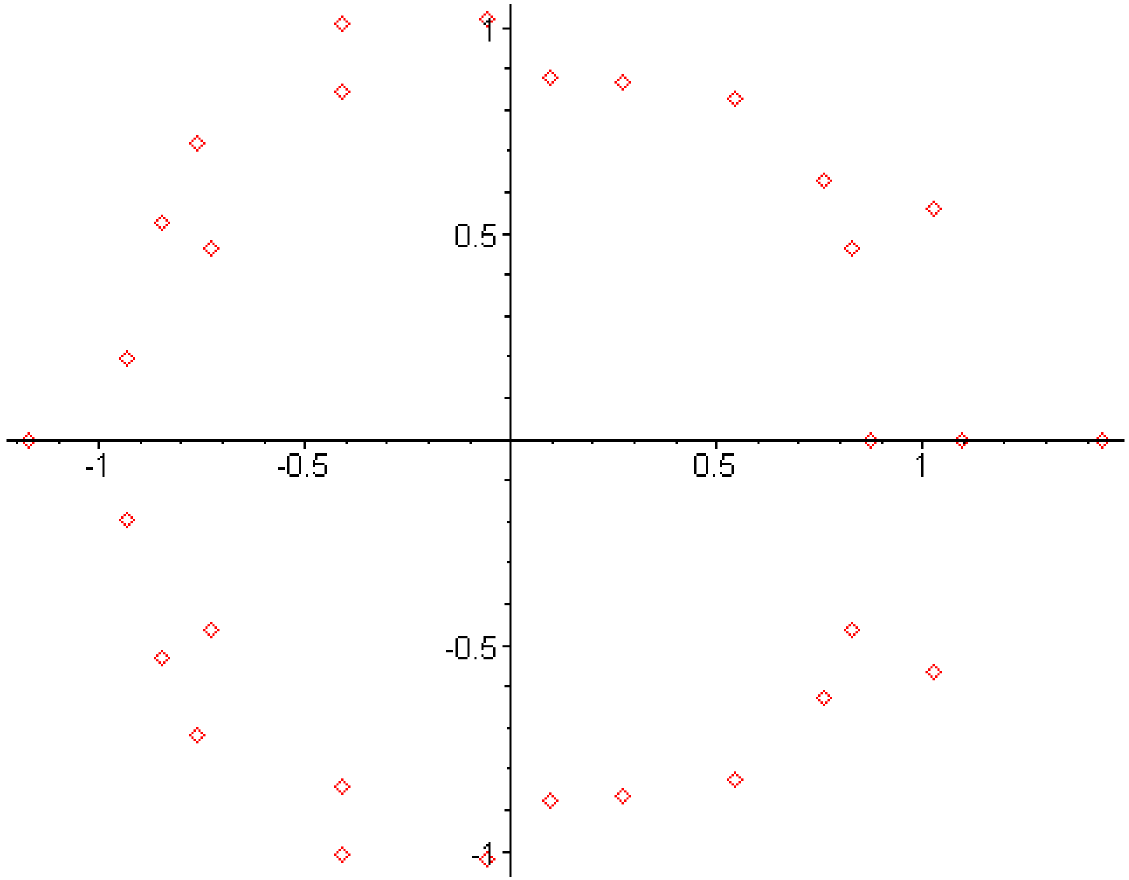
$d = 10$

$f = -x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 - x^4 + x^3 - x^2 + x - 1$



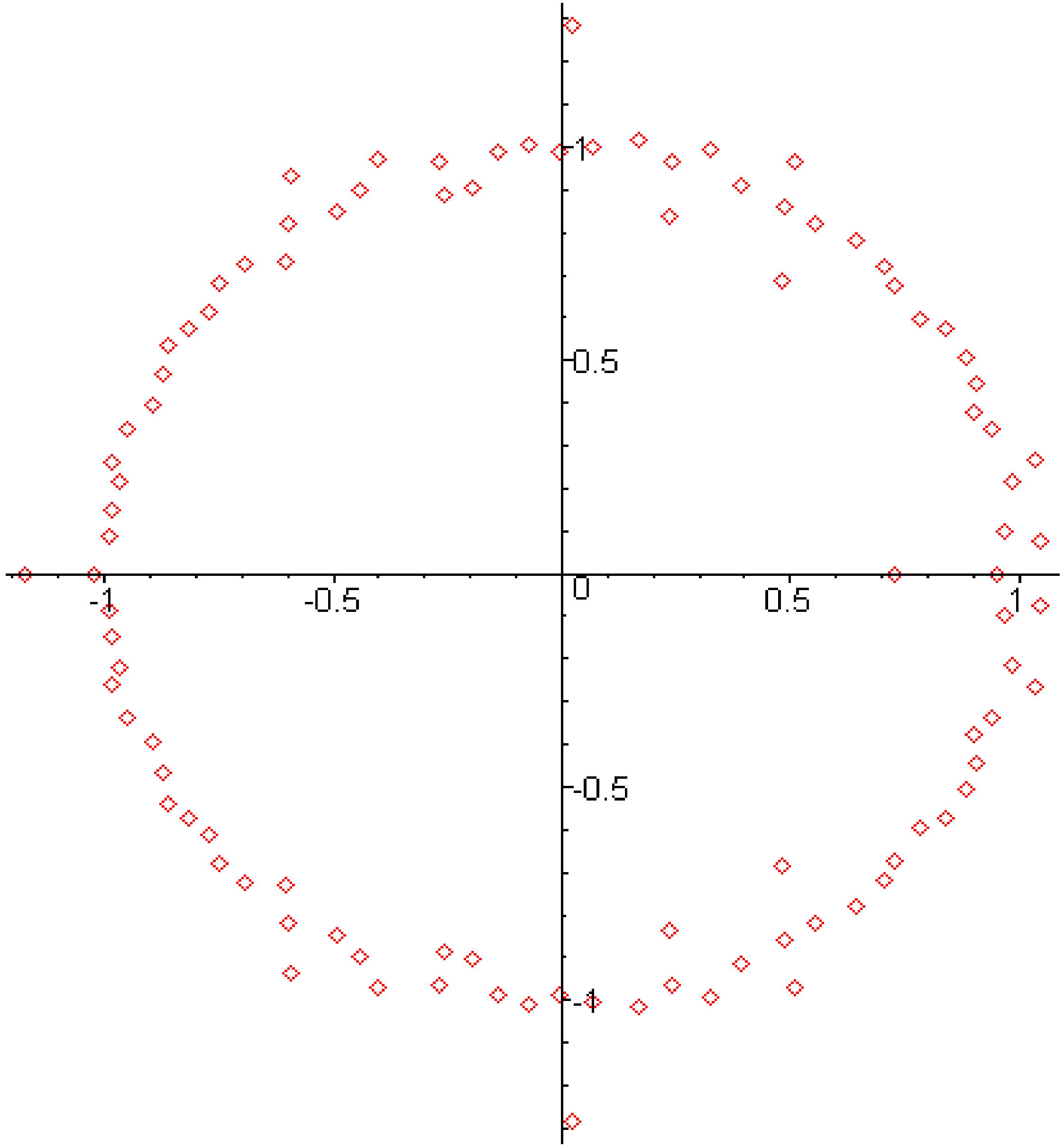
$$d = 30$$

$$f = x^{30} - x^{29} - x^{28} + x^{26} + x^{25} - x^{24} - x^{23} - x^{22} + x^{21} - x^{20} + x^{19} + \cdots$$

