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TENT SPACES BASED ON WEIGHTED LORENTZ SPACES. CARLESON MEASURES

by
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A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 1990
Saint Louis, Missouri

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Acknowledgments

I am very grateful to professors M. Taibleson and G. Weiss for their continual encouragement and support during my research. I can not put into words how much I owe to them. Their friendly talks and good advice is something I will never forget. I have also enjoyed the many conversations I have had with other members of the department. To all of them I extend my deepest gratitude.

Thanks also to E. Hernández, to whom I owe much of the motivation in the proof of the reitaration result of Chapter II.

I also want to thank all of my friends in and out the department for the times we have shared together.

¿Cómo expresar en estas líneas todo el cariño y afecto recibido de mis padres y hermanos durante estos cuatro años?. Hay tantos detalles que querría agradecerles, tantas son las cosas que han hecho por mí que no puedo mas que reconocer lo orgulloso que me siento de todos ellos, de mi familia. Espero que algún día pueda estar yo también a su misma altura.

El punto final lo cierro con María Jesús. Juntos empezamos un nuevo párrafo.

Introduction

In a sequence of recent papers (see [**CO-ME-ST 1-2**]), Coifman, Meyer and Stein have introduced a new family of function spaces, the Tent spaces. These spaces are well adapted for the study of a variety of questions related to harmonic analysis and its applications, for example, the Cauchy integral on Lipschitz curves, maximal functions and atomic decompositions. Tent spaces have been also studied by other authors (see, for example, Bonami and Johnson [**BO-JO**], Alvarez and Milman [**AL-MI 1-2**]), where they are generalized to also consider norms based on the classical Lorentz spaces $L^{p,q}$. These new spaces show up in a natural way as the interpolated spaces for the real interpolation method of Lions and Peetre (see [**LI-PE**]).

Our present work continues this line of development. We consider the theory of Tent spaces related to a class of weights on $\mathbf{R}^+$, namely, positive and measurable functions that satisfy Hardy's inequalities (see [**MU**]). The main motivation for our study comes from the method of interpolation with a function parameter, due to Kalugina and Gustavsson (see [**KA**] and [**GU**]). The function parameters we use allow us to prove new interpolation results for some weighted spaces; for example $L^p(\log L)^\gamma$. Our goal is to give a through account of the properties of the Tent spaces defined by weights related to a function parameter, extend their atomic decomposition and the characterization of the dual spaces, (which brings in the theory of Carleson measures), to the new setting of weighted spaces, and develop the techniques necessary to obtain such results. As a consequence, we are able to apply some of these results to the theory of maximal Hardy spaces, weighted inequalities for fractional maximal operators, and related topics.

We now give a brief review of the topics in each of the chapters:

In Chapter I, after introducing two classes of function parameters and giving a detailed list of properties that the functions in these classes satisfy, we show, in Theorem 1.2.7, that the family of weights we are dealing with make both the Hardy operator and its adjoint, bounded operators on some weighted L^p-spaces; that is, they satisfy Hardy's inequalities. This fact will turn out to be one of the main tools in the remaining chapters. Theorem 1.2.7 appeared in a joint work with E. Hernández (see [**HE-SO**]).

In Chapter II we define in (§1) the weighted Lorentz spaces in terms of the non-increasing rearrangement of a function and find a new description involving the distribution function (see Theorem 2.1.6). We analyze other properties, as duality and embbedings, depending on the several indices defining the spaces. In (§2) we exhibit a discrete decomposition of functions in some of the weighted Lorentz spaces. In (§3) we take up the study of the interpolation results, using the theory of complex interpolation of Banach spaces, as in [**CCRSW 1-2**]. Our main technique is a new reiteration theorem that relates the interpolation of a couple of Banach spaces with a family of function parameters, which can then be interpolated using the method of CCRSW. The reiteration result, (Theorem 2.3.7), was first proved by the author in the work mentioned above of [**HE-SO**] and generalizes some previous results of the first named author (see [**HE 2**]), and Karadžov and Berg (see [**BE-LO**]). Finally in (§4) we answer a question of Y. Sagher regarding the interpolation with a function parameter of some endpoint spaces, as H^1 and C_0 or similarly the space of finite Borel measures and BMO. The result in all cases is that we always obtain the weighted Lorentz spaces as the intermediate spaces.

In Chapter III, we give the definition of the weighted Tent spaces, whose motivation can be found in the Littlewood-Paley theory and we prove in (§3) their atomic decomposition, using a new discretization of the norm in terms of the distribution function. As is shown in the next chapter, this decomposition allows us to easily get the boundedness of certain maximal operators. In (§4) we find the dual spaces of the weighted Tent spaces, and for a particular case, we obtain a new class of Carleson measures. In (§5) we investigate the extension of the previous results to the case where the functionals defining the norms of the Tent spaces are replaced by functionals over general domains, instead of the standard cones. In (§6) we study the interpolation results for the Tent spaces and our new spaces of Carleson measures, using the real interpolation method. Again, as in Chapter II, we obtain the corresponding results for the method of CCRSW by means of our reiteration theorem.

