Fourier Series and Recent Developments in Analysis

Karlstad, June 2003

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Jean Baptiste Joseph Fourier (1768-1830)

It was around 1804 that Fourier did his important mathematical work on the theory of heat. By 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies*, which was read to the Paris Institute on December 21, 1807 and a committee consisting of Lagrange, Laplace, Monge and Lacroix was set up to report on the work.

The Institute set as a prize competition subject the propagation of heat in solid bodies for the 1811 mathematics prize. Fourier submitted his 1807 memoir together with additional work on the cooling of infinite solids and terrestrial and radiant heat. Only one other entry was received and the committee set up to decide on the award of the prize, Lagrange, Laplace, Malus, Haüy and Legendre, awarded Fourier the prize. The report was not however completely favorable and states:

"... the manner in which the author arrives at these equations is not exempt of difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigor."

`The MacTutor History of Mathematics archive:`
`http://turnbull.mcs.st-and.ac.uk/history`

Jean Baptiste Joseph Fourier (1768-1830)

Heat flow in a circular ring

Fourier formulated that the heat equation $\boxed{u_t = u_{xx}}$ describes the time evolution of the temperature of a circular ring (assuming that $u(x,t)$ is 2π-periodic function of x, but not of t). By using the classical method of separation of variables and linearity, it is easy to arrive at a solution of the form

$$\boxed{u(x,t) = \sum_{n=0}^{N} (a_n \cos nx + b_n \sin nx) e^{-n^2 t}}$$

This will fit an initial temperature $f(x)$ (at time $t = 0$) only if f can be expressed as a finite trigonometric sum:

$$\boxed{f(x) = \sum_{n=0}^{N} (a_n \cos nx + b_n \sin nx)}$$

Thus, if we let the number of coefficients N go to infinity, we would like to know which 2π-periodic functions f can be written as an **infinite** trigonometric series, and how can the coefficients a_n and b_n be calculated in terms of f?

It turns out that the trigonometric system enjoys very nice orthogonality relations:

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0, \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0, \qquad \text{all } m, n$$

Thus, if $f(x) = \sum_{n=0}^{N} (a_n \cos nx + b_n \sin nx)$ then

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx}$$

$$\boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad n = 1, 2, \cdots}$$

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \qquad n = 1, 2, \cdots}$$

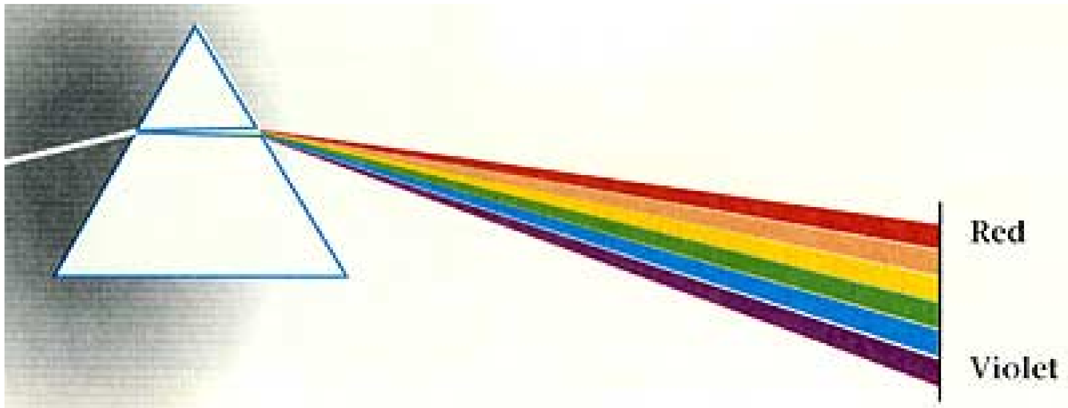
Main Question

Given a 2π-periodic function f, we want to know when the equality

$$\boxed{f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

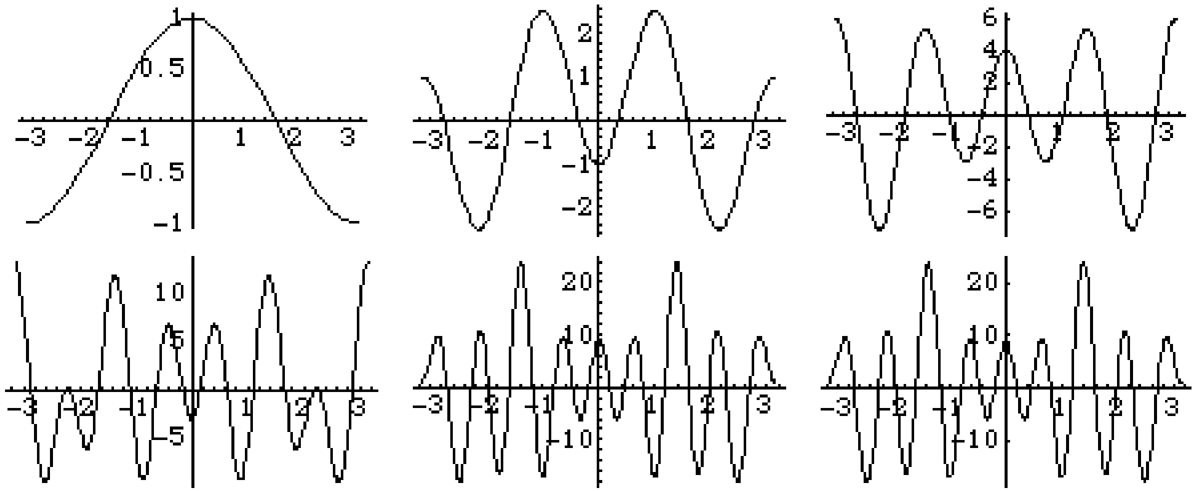
holds, where a_n and b_n are defined as above. Also we want to study the convergence properties of the series.