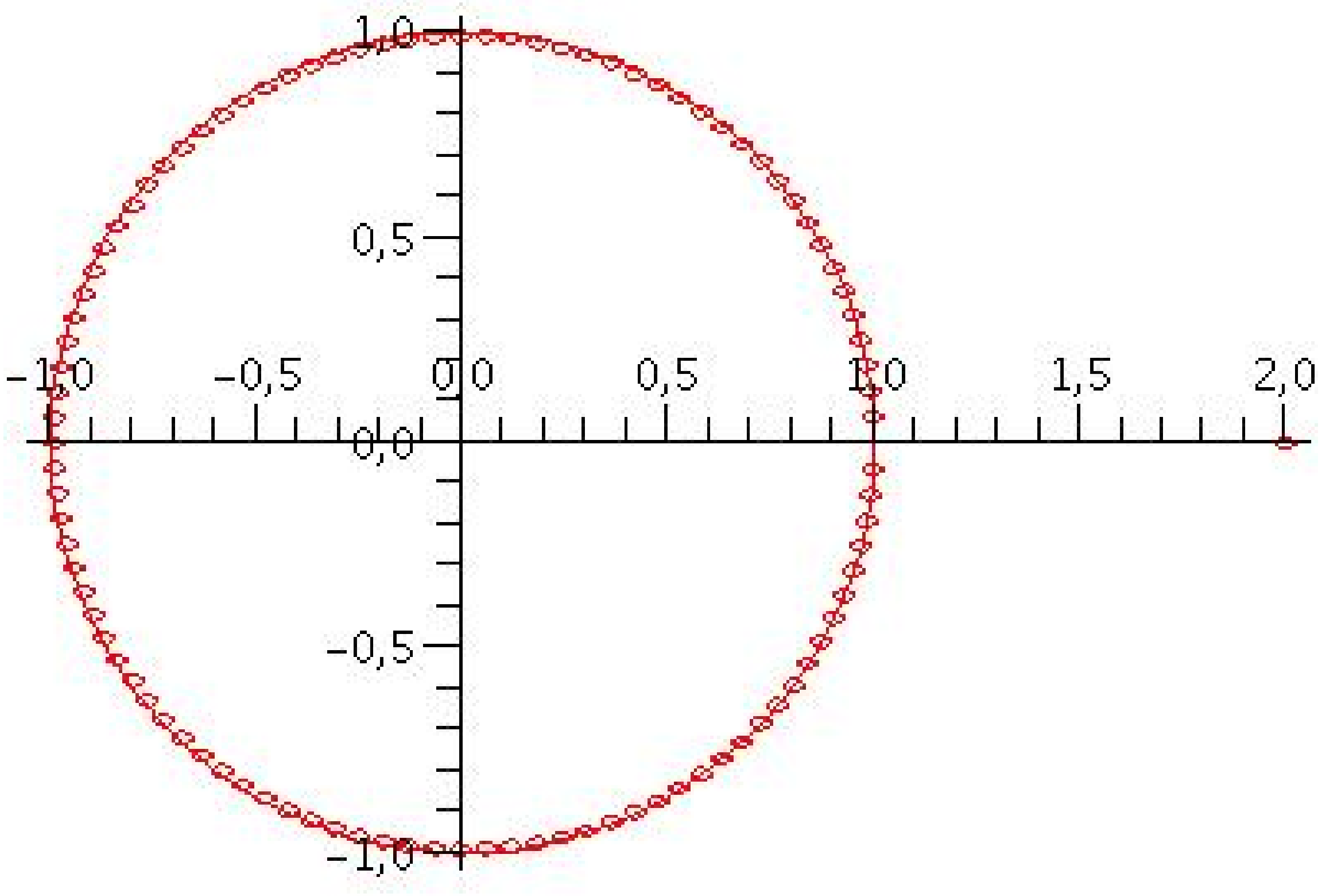


$d = 100$

$f = -x^{100} - x^{98} + x^{96} + x^{94} - x^{93} + x^{92} - x^{91} - x^{90} + x^{88} - x^{84} + \ldots$



$$f = x^{100} - x^{99} - x^{98} - x^{97} - \dots - 1$$



THE ERDÖS-TURÁN THEOREM (1948)

LET $f = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \dots + \alpha_0 \in \mathbb{C}[x]$ with $\alpha_d \alpha_0 = 1$

$$V(f) = \{\xi_1, \dots, \xi_d\}$$

THEN $\forall\ 0 \leq \theta_1 \leq \theta_2 \leq 2\pi$

$$\left| \frac{\#\{j \mid \theta_1 \leq \arg(\xi_j) < \theta_2\}}{d} - (\theta_2 - \theta_1) \right| \leq 3 \left(\frac{1}{d} \log\left(\sum_i |\alpha_i| \right) \right)^{1/2}$$

AND $\forall\ 0 < \varepsilon \leq 1$

$$\#\left\{ j \mid |\xi_j| < 1 - \varepsilon \text{ or } |\xi_j| > \frac{1}{1-\varepsilon} \right\} \leq \frac{2}{\varepsilon d} \log\left(\sum_i |\alpha_i| \right)$$

COR: LET $f_d \in \mathbb{Z}[x]$, $d \geq 1$, with $\log \|f_d\|_1 = o(d)$

THEN

$$\frac{1}{d} \sum_{\xi \in V(f_d)} \delta_\xi \xrightarrow[\text{weak-}*]{} \mathrm{Haar}_{S^1}$$

THE DISTRIBUTION OF THE ZEROS DEPENDS ON THE DISTRIBUTION OF THE COEFFICIENTS

$$f = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \dots + \alpha_0 \in \mathbb{C}[x]$$

$$\|f\|_{FS} = \left(\sum_i \binom{d}{i}^{-1} |\alpha_i|^2 \right)^{1/2}$$ FUBINI-STUDY NORM

BASIC PROPERTY:

INVARIANCE UNDER UNITARY CHANGES OF COORDINATES

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2) \Rightarrow \left\| f\left(\frac{ax+b}{cx+d}\right) \right\|_{FS} = \| f(x) \|_{FS}$$

$\mathcal{S}_d = \{ f \in \mathbb{C}[x]_d \mid \|f\|_{FS} = 1 \}$ UNIT SPHERE

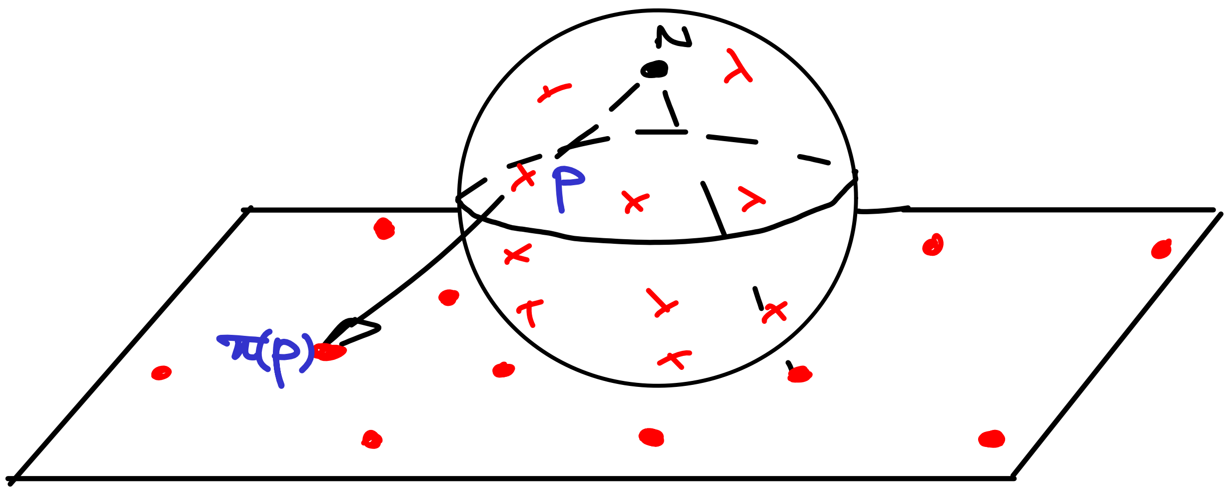
LET $f \in \mathcal{S}_d$ FOLLOWING THE UNIFORM DISTRIBUTION

