The greatness of small points

An excursion into arithmetic equidistribution theory

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UAB mathematics colloquium

20 May 2026

Polynomial equations in roots of unity

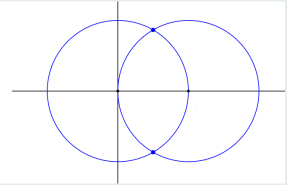
Let $\mu_\infty \subset \mathbb{C}$ the group of roots of unity

For $P \in \mathbb{Z}[X_1, \ldots, X_n]$ what can be said about the solution set of

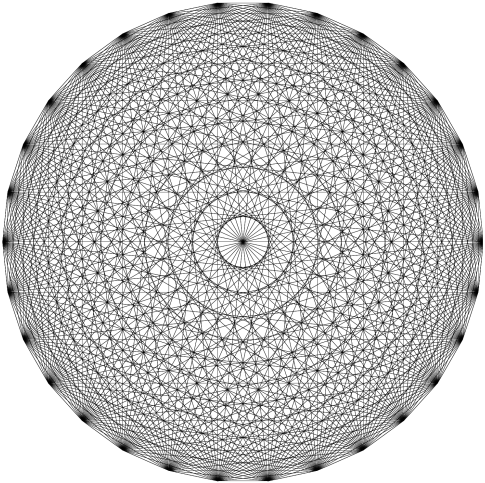
$$P(\omega_1, \ldots, \omega_n) = 0 \quad \text{with } \omega_i \in \mu_\infty?$$

Example

$\omega_1 + \omega_2 = 1$ with $\omega_1, \omega_2 \in \mu_\infty$ iff $(\omega_1, \omega_2) = (e^{\pm i\pi/3}, e^{\mp i\pi/3})$



How many intersection points?



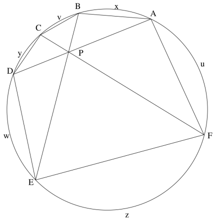
$I(30) = 16.801$

From B. Poonen and M. Rubinstein, SIAM J. Discrete Math. 1998

Vanishing sums of 12 roots of unity

The computation of the number of intersection points $I(N)$ reduces the solving of

$$\omega_1 + \cdots + \omega_{12} = 0 \quad \text{with } \omega_i \in \mu_\infty$$



Each vanishing sum breaks down as a sum of *rotations* of minimal ones.
For instance if

$$\omega_1 + \cdots + \omega_5 = 0 \quad \text{and} \quad \omega_6 + \cdots + \omega_{12} = 0$$

then

$$\eta_1(\omega_1 + \cdots + \omega_5) + \eta_2(\omega_6 + \cdots + \omega_{12}) = 0 \quad \text{for all } \eta_i \in \mu_\infty$$

Theorem (Poonen and Rubinstein 1998)

There are 107 *minimal vanishing sums (up to rotations)*

$$\sum_{i=1}^{k} \omega_i = 0 \quad \textit{with } \omega_i \in \mu_\infty \textit{ and } k \leqslant 12.$$

Many torsion points

The *algebraic torus* $(\mathbb{C}^\times)^n$ is an algebraic group under the component-wise multiplication

Torsion group μ_∞^n

A *torsion coset* is a subvariety of the form

$$H = \omega \cdot \rho((\mathbb{C}^\times)^r)$$

with $\omega \in \mu_\infty^n$ and $\rho\colon (\mathbb{C}^\times)^r \to (\mathbb{C}^\times)^n$ a group homomorphism

Note that

$$H \cap \mu_\infty^n = \omega \cdot \rho(\mu_\infty^r)$$

is Zariski dense in H

The toric Manin-Mumford conjecture

Let $X \subset (\mathbb{C}^{\times})^n$ a subvariety. Which is the structure of $X \cap \mu_{\infty}^n$?

Theorem (Laurent 1987)

If $X \cap \mu_{\infty}^n$ is Zariski dense in X then X is a torsion coset

Equivalent formulation:

There are torsion cosets $H_1, \dots, H_s \subset X$ such that

$$X \cap \mu_{\infty}^n = \bigcup_{j=1}^{s} (H_j \cap \mu_{\infty}^n)$$

If X is a curve that is not a torsion coset then $X \cap \mu_{\infty}^n$ is finite

A test

Is $\alpha = \dfrac{1}{100000000000000000000000000000} \in \mathbb{Q}$ big or small?