This series is called the **Fourier Series** of f, and can be viewed as the spectral decomposition of a signal, in terms of pure frequencies (in this case the sine and cosine functions): If we think of these functions as representing the "pure" colors (red, green, blue, etc.), then if f is a light ray, the coefficients represent the intensity of each of the "pure" colors.



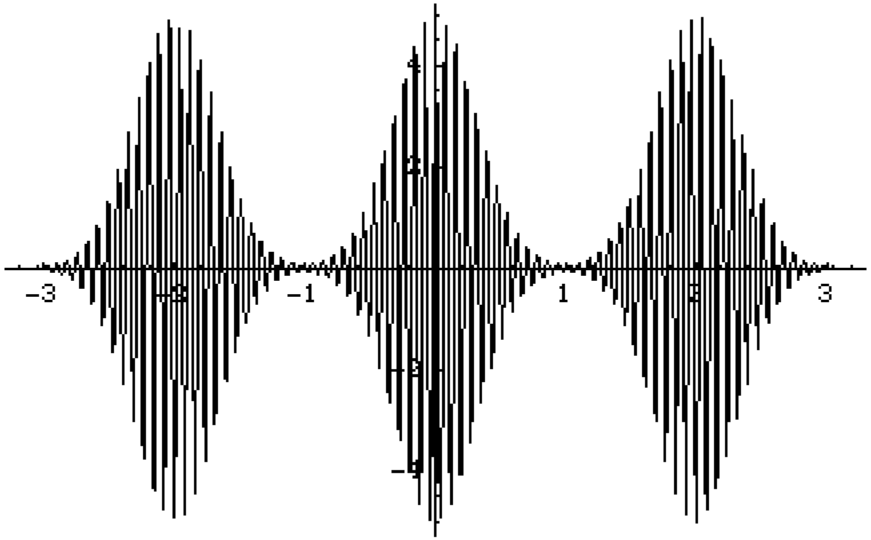
Examples of Fourier Series

$$\cos(x) - 2\cos(3x) + 5\cos(4x) - 7\cos(7x) + 12\cos(9x) + 0.1\cos(23x)$$

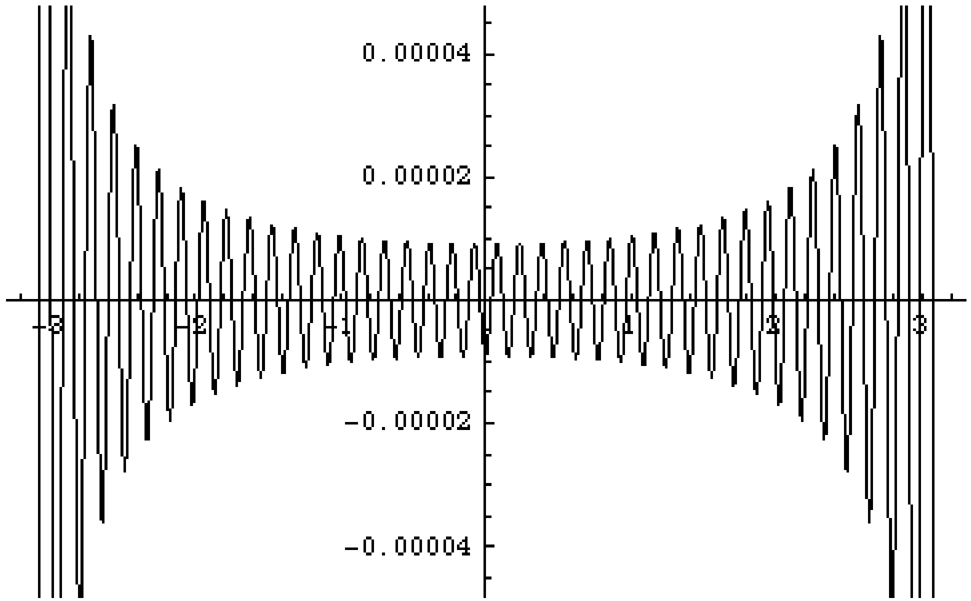


Sums with $N = 1, 3, 4, 7, 9, 23$ coefficients

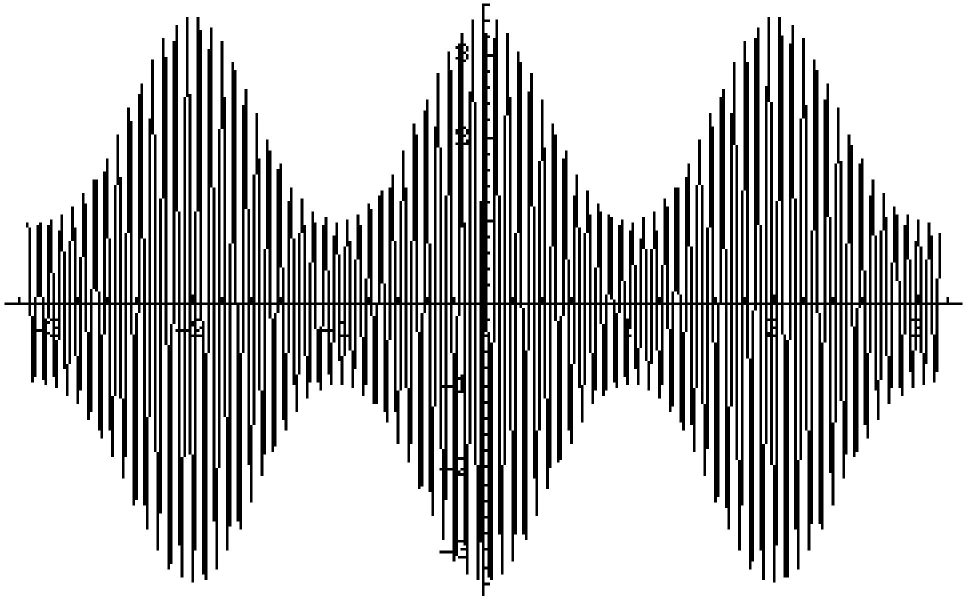
Blips



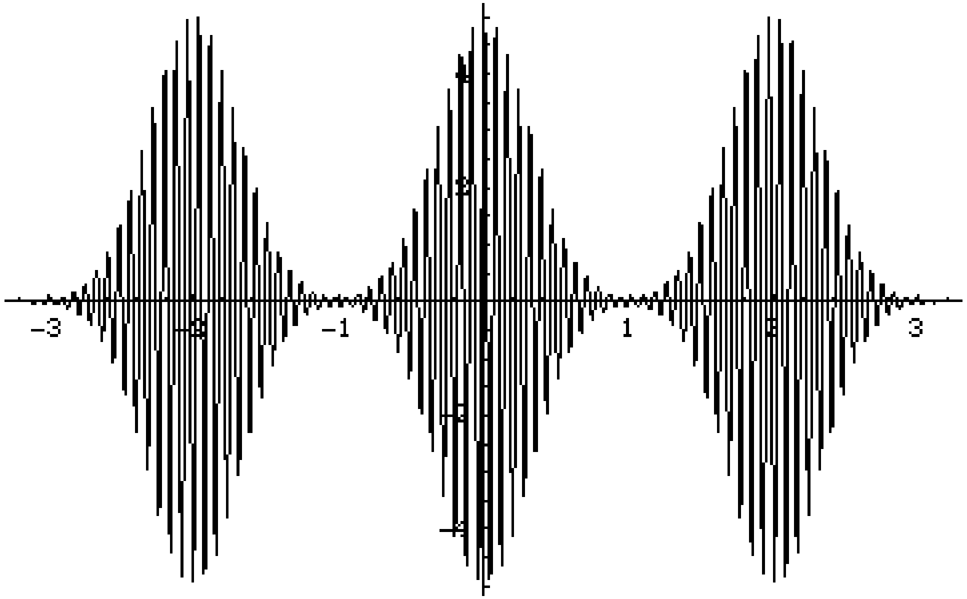
$$5\cos(80x)e^{-5x^2} + 5\cos(80x)e^{-5(x+2)^2} + 5\cos(80x)e^{-5(x-2)^2}$$



Fourier Series with 40 coefficients



Fourier Series with 80 coefficients



Fourier Series with 120 coefficients

"The problem of convergence of the Fourier Series of a function f has been one of the most productive in analysis, and many basic notions and results in mathematics have been developed by mathematicians working in this field. The modern concept of function was first introduced by Dirichlet while studying this convergence. The Riemann and, later, the Lebesgue integrals were originally introduced in works dealing with harmonic