THEN THE "EXPECTED ZERO DENSITY" OF f IS GIVEN BY

$$\mathbb{E}(Z(f)) = \pi_*(\mu_{S^2})$$

WITH μ_{S^2} UNIFORM DISTRIBUTION ON S^2

$\pi : S^2 \setminus \{\text{north pole}\} \to \mathbb{C}$ STEREOGRAPHIC

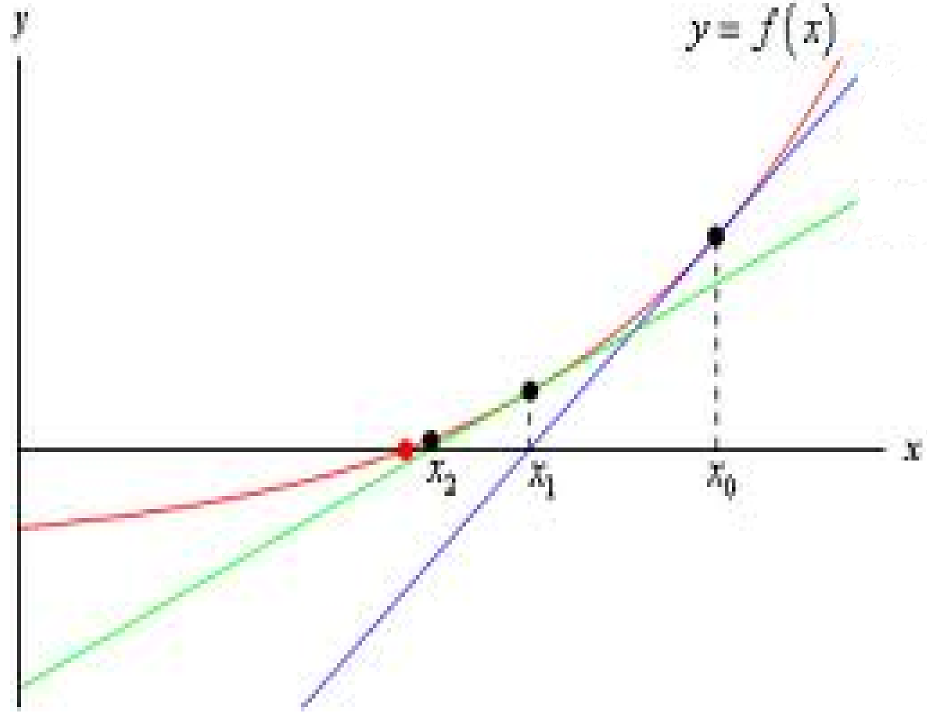


Newton's Method

$f \in \mathbb{C}[x]$

$z_0 \in \mathbb{C}$ starting point

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}$$



If z_0 close to a zero $\xi \in V(f)$

$$\log_2 |z_k - \xi| \leq 2^k \log_2 |z_0 - \xi|$$

"Quadratic convergence"

NEWTON OPERATOR AS A DYNAMICAL SYSTEM

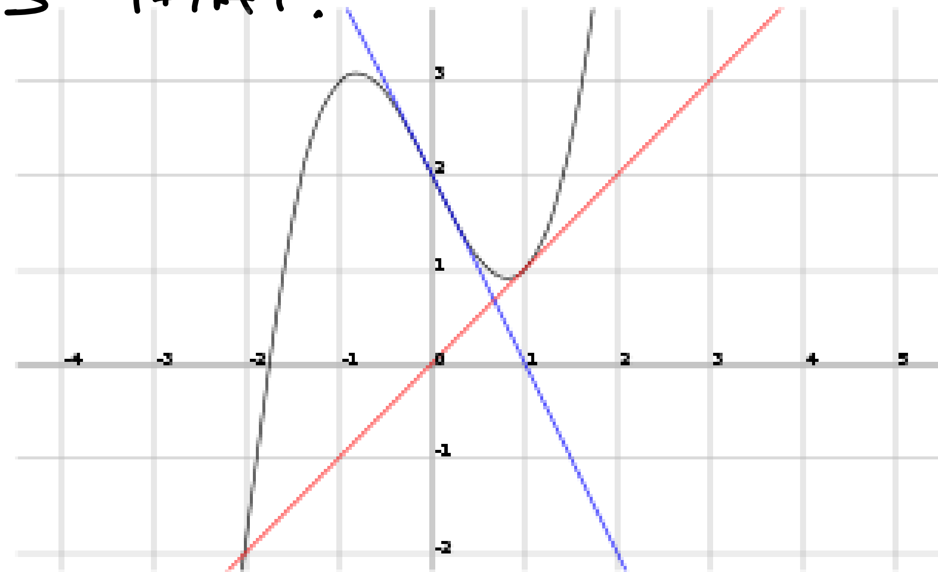
$$\rho_f : \mathbb{C} \dashrightarrow \mathbb{C}, \quad x \mapsto x - \frac{f(x)}{f'(x)}$$

- THE ROOTS OF f ARE ATTRACTING FIXED POINTS
- IF $\rho_f^{k_0}(z_0) = \overbrace{\rho_f \circ \cdots \circ \rho_f}^{k_0}$ ENTERS THE BASSIN OF OF $\xi \in V(f)$ THEN $\rho_f^{k}(z_0)$ CONVERGES QUADRATICALLY TO ξ FOR $k \geq k_0$
- NOT EVERY $z_0 \in \mathbb{C}$ DOES THAT:

$$f(x) = x^3 - 2x + 2$$

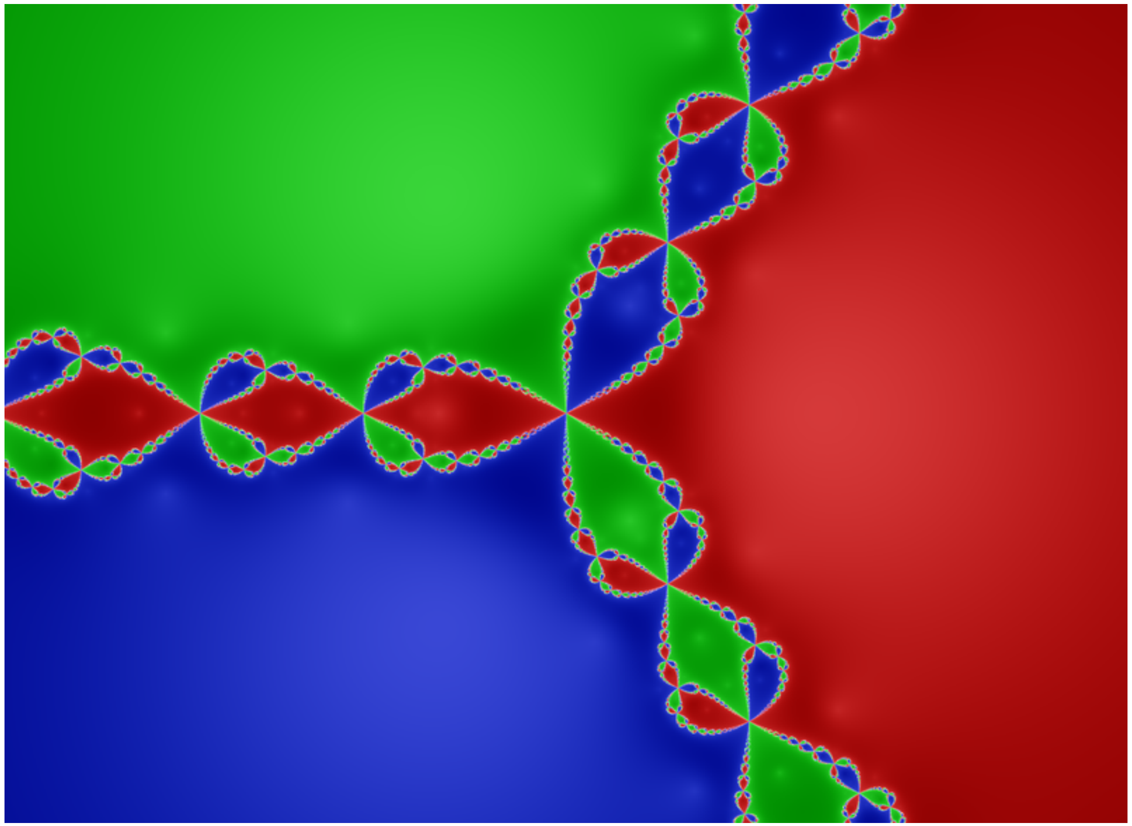
$$\rho_f(0) = 1 \qquad \rho_f(1) = 0$$

$\rightsquigarrow$ ATTRACTING CYCLE OF LENGTH 2

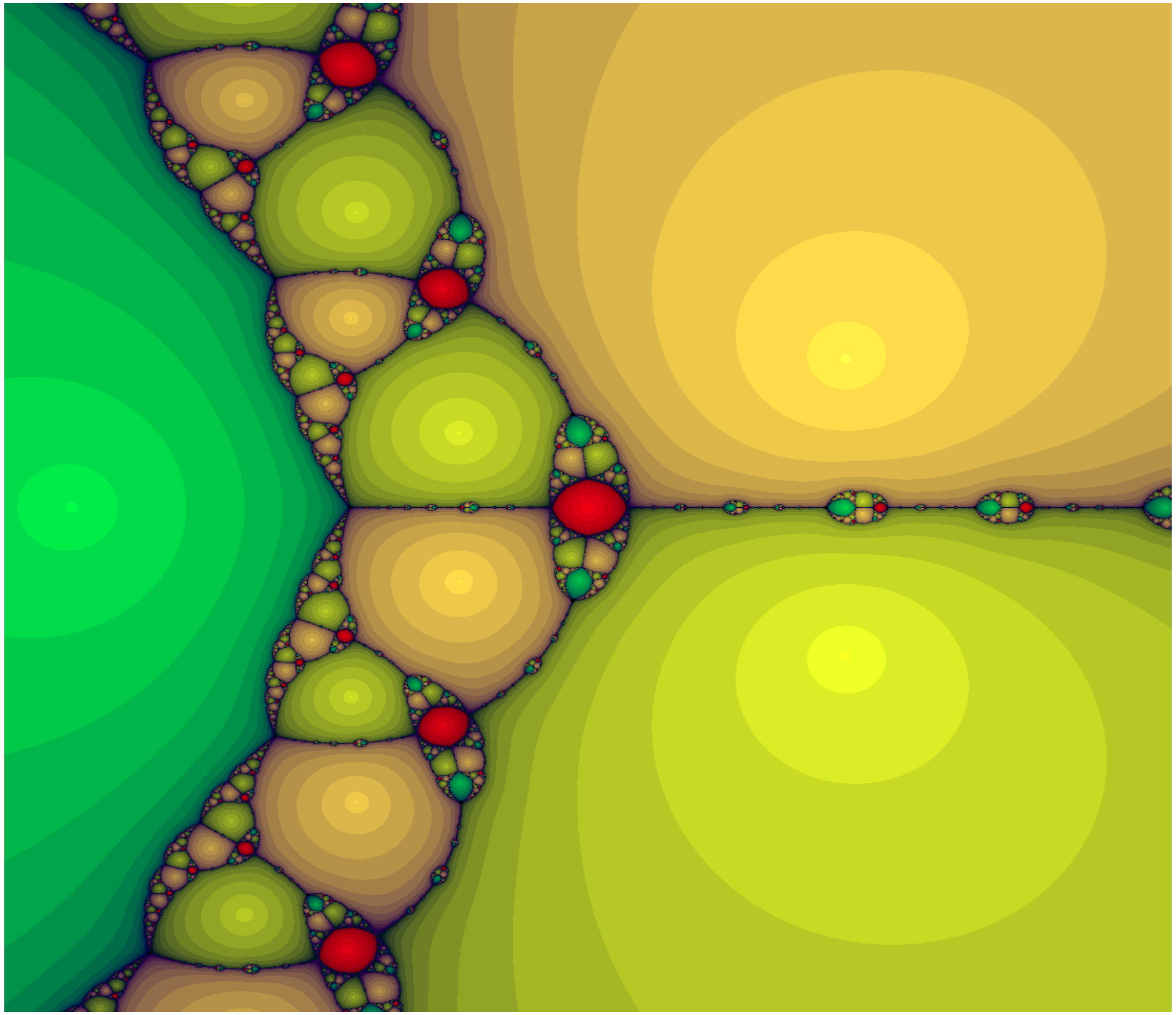


The Newton Fractal

The Julia set of Q_f divides $\mathbb{C}$ into regions according to the roots of f (and attracting cycles of length ≥ 2)



$$f(x) = x^3 - 1$$



$$f(x) = x^3 - 2x + 2$$

SOLVING POLYNOMIAL EQUATIONS (NUMERICALLY)

$z_0 \in \mathbb{C}$ IS AN "APPROXIMATE ZERO" OF f IF
$\exists\ \xi \in V(f)$ S.T.

$$\log_2 |z_k - \xi| \leq 2^k \log_2 |z_0 - \xi| \qquad (\forall k \geq 1)$$

THE α-THEOREM (SMALE 1981): A CRITERION TO DECIDE IF z_0 IS AN APPROXIMATE ZERO:

SET
$$\alpha(f,z) = \left|\frac{f'(z)}{f(z)}\right| \sup_{i \geq 2} \left|\frac{f^{(i)}(z)}{i!\, f'(z)}\right|$$

$\exists\ \alpha_0$ ($\alpha_0 = 0{,}03$ SUFFICES) S.T. IF $\alpha(f, z_0) < \alpha_0$
THEN z_0 IS AN APPROXIMATE ZERO.