Bit-size $\approx \log_2(1) + \log_2(10^{29}) = 29\log_2(10) \approx 96.33$

Also measured by *absolute values*: indeed $|\alpha| = 10^{-29} = 0.000\ldots$ but

$$|\alpha|_2 = 2^{29} \quad \text{and} \quad |\alpha|_5 = 5^{29}$$

The Weil height

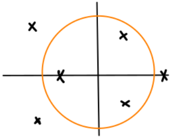
Let $\alpha \in \overline{\mathbb{Q}}$ with minimal polynomial

$$P_\alpha = c_d X^d + \cdots + c_0 = c_d \prod_{\beta \in \mathrm{Gal}(\alpha)} (X - \beta) \in \mathbb{Z}[X]$$

Here $d = \deg(\alpha)$ and $\mathrm{Gal}(\alpha)$ are the *degree* and the *Galois orbit* of α

The *Weil height* of α is

$$h_{\mathrm{W}}(\alpha) = \frac{1}{\deg(\alpha)} \left(\log |c_d| + \sum_{\beta \in \mathrm{Gal}(\alpha)} \log^+ |\beta| \right)$$



with $\log^+ |z| = \log \max(1, |z|)$

If $\alpha = \frac{a}{b} \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$ coprime then

$$h_{\mathrm{W}}(\alpha) = \log \max(|a|, |b|)$$

Basic properties

- $h_W(\alpha) = \dfrac{1}{d} \log \max_j |c_j| + O(1)$ *(arithmetic complexity)*
- $h_W(\alpha) \geqslant 0$ *(nonnegativity)*
- $h_W(\alpha^k) = k\, h_W(\alpha)$ for all $k \in \mathbb{Z}$ *(functoriality)*
- for all $c_1, c_2 > 0$ the set

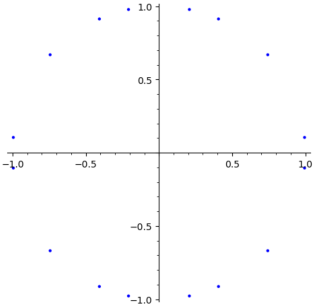
$$\{\alpha \in \overline{\mathbb{Q}} \mid \deg(\alpha) \leqslant c_1,\ h_W(\alpha) \leqslant c_2\}$$

  is finite *(Northcott property)*
- $h_W(\alpha) = 0$ iff $\alpha \in \{0\} \cup \mu_\infty$ *(Kronecker's theorem)*

Galois orbits of points of minimal height

Let α primitive N-th root of unity. Its Galois orbit is

$$\mathrm{Gal}(\alpha) = \{e^{i\,2\pi k/N} \mid \gcd(k, N) = 1\}$$

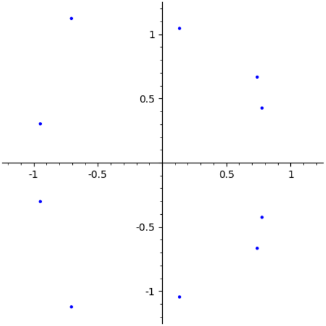


$N = 60$

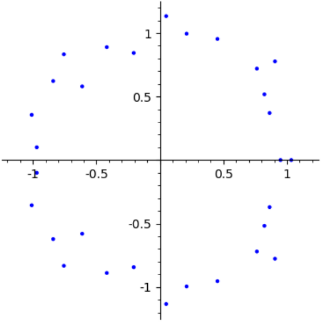
Galois orbits of points of small height

Let $\alpha \in \overline{\mathbb{Q}}$ with minimal polynomial of degree d and coeffs in $\{0, \pm 1\}$. Then

$$h_W(\alpha) \leqslant \frac{\log(d+1)}{d} \longrightarrow 0 \quad \text{as } d \longrightarrow \infty$$



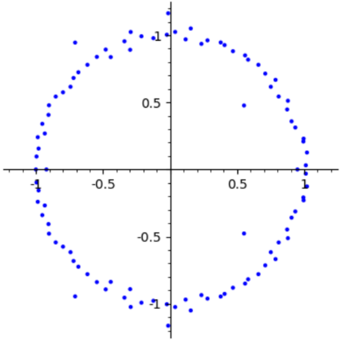
(a) $\ell = 10$



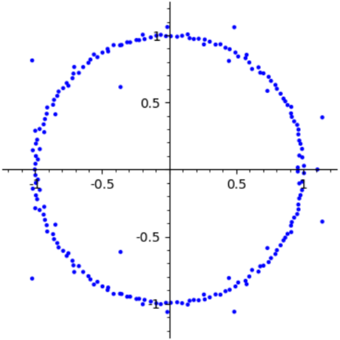
(b) $\ell = 30$

from R. Gualdi, Gaceta RSME 2023

Galois orbits of points of small height (cont.)