analysis. Infinite cardinal and ordinal numbers, probably the most original and striking notions of modern mathematics, were developed by Cantor in his attempts to solve a delicate real-variable problem involving trigonometric series."

Guido Weiss: "Harmonic Analysis", a chapter in the book MAA Studies in Math. vol. III, AMS, 1964

There are several easy results concerning the convergence of

(CFS) $$\boxed{f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

* If $\sum_n |a_n| < \infty$ and $\sum_n |b_n| < \infty$ then $\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges absolutely and uniformly. Hence if it converges to f then f has to be a continuous function.
* If f is a smooth function of class $\mathcal{C}^1$ then (CFS) holds. In fact it suffices a very small regularity condition on f (Lipschitz regularity).

One of the first striking negative results on (CFS) was given by P. du Bois Reymond in 1873, who showed the existence of a **continuous** function for which (CFS) fails at a point.

Not so easy results about (CFS) are the following:

* If f satisfies a Dini condition at x:

$$\int_0^\delta \frac{|f(x+t)+f(x-t)-2f(x)|}{t}\,dt<\infty,$$

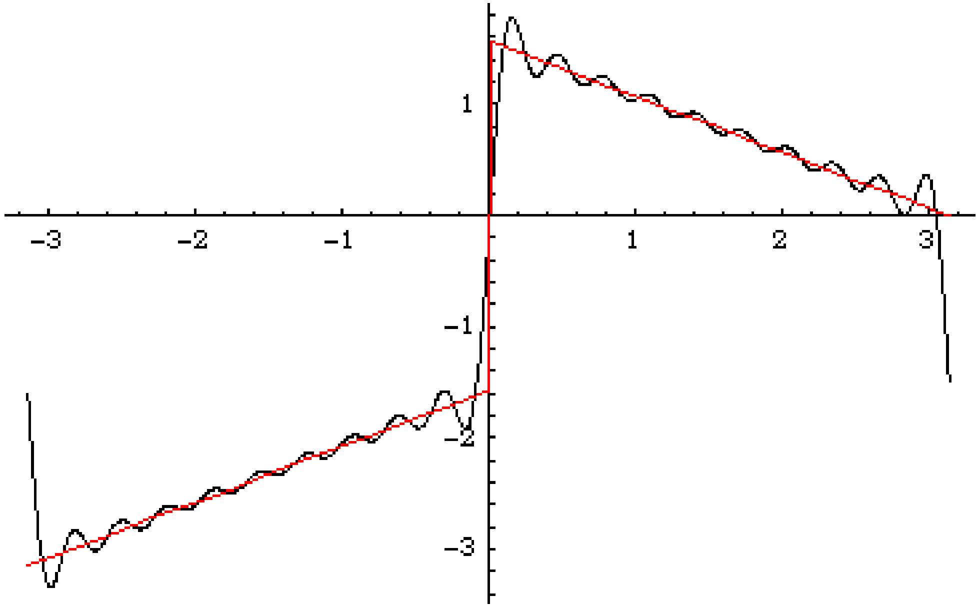
  then (CFS) holds (thus (CFS) is a **local property**, even though it involves the values of f in all the domain).