Homotopy Method

PB: Given $f \in \mathbb{C}[x]$ and $\varepsilon > 0$, find $\bar{z} \in \mathbb{C}$ st $|f(\bar{z})| < \varepsilon$

Approach: Choose $z_0 \in \mathbb{C}$ and set

$$f_t(z) = f(z) - (1-t) f(z_0) \qquad 0 \leq t \leq 1$$

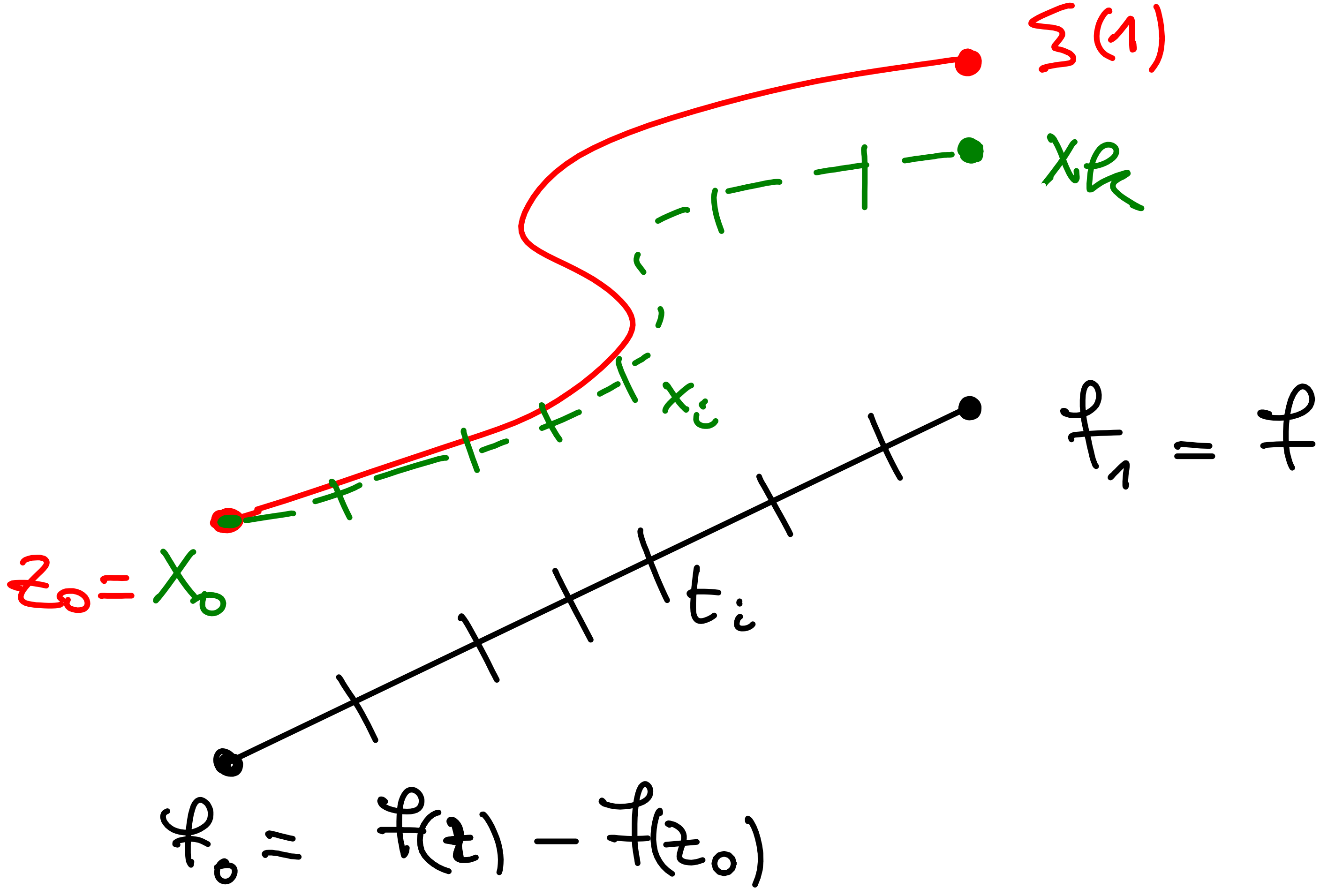
Let $\xi : [0,1] \longrightarrow \mathbb{C}$ a curve s.t.

$$\xi(0) = z_0 \qquad f_t(\xi(t)) = 0 \qquad 0 \leq t \leq 1$$

We follow $\xi(t)$ using Newton's method:
choose $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ and set

$$x_0 = z_0$$

$$x_i = x_{i-1} - \frac{f_i\,(x_{i-1})}{f_i'\,(x_{i-1})} \qquad 0 \leq i \leq k$$



FOR A SUITABLE CHOICE OF z_0 AND t_i'S THE PB CAN BE SOLVED WITH AVERAGE COMPLEXITY $\mathcal{O}\left(d^2\left(d + \log\frac{1}{\varepsilon}\right)\right)$

(SHUB-SMALE 1993)

SYSTEMS OF MULTIVARIATE POLYNOMIAL EQUATIONS

PB: GIVEN $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$, FIND $z \in \mathbb{C}^n$ S.T.

$$f_1(z) = \dots = f_s(z) = 0$$

1) A TRIGONOMETRIC EQUATION

$$\sin(x)^3 + \cos(3x) = 0$$

EQUIVALENT TO THE SYSTEM

$$\begin{cases} s^3 + 4c^3 - 3c = 0 \\ s^2 + c^2 - 1 = 0 \end{cases}$$

2) HEXAPOD

The Multivariate FTA

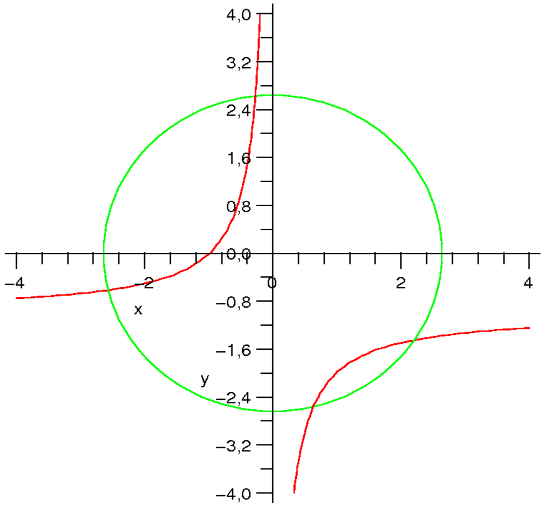
<u>Thm</u> (Bézout 1764)

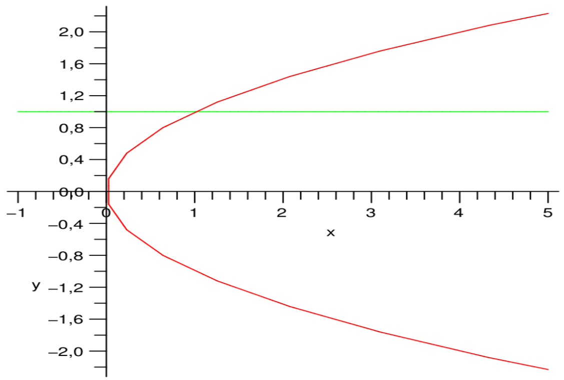
$f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$ st $\dim(V(f_1, \dots, f_n)) = 0$

Then $\# V(f_1, \dots, f_n) \leq \prod_i \deg(f_i)$

<u>Ex:</u>

$$\begin{cases} x_1 x_2 + x_1 - 1 = 0 \\ x_1^2 + x_2^2 - 7 = 0 \end{cases} \qquad \qquad \begin{cases} x_1 - x_2^2 = 0 \\ x_2 - 1 = 0 \end{cases}$$





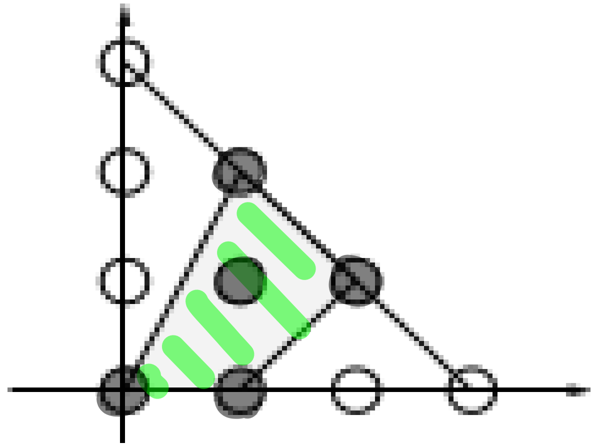
"Sparse" Polynomials

Let $f = \sum_{j=1}^{N} \alpha_j x^{a_j} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