(c) $\ell = 100$



(d) $\ell = 200$

from R. Gualdi, Gaceta RSME 2023

Equidistribution of points of small height

A sequence $(\alpha_n)_n$ in $\overline{\mathbb{Q}}$ is *generic* if it has no infinite repetitions

Theorem (Bilu 1997)

Let $(\alpha_n)_n$ be a generic sequence in $\overline{\mathbb{Q}}$ with $\lim_n h_{\mathrm{W}}(\alpha_n) = 0$. Then

$$\lim_n \delta_{\mathrm{Gal}(\alpha_n)} = \lambda_{S^1}$$

with $\delta_{\mathrm{Gal}(\alpha_n)}$ the uniform probability measure on the Galois orbit and λ_{S^1} the Haar probability measure of the unit circle

The n-dimensional version implies the toric Manin-Mumford conjecture (and more!)

More general heights

A *Green function* is a continuous function $g\colon \mathbb{C} \longrightarrow \mathbb{R}$ such that $g(\overline{z}) = g(z)$ and

$$g(z) = \log|z| + a + o(1) \qquad \text{as } z \longrightarrow \infty \qquad (a \in \mathbb{R})$$

For $\alpha \in \overline{\mathbb{Q}}$ with minimal polynomial $P_\alpha = c_d X^d + \cdots + c_0$ we set

$$h_g(\alpha) = \frac{1}{\deg(\alpha)}\left(\log|c_d| + \sum_{\beta \in \mathrm{Gal}(\alpha)} g(\beta)\right)$$

Setting also $h_g(\infty) = a$ (point at infinity) gives

$$h_g\colon \mathbb{P}^1(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$$

For $g(z) = \log^+|z|$ we have $h_g(\alpha) = h_{\mathrm{W}}(\alpha)$ and $h_g(\infty) = 0$

Small points

The *absolute minimum* is $\mathrm{abs}(h_g) = \inf_x h_g(x)$
The *essential minimum* is

$$\mathrm{ess}(h_g) = \inf \left\{ \liminf_n h_g(x_n) \mid (x_n)_n \text{ generic sequence in } \mathbb{P}^1(\overline{\mathbb{Q}}) \right\}$$

A generic sequence $(x_n)_n$ is *small* if $\lim_n h_g(x_n) = \mathrm{ess}(h_g)$

Example (Weil height)

$\mathrm{abs}(h_{\mathrm{W}}) = \mathrm{ess}(h_{\mathrm{W}}) = 0$

Example (Zhang-Zagier height)

For $\alpha \in \overline{\mathbb{Q}}$ set $h_{\mathrm{ZZ}}(\alpha) = h_{\mathrm{W}}(\alpha) + h_{\mathrm{W}}(1-\alpha)$ and $h_{ZZ}(\infty) = 0$

We have

$$h_{\mathrm{ZZ}}(\alpha) \geqslant \frac{1}{2} \log\left(\frac{1+\sqrt{5}}{2}\right) = 0.2406\ldots \quad \text{(Zagier 1993)}$$

unless $x \in \{0, 1, \infty, e^{\pm i\pi/3}\}$.
Hence

Equidistribution of points of small height revisited

Theorem

Assume that g is subharmonic and $\mathrm{ess}(h_g) = \mathrm{abs}(h_g)$.
Then for every small generic sequence $(x_n)_n$ *we have*

$$\lim_n \delta_{\mathrm{Gal}(x_n)} = \Delta g \quad \textit{(Laplacian measure)}$$

There are also versions for (non exhaustive list)

- abelian and semiabelian varieties (Szpiro, Ullmo and Zhang 1997, Kühne 2022)
- non-archimedean places (Favre and Rivera-Letelier 2006)
- higher dimensional varieties (Yuan 2009, Yuan and Zhang 2026)
- weaker positivity (Berman and Boucksom 2010, Chen 2011)
- more general heights (Burgos, Philippon, Rivera-Letelier and S 2019, Ballaÿ and S 2025)

Proof.

Arakelov geometry □

The Fubini-Study height

Let $h_{\mathrm{FS}} \colon \mathbb{P}^1(\overline{\mathbb{Q}}) \to \mathbb{R}$ defined by

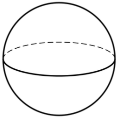
$$h_{\mathrm{FS}}(\alpha) = \frac{1}{\deg(\alpha)} \left(\log |c_d| + \sum_{\beta \in \mathrm{Gal}(\alpha)} \log \sqrt{1 + |\beta|^2} \right)$$

and $h_{\mathrm{FS}}(\infty) = 0$.
Then

$$\mathrm{abs}(h_{\mathrm{FS}}) = 0 \quad \text{and} \quad \mathrm{ess}(h_{\mathrm{FS}}) = \log \sqrt{2}$$

and for every small generic sequence $(x_n)_n$ we have

$$\lim_n \delta_{\mathrm{Gal}(x_n)} = \lambda_{S^1} \neq \Delta \log \sqrt{1 + |z|^2}$$



(Burgos, Philippon, Rivera-Letelier and S 2019)