* If f is differentiable at x_0 then (CFS) holds at x_0.

* **Dirichlet-Jordan** If f is a function of bounded variation (i.e., it can be written as the difference of two monotone and bounded functions) then the Fourier Series converges at the middle point of f at x (and hence (CFS) holds if f is continuous at that point).

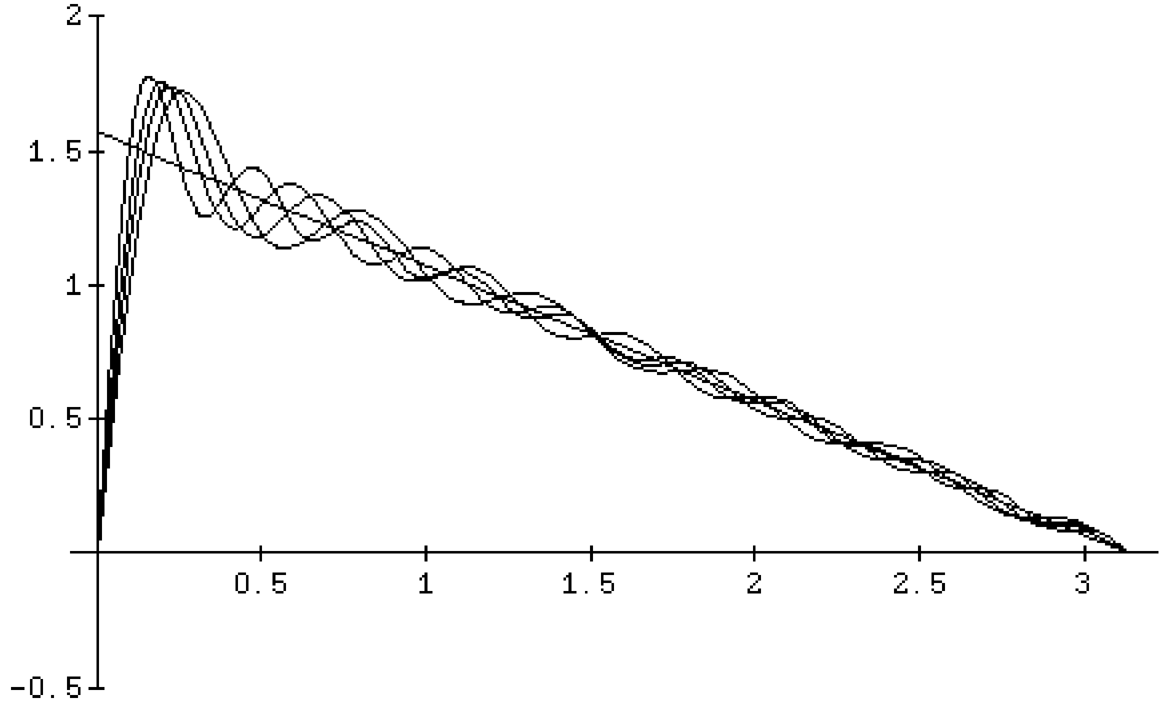
* If f is absolutely continuous, then (CFS) holds everywhere.

However, A. N. Kolmogorov showed in 1922 the existence of a function f for which (CFS) fails at **any** point.



$$f(x) = \begin{cases} (x-\pi)/2, & -\pi < x < 0 \\ (\pi - x)/2, & 0 < x < \pi \end{cases}$$

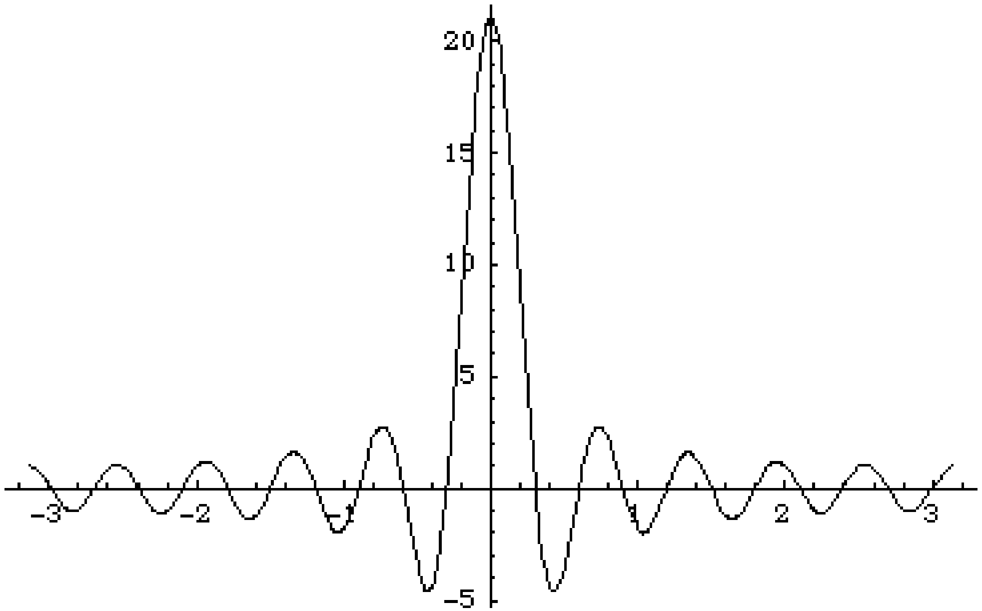
Dirichlet-Jordan Theorem



Gibbs Phenomenon

On jump discontinuities the overshot at the point always exceeds a fix amount (about 18%) the lateral limit point.