The Newton polytope is $N(f) = \mathrm{conv}(a_1, \dots, a_N) \subset \mathbb{R}^n$

Ex:

$$f = 2 - x_1 + 3x_1x_2 - x_1x_2^2 + x_1^2x_2$$



The mixed volume of convex bodies $\Delta_1, \Delta_2 \subset \mathbb{R}^2$ is

$$MV(\Delta_1, \Delta_2) = \mathrm{vol}(\Delta_1 + \Delta_2) - \mathrm{vol}(\Delta_1) - \mathrm{vol}(\Delta_2)$$

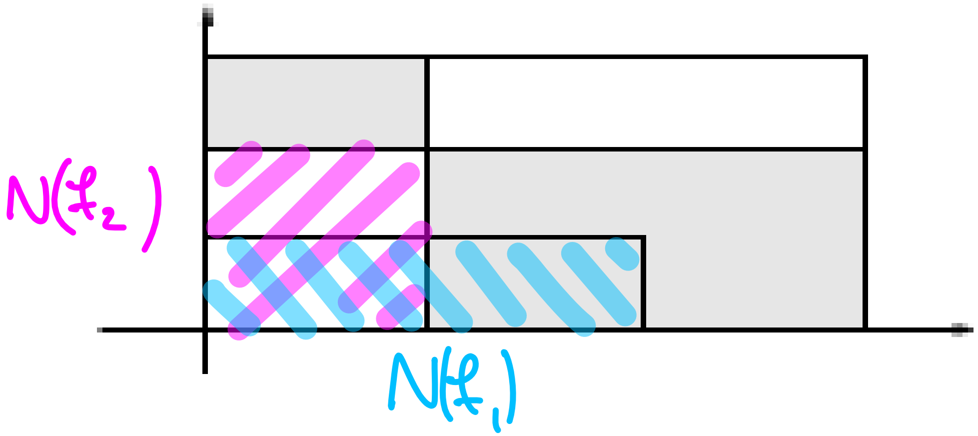
The Bernstein-Kushnirenko Theorem (1975)

Let $f_1, f_2 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ s.t. $\dim(V(f_1, f_2)) = 0$