Dynamical heights

Let $F = \frac{P}{Q}$ with $P, Q \in \mathbb{Z}[X]$ coprime and consider the associated dynamical system $F\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$

Assuming that $q \coloneqq \max(\deg(P), \deg(Q)) \geqslant 2$ it induces a height

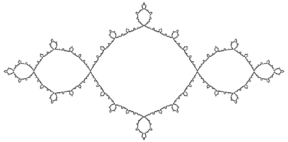
$$h_{\mathrm{dyn}}\colon \mathbb{P}^1(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geqslant 0}, \qquad x \longmapsto \lim_k \frac{h_W(F^{\circ k}(x))}{q^k}$$

Then $h_{\mathrm{dyn}}(F(x)) = q\, h_{\mathrm{dyn}}(x)$ for all x and $h_{\mathrm{dyn}}(x) = 0$ if and only if x is preperiodic for F

Corollary

Let $(x_n)_n$ be a generic sequence of preperiodic points for F. Then

$$\lim_n \delta_{\mathrm{Gal}(x_n)} = \mu_F \qquad \textit{(equilibrium measure)}$$



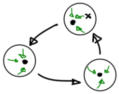
The Julia set of $F = z^2 - 1$ (from Wikipedia commons)

Families of dynamical systems

For $d \geqslant 2$ consider the family of dynamical systems

$$F_\lambda \colon \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C}), \quad z \longmapsto 1 + \frac{\lambda}{z^d} \qquad (\lambda \in \mathbb{C}^\times)$$

It is possible to define a height (Rivera-Letelier and S 2019)



$$h_{\mathrm{bif}} \colon \mathbb{P}^1(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geqslant 0}$$

such that $h_{\mathrm{bif}}(x) = 0$ if and only if F_x is *post-critically finite*

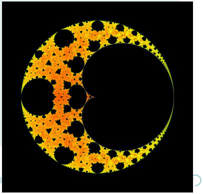
For $\lambda \in \overline{\mathbb{Q}}^\times$ it is $h_{\mathrm{bif}}(\lambda) = \lim_k \frac{h_{\mathrm{W}}(F_\lambda^k(0))}{\deg(F_\lambda^k)}$

Corollary

Let $(x_n)_n$ be a generic sequence s.t. F_{x_n} is PCF for all n. Then

$$\lim_n \delta_{\mathrm{Gal}(\alpha_n)} = \nu$$

(bifurcation measure of $\lambda \mapsto F_\lambda$)



The bifurcation locus for $d = 2$ (figure by N. Fagella)

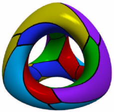
Rational points in curves

"For introducing powerful tools in arithmetic geometry and resolving the long-standing diophantine conjectures of Mordell and Lang"

Laudatio of the 2026 Abel prize awarded to Gerd Faltings

Theorem (Faltings 1983)

Let C be a curve of genus $g(C) \geqslant 2$. Then $\#C(\mathbb{Q}) < \infty$

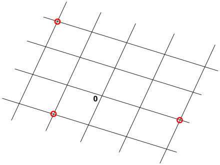
Klein's quartic curve $C = (X^3Y + Y^3Z + XZ^3 =$ 

The uniform Mordell conjecture

Let C be a curve and consider its immersion

$$C \hookrightarrow \mathrm{Jac}(C) \quad \text{(Jacobian variety)}$$

By Mordell-Weil $\mathrm{Jac}(C)(\mathbb{Q})$ is a finitely generated abelian group



$C(\mathbb{Q}) \hookrightarrow \mathrm{Jac}(C)(\mathbb{Q})$

Theorem (Dimitrov, Gao and Habegger 2021 plus Kühne 2021)

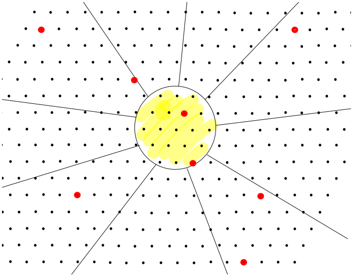
Let $g \geqslant 2$ and C a curve with $g(C) = g$. Then

$$\#C(\mathbb{Q}) \leqslant c(g)^{1+\mathrm{rank}(\mathrm{Jac}(C)(\mathbb{Q}))}$$

with $c(g) > 0$ depending only on g

The approach

Based on the Vojta-Bombieri method, which splits $C(\mathbb{Q})$ into large and small points with respect to the Néron-Tate height:



$C(\mathbb{Q}) \hookrightarrow \mathrm{Jac}(C)(\mathbb{Q})$

- Large points are controlled by the Dimitrov-Gao-Habegger "new gap principle"
- Kühne's *equidistribution for small points* in the universal family of Jacobians of curves of genus g implies the *uniform Bogomolov property* allowing to deal with the small points

Gràcies!