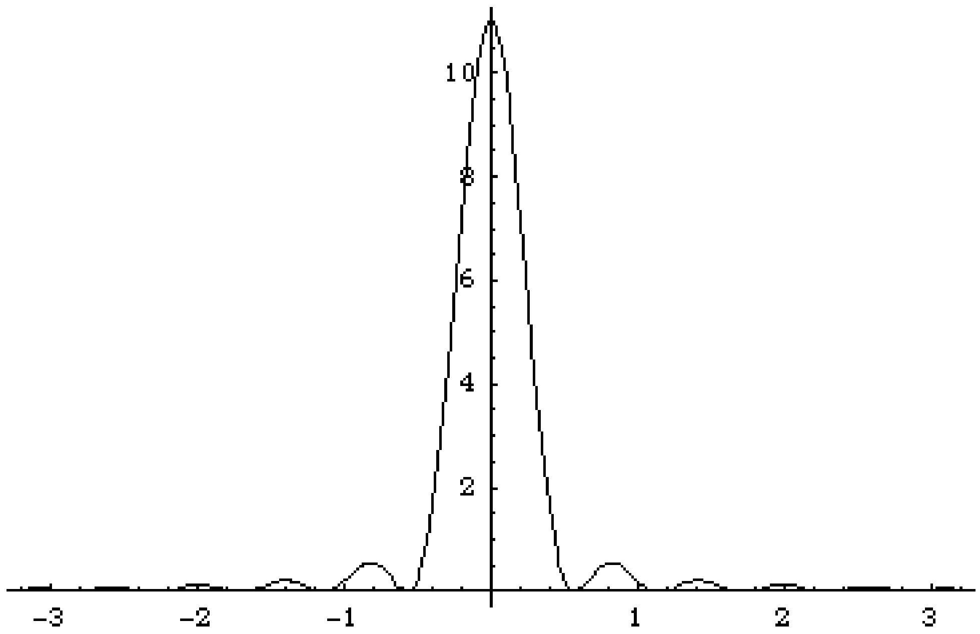
Most of the results concerning (CFS) are based on the integral representation of the partial sums of the Fourier Series, in terms of the convolution with the Dirichlet kernel:



$$D_k(x) = \frac{\sin(k + 1/2)x}{\sin x/2}$$

This family of kernels has a lot of oscillations and it is hard to control.

However, if we consider the problem of convergence in means, which is a more regular approach, then the kernel that shows up is much nicer:



Fejér's kernel: $K_n(x) = \dfrac{1}{n+1}\left(\dfrac{\sin(n+1)x/2}{\sin x/2}\right)^2$

This modified convergence ($(C,1)$-convergence) corresponds to the so called Summability Methods. In particular, now the (CFS) holds **uniformly** for continuos functions (this is just a particular case of an Approximation of the Identity).

The Nice L^2 Theory

The behaviour of the Fourier Series for a function in L^2, i.e.,

$$\|f\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \right)^{1/2} < \infty$$

can be easily controlled by using the fact that the trigonometric system is an orthonormal basis in L^2, and hence we can have all the functional properties known for Hilbert spaces. In particular, we have the generalization of Pythagoras' Theorem, which is called Parseval's Theorem:

$$\|f\|_2 = \left(\sum_{k=0}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} |b_k|^2 \right)^{1/2}$$

From here one can show that (CFS) holds in the L^2-norm. By using some other techniques based on the conjugate Fourier Series one can show that in fact (CFS) holds in L^p for all $1 < p < \infty$ (the cases $p = 1, \infty$ are false).

It is know that the norm convergence implies the existence of a subsequence which also converges pointwise, almost everywhere. But, what is the best we can say about (CFS) in the pointwise sense?

In 1966, L. Carleson proved what is considered the most important (and possibly the most difficult to read!) result on Fourier Series:

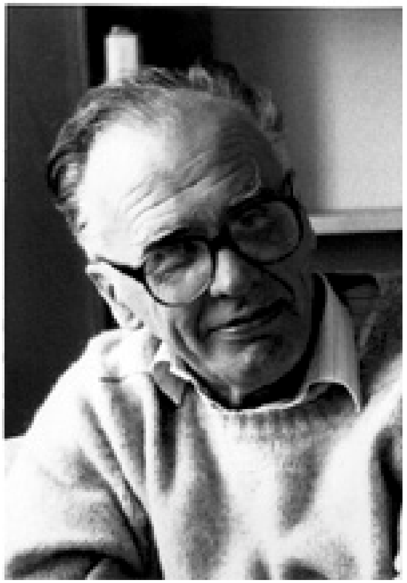
$$\boxed{\text{If } f \in L^2, \text{ then } f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx), \text{ almost everywhere}}$$

This conjecture was posed by N. Lusin in 1915. The proof of Carleson was so complicated because he was trying, in fact, to build a counterexample. After cutting the bad set in many pieces, he realized that he was able to show that the measure of the exceptional set was zero...

In 1968, R. Hunt extended the pointwise convergence to all functions in L^p, $1 < p$. There is yet an **open question**:

$$\boxed{\text{What is the largest space in } L^1 \text{ for which (CFS) holds a. e.?}}$$

Up to now (2003) the largest known is $L(\log L)(\log\log\log L)$.