Then $$\# V(f_1, f_2) \leq MV(\Delta_1, \Delta_2)$$

Ex: Bihomogeneous Bézout: Let $f_i \in \mathbb{C}[x_1, x_2]$

Then $$\# V(f_1, f_2) \leq \deg_x(f_1) \deg_y f_2 + \deg_y(f_1) \deg_x(f_2)$$



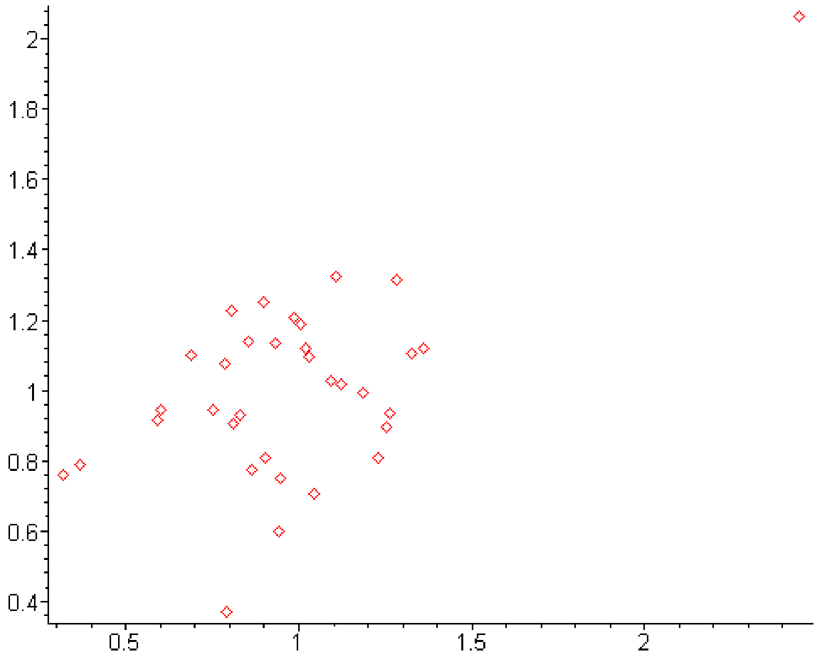
The Distribution of the Zeros

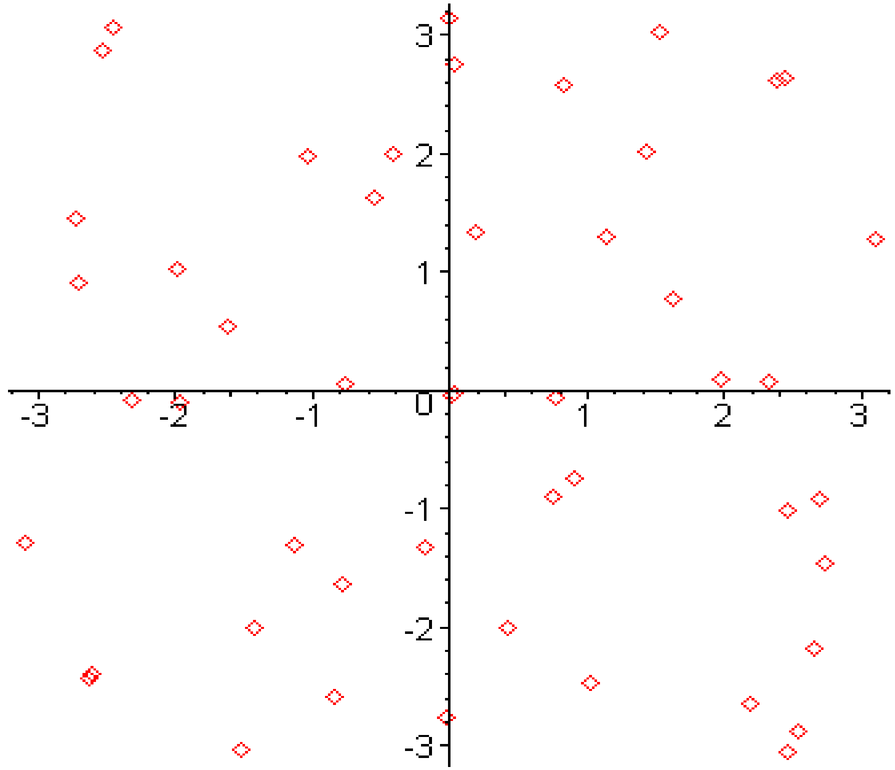
$d = 7$

$f = x^7 + x^6 y + x^5 y^2 + \cdots$

$g = -x^7 - x^5 y^2 + \cdots$

The joint modulus & argument of $f = g = 0$ plot as



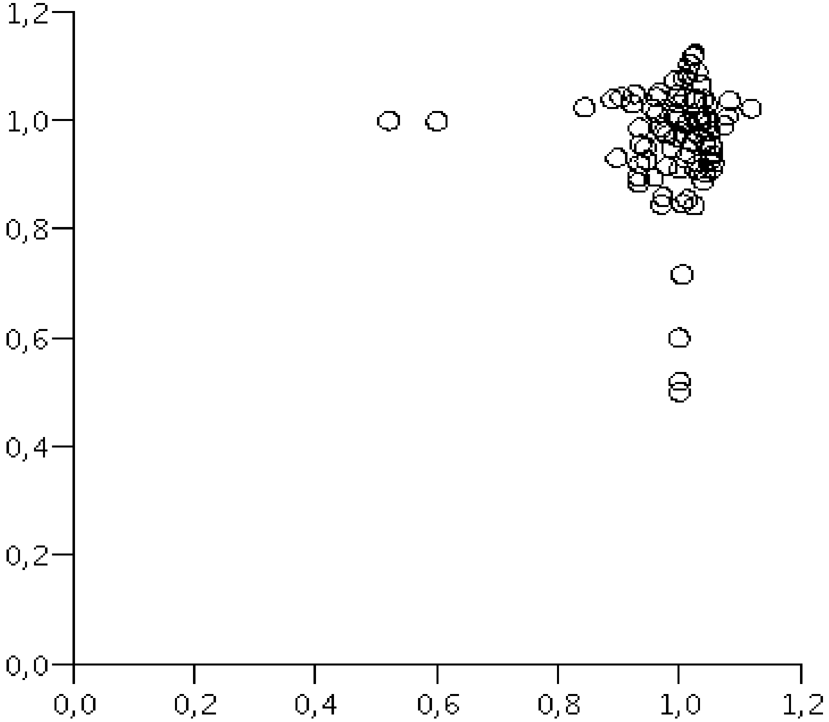


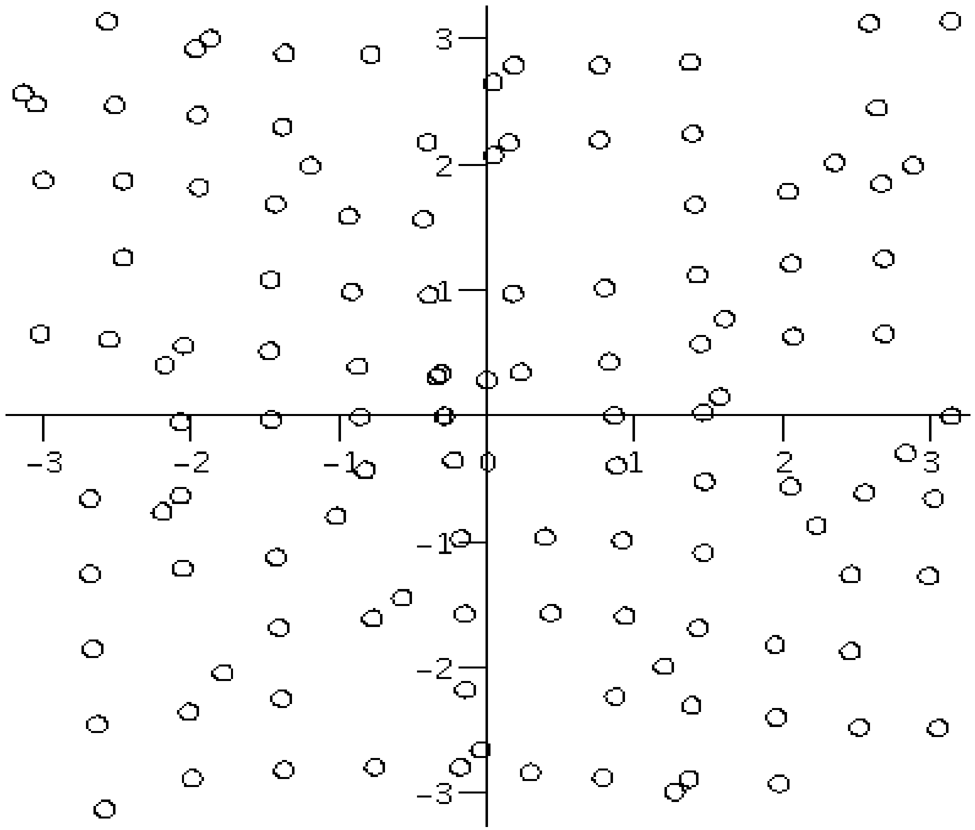
$d = 13$

$f = x^{13} + xy^{12} + y^{13} + 1$

$g = x^{12}y - y^{13} - xy + 1$

THE JOINT MODULUS & ARGUMENT OF $f = g = 0$ PLOT AS





A MULTIVARIATE E-T THM (D'ANDREA - GALLIGO - S 2011)

$f_i \in \mathbb{Z}[x_1, x_2]$ $(i=1,2)$ $\qquad d_i = \deg(f_i)$

$$\{\underline{x} \in \mathbb{C}^2 \mid f_i(\underline{x}) = 0,\ i=1,2\} = \{\xi_j\}_{j=1,\dots,D}$$

SUPPOSE THAT $D = d_1 d_2$

LET $R \subset [0, 2\pi]^2$

$$\left| \frac{\#\{j \mid \arg(\xi_j) \in R\}}{d_1 d_2} - \mathrm{vol}(R) \right| \leq 2^8 \cdot \left(\frac{\log \|f_1\|_1}{d_1} + \frac{\log \|f_2\|}{d_2} \right)^{1/4}$$

AND $0 < \varepsilon \leq 1$

$$1 - \frac{\#\{j \mid 1-\varepsilon < |\xi_j| < \frac{1}{1-\varepsilon}\}}{d_1 d_2} \leq \frac{4}{\varepsilon} \left(\frac{\log \|f_1\|_1}{d_1} + \frac{\log \|f_2\|}{d_2} \right)$$

RANDOM SPARSE POLYNOMIALS

THM (DGS 2011)

$\Delta_i \subset \mathbb{R}^2$ LATTICE POLYTOPE, $d \geq 1$

$f_i \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ $(i=1,2)$ RANDOM WITH $N(f_i) \subset d\Delta_i$

AND INDEPENDENT STANDARD GAUSSIAN COEFFS,

THEN

$$\frac{1}{d^2 MV(\Delta_1, \Delta_2)} \sum_{\xi \in V(\underline{f})} \delta_\xi \longrightarrow \mathrm{Haar}_{S^1 \times S^1}$$

COMPUTATIONAL COMPLEXITY

THE COMPLEXITY OF AN ALGORITHM $\mathcal{A}$ IS

$$c_{\mathcal{A}} : \mathcal{U} \to \mathbb{N}$$

$$f \mapsto \#\{\text{OPS OF } \mathcal{A} \text{ ON THE INPUT } f\}$$

$\mathcal{A}$ ~~HAS~~ POLYNOMIAL COMPLEXITY IF

$\exists k_i$ S.T. $\forall m \geq 1$, $f \in \mathcal{U}$ OF SIZE $\leq m$

$$c_{\mathcal{A}}(f) \leq k_1 m^{k_2}$$

HILBERT NULLSTELLENSATZ

HN: GIVEN $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$

DECIDE IF $V(f_1, \dots, f_s) = \emptyset$

LET $\deg(f_i) \leq d$. THE INPUT SIZE IS $N = s\binom{d+n}{n}$

HILBERT (1890): $V(f_1, \dots, f_s) = \emptyset \Leftrightarrow (\exists g_i)\ 1 = \sum_{i=1}^{s} g_i f_i$

BROWNAWELL, KOLLÁR (1987): IS POSSIBLE TO FIND g_i S.T.