Lennart Carleson, Royale Institute of Technology (KTH), Stockholm, Sweden.

Discrete and Fast Fourier Transform

There is a basic tool for computation of the Fourier Series, the *Discrete Fourier Transform (DFT)*. The computer algorithm for this computation is called *Fast Fourier Transform (FFT)*. The main idea is to discretize the integrals defining the Fourier coefficients of the function f, by means of a Riemann sum: Fix a big number N (usually a power of 2), and use the approximation:

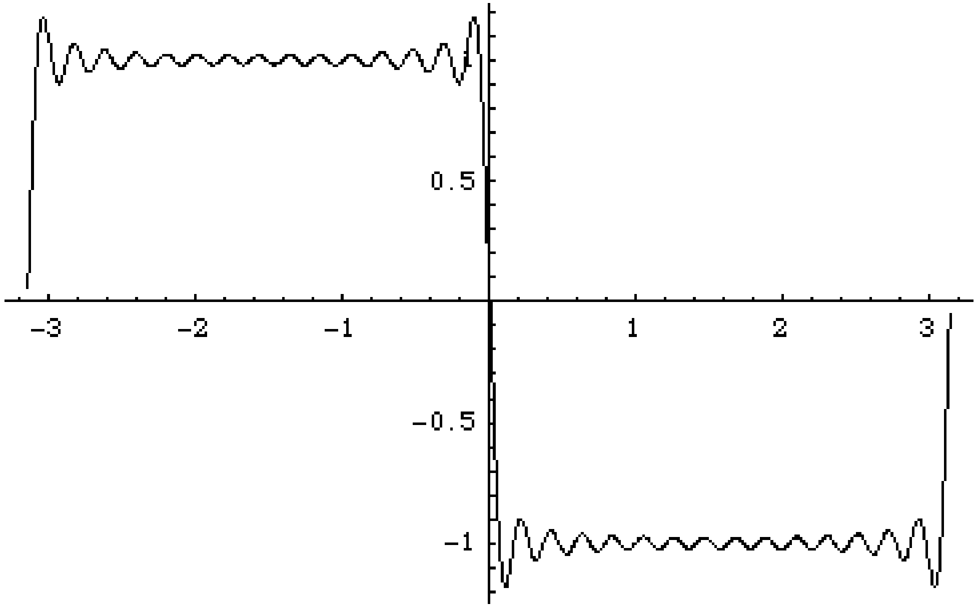
$$a_n \approx \frac{2}{N} \sum_{j=0}^{N} f(2j\pi/N) \cos(2j\pi n/N)$$

$$b_n \approx \frac{2}{N} \sum_{j=0}^{N} f(2j\pi/N) \sin(2j\pi n/N)$$

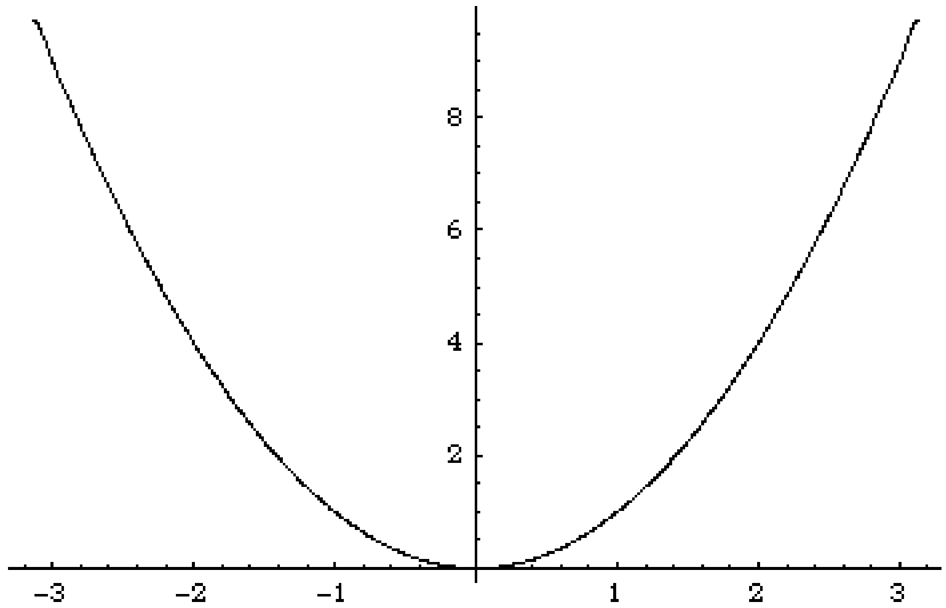
This discrete transformation is now used to compute (approximately) the Fourier Series of f. It turns out that this calculation involves the multiplication of $N \times N$ trigonometric matrices, which requires the order of N^2 multiplications: If $N = 1024$ then this would amount up to 1,048,576 products.

By using an optimization method to carry over this product, the FFT allows a tremendous saving: we can get the same result but only using $N \log N$ products. In the previous case, we reduce the products from 1,048,576 to 5,120.

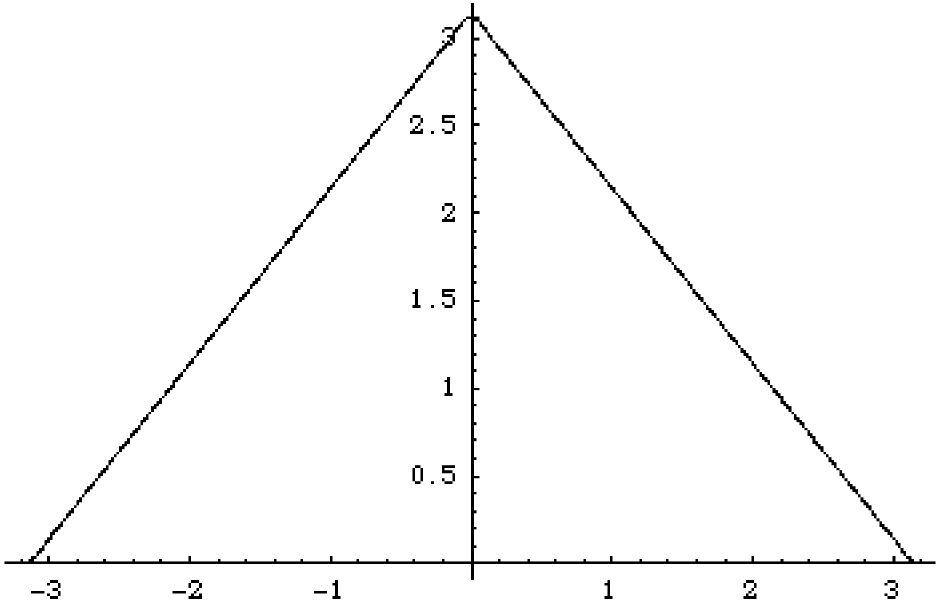
Examples using FFT with $N = 1024$ and 30 coefficients



$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases} \qquad \text{(0.06 seconds)}$$



$f(x) = x^2$ (0.08 seconds)



$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases} \qquad \text{(0.06 seconds)}$$

How to optimize multiplication

Consider two (BIG) numbers:

$$A = 4384712398574181485149 85^{100000}$$

$$B = 7584375834758347568347 56^{100000}$$

(A has $2,364,195$ digits). We would like to calculate the following:

$$\boxed{A^2 + 2AB + B^2}$$

We observe that this operation requires 3 products and 3 sums ($A^2 + 2AB + B^2 = A^2 + AB + AB + B^2$). Using *Mathematica* we measure the time needed for this calculation:

17.599999999976717 seconds

We now observe that we can also write the above number as follows:

$$\boxed{(A + B)(A + B)}$$

which only needs 1 product and 1 sum. Now, to calculate this we only need

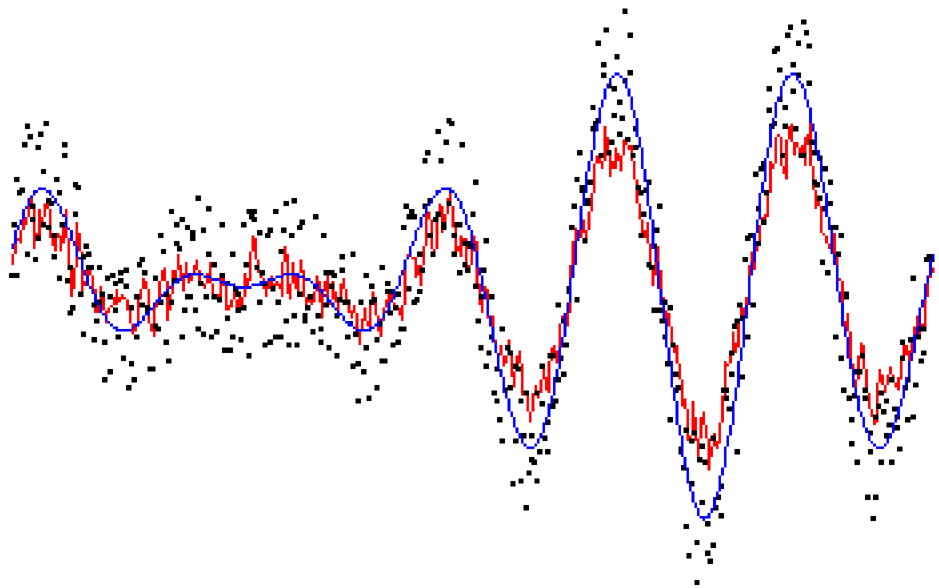
5.46667 seconds

which is a BIG improvement.

Applications of Fourier Series

* Prediction of Tides (Lord Kelvin, 1871).
* Isoperimetric Problem: The circle is the figure with maximum area for a fixed perimeter (J. Steiner, 1841. A. Hurwitz, 1900)
* The age of the Earth: Kelvin's model of a cooling earth (Rough approximation: 100-400 million years! Radioactivity was not yet discovered...)
* Analog telephone.
* Spectral Analysis: Atomic, Infrared, Cosmic.
* Physical Models described by some Partial Differential Equations: Motions of the Planets, Heat Flow in Solids, etc.

* Denoising: Filtering "noises" in a signal:



* Compact Discs: How music is stored on CD's and then reproduced? (Shannon Sampling Theorem.)

However, there are many cases in which Fourier methods are not always a good tool to analyze a signal, in particular if it is highly nonsmooth:

* The Fourier Series behaves badly at discontinuity points.
* What can we say about the regularity of a function by looking at the size of its coefficients (some information can be recovered but there is always a loss).
* Can we characterize other spaces than L^2 by the Fourier coefficients? (It is not true for L^p if $p \neq 2$)

It was in the 1980's that several people (R. Coifman, I. Daubechies, M. Frazier, A. Grossman, B. Jawerth, S. Mallat, Y. Meyer, J.O. Strömberg, G. Weiss) realized that a new method for discretizing functions (analyzing signals) could be used in a more efficient way.