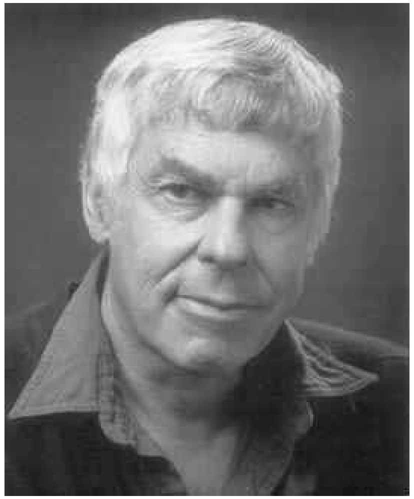
$$\deg(g_i f_i) \leq d^n$$

$\Rightarrow$ HN REDUCES TO A LINEAR SYSTEM / $\mathbb{C}$ OF SIZE

$s\binom{d^n+n}{n} \times \binom{d^n+n}{n}$, NOT POLYNOMIAL IN N

INDEED, HN IS NP-HARD

SMALE 17TH PROBLEM



"CAN A ZERO OF n COMPLEX EQUATIONS IN n UNKNOWNS BE FOUNDED APPROXIMATIVELY, IN THE AVERAGE, IN POLYNOMIAL TIME WITH A UNIFORM ALGORITHM?"

- $\mathcal{H}_d = \left(\mathbb{C}[x_0, \dots, x_n]_d\right)^n$ SQUARE SYSTEMS OF HOMOGENEOUS POLYS OF DEGREE d
- $N_d = \dim_{\mathbb{C}}(\mathcal{H}_d) = n\binom{d+n}{n}$
- $\|\cdot\|_{FS}$ FUBINI-STUDY NORM ON $\mathcal{H}_d$
- $\mathbb{S}_d$ UNIT SPHERE OF $(\mathcal{H}_d, \|\cdot\|_{FS})$
- $\underline{f} = (f_1, \dots, f_n) \in \mathbb{S}_d$

UNDERSTANDING THE STATEMENT

- "APPROXIMATIVELY": THE ALGORITHM $\mathcal{A}$ FINDS AN APPROXIMATE ZERO OF $\underline{f}$
- "IN POLYNOMIAL TIME, IN THE AVERAGE":

$$\mathbb{E}(c_{\mathcal{A}}) = \int_{\mathcal{S}_d} c_{\mathcal{A}}(\underline{f}) \, d\mu_{FS}(\underline{f}) \leqslant k_1 N_d^{k_2}$$

AVERAGE COMPLEXITY ON $\mathcal{S}_d$ WR UNIFORM DISTRIBUTION

- "UNIFORM ALGORITHM": DETERMINISTIC BBS MACHINE THAT WORKS FOR ALL d AND $\underline{f}$ OUTSIDE THE DISCRIMINANT VARIETY

Multivariate Homotopy Method

$$\mathcal{V} = \{ (\underline{f}, \underline{z}) \in \mathcal{S}_d \times \mathbb{P}^n \mid \underline{f}(\underline{z}) = 0 \}$$

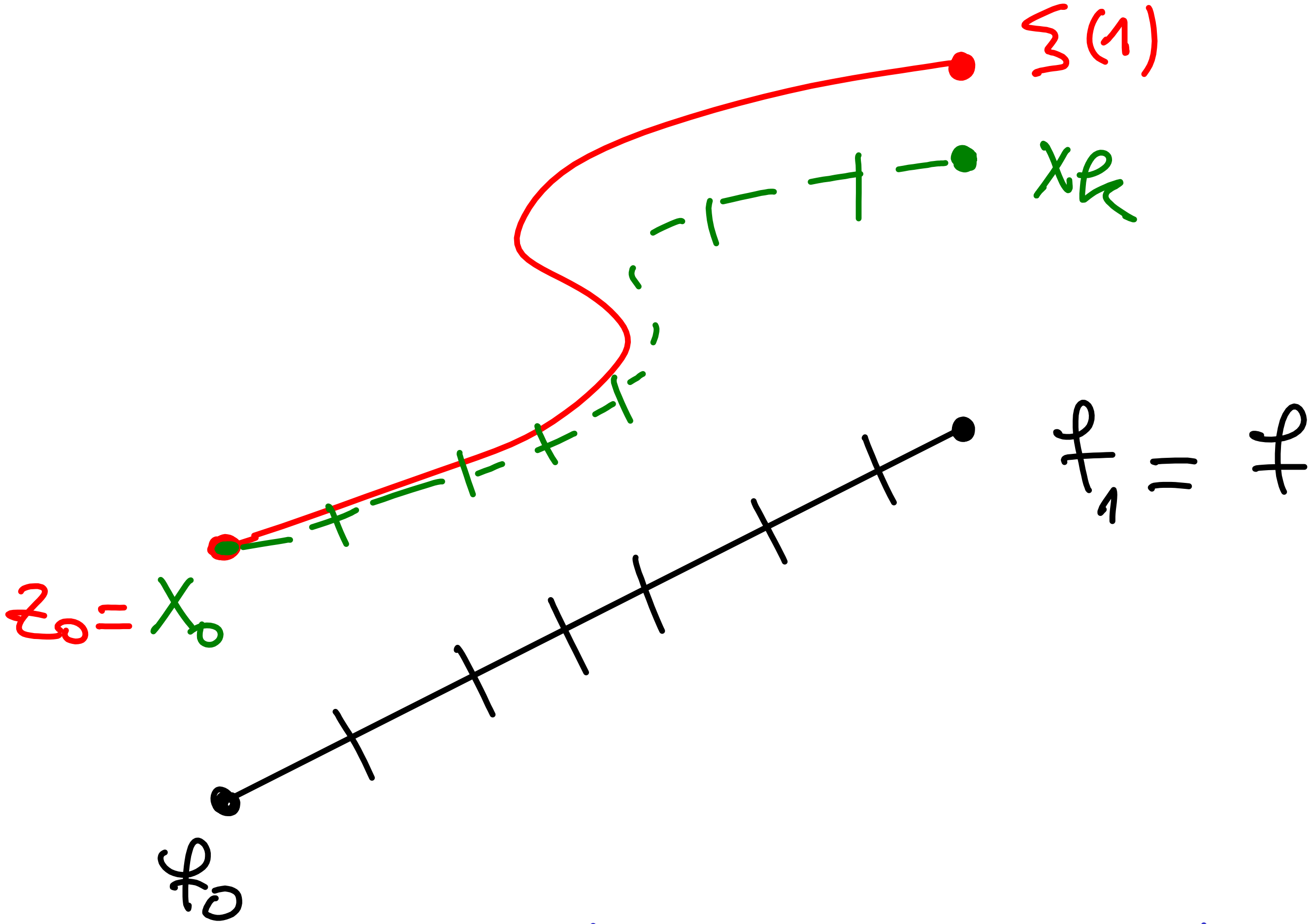
SOLUTION VARIETY

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{S}_d & & \mathbb{P}^n \end{tikzcd}$$

$$\Sigma = \pi_1 \left(\text{critical poins of } \pi_1|_{\mathcal{V}} \right) \subset \mathcal{S}_d$$

DISCRIMINANT VARIETY

$$\underline{f} \notin \Sigma \implies \# V(\underline{f}) = d^n$$



MAIN ADVANCES (FOR THE MOMENT!)

- NON-UNIFORM VERSION SOLVED BY SHUB-SMALE (1985-1994)
- PROBABILISTIC ALGORITHM BY BELTRAN-PARDO 2007
- DETERMINISTIC "ALMOST" POLYNOMIAL ALGORITHM BY BÜRGISSER-CUCKER (2010)

THANK YOU!