WAVELETS

A Wavelet is just an L^2 function Ψ which satisfies the following property:

If we dilate and translate the function, $\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$, then

$$\boxed{\left\{\Psi_{j,k}\right\}_{j,k} \text{ is an orthonormal basis of } L^2}$$

Hence, we can now define the wavelet transform by assigning to each function f the coefficients

$$f_{j,k} = \int f(x)\Psi_{j,k}(x)\,dx$$

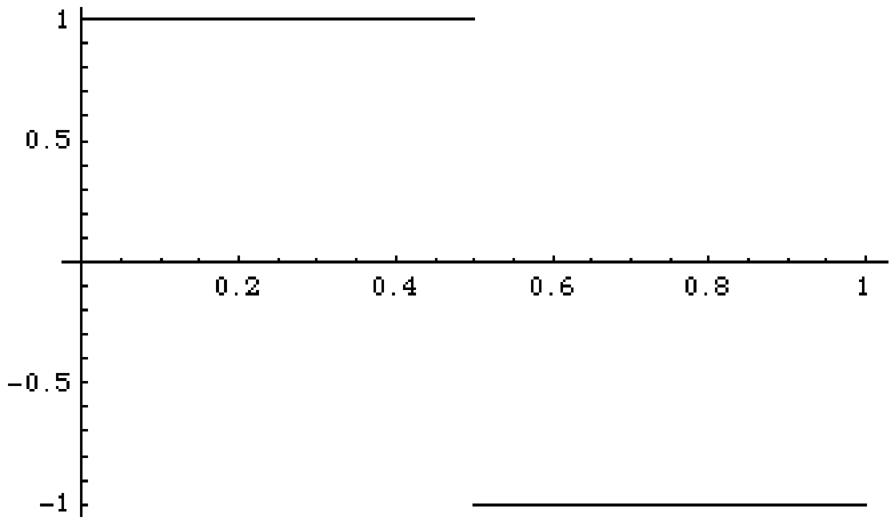
and we can recover the function back by means of the series

$$f(x) = \sum_{j,k} f_{j,k}\Psi_{j,k}(x)$$

if $f \in L^2$.

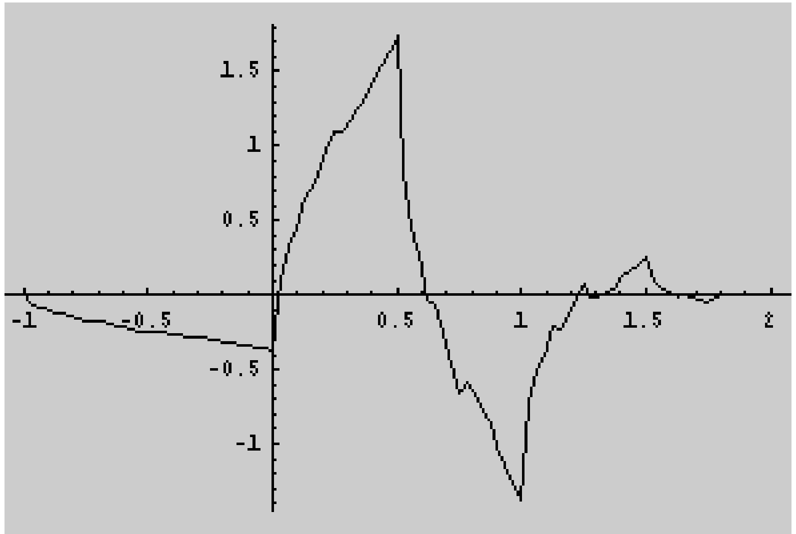
Examples

Haar wavelet



$$f(x) = \begin{cases} 1, & 0 < x < 1/2 \\ -1, & 1/2 < x < 1 \end{cases}$$

Daubechies wavelet



Continuous compactly supported wavelet

Good Features of the Wavelet Transform

* We can choose the "right" wavelet for each case: we can either fix the smoothness, or the support, etc. Hence we can completely determine the regularity properties of a function by measuring the size of the discrete sequence of coefficients.

* In general, most of the functional spaces in Analysis (Lebesgue, Sobolev, Triebel-Lizorkin, etc.) can be discretized using wavelets.

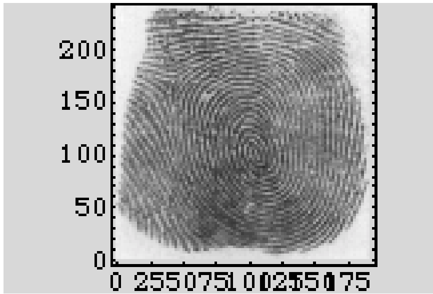
* All wavelets are characterized by the simple equations:

$$\sum_j |\hat{\Psi}(2^j \xi)|^2 = 1, \qquad \text{a.e.}$$

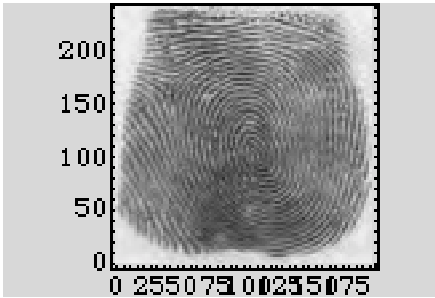
$$\sum_{j=0}^{\infty} \hat{\Psi}(2^j \xi)\overline{\hat{\Psi}(2^j(\xi + 2m\pi))} = 0 \qquad \text{a.e.}$$

  for every odd integer m.

* A small number of wavelet coefficients are needed to accurately represent a function:



Original fingerprint



Using 7% of the wavelet coefficients

The full FBI fingerprints database (more than 200 million records, about 2,000 terabytes in size) has been compressed about 15:1 by using a wavelet type image coding.

References

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