In Chapter IV we give several applications of the theory developed in the previous chapters. In (§1) we consider the study of the maximal Hardy spaces, relative to a not necessarily smooth kernel, as in **[WE]**. Our main theorem is the extension to $\mathbf{R^n}$ of the minimality property, proved in one dimension by Uchiyama and Wilson in **[UC-WI]**, of the space of special atoms with respect to this family of Hardy spaces. We also characterize a class of Carleson measures, satisfying another minimality condition and using the recent discrete decomposition of the Triebel-Lizorkin spaces, $\dot{F}_p^{\alpha,q}$, due to Frazier and Jawerth (see **[FR-JA 2]**), we find necessary and sufficient conditions on a kernel so that the corresponding Hardy space contains a particular $\dot{F}_1^{0,p}$, $1 \leq p \leq 2$. In (§2) we show several results dealing with pointwise estimates for a class of operators, which gives another proof of the pointwise boundedness of the Fourier transform of distributions in H^p and we improve the Féjer-Riesz inequality for harmonic extensions of H^p functions. Finally in (§3) we study the boundedness of a fractional maximal function on a weighted Lorentz space. The proof is based on the atomic decomposition of a certain Tent space.

Chapter I

Preliminaries

(§1) Function parameter.

Function parameters were first introduced by T. F. Kalugina (see [**KA**]) in the real interpolation setting in order to construct new interpolation spaces.

In this section we give the definition of two classes of function parameters, show in a precise way that they are equivalent, and give a detailed list of properties that the functions in theses classes satisfy.

DEFINITION 1.1.1. *Suppose* $f : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$. f *belongs to the class* B_K *if:*

$$f \text{ is continuous, non-decreasing and } f(1) = 1\cdot \tag{1}$$

$$\overline{f}(s) = \sup_{t>0} \frac{f(st)}{f(t)} < \infty, \text{ for every } s > 0\cdot \tag{2}$$

$$\int_0^\infty \min(1, 1/t)\overline{f}(t) \, \frac{dt}{t} \; < \; \infty\cdot \tag{3}$$

EXAMPLES 1.1.2:

(a) If $0 < \alpha < 1$ then $f(t) = t^\alpha$ belongs to B_K and $\overline{f}(t) = t^\alpha$.

(b) If $0 < \alpha < \beta < 1$ then $f(t) = \log 2 \, \dfrac{t^\beta}{\log(1 + t^\alpha)}$ belongs to B_K and we have that $\overline{f}(t) = \max(t^{\beta-\alpha}, t^\beta)$.

DEFINITION 1.1.3. *The class* B_Ψ *consists of all continuously differentiable functions* $f : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *such that* $f(1) = 1$ *and*

$$0 < \alpha_f = \inf_{t>0} \frac{tf'(t)}{f(t)} \leq \sup_{t>0} \frac{tf'(t)}{f(t)} = \beta_f < 1. \tag{4}$$

EXAMPLE 1.1.4:

If $0 < \alpha < 1$, $\theta \in \mathbf{R}$, then

$$f(t) \;=\; \frac{1}{(\log 2)^{\theta}}\; t^{\alpha}\left(\log(1+t^{\gamma})\right)^{\theta}$$

belongs to B_Ψ, if γ is small enough. In fact

$$\alpha_f = \min(\alpha, \alpha+\theta\gamma), \qquad \beta_f = \max(\alpha, \alpha+\theta\gamma).$$

The following two results show that the classes B_K and B_Ψ are equivalent.

PROPOSITION 1.1.5. ([**GU**], Prop. 1.2)

The class of functions B_Ψ is contained in B_K.

PROPOSITION 1.1.6. ([**GU**], Prop. 1.3)

If $f \in B_K$, there is a function $g \in B_\Psi$ such that f and g are equivalent; i.e., there are two positive constants C_1 and C_2 such that

$$C_1\; g(t) \le f(t) \le C_2\; g(t), \qquad t > 0.$$

In fact, we can choose

$$g(s) = C_f \int_0^\infty \min(1, \frac{s}{t})\, f(t)\, \frac{dt}{t}, \tag{5}$$

where C_f is chosen so that $g(1) = 1$.

There is a nice way of constructing all functions in the class B_Ψ:

PROPOSITION 1.1.7. *Suppose*

$$\mathcal{C} = \left\{ h:\ \mathbf{R}^+ \longrightarrow (0,1)\ :\ h \text{ continuous},\ 0 < \inf_{t>0} h(t) \le \sup_{t>0} h(t) < 1 \right\}.$$

Then there exists a one to one correspondence T from $\mathcal{C}$ to B_Ψ given by

$$\begin{aligned} &\mathcal{C} \xrightarrow{T} B_\Psi \\ &h \longrightarrow T(h)(t) = \exp\left(\int_1^t \frac{h(s)}{s}\, ds\right). \end{aligned}$$

In fact,

$$T^{-1}(g)(t) = \frac{tg'(t)}{g(t)}\cdot$$

PROOF: The proof is an easy exercise in differential equations. To see that T is well defined we observe that

$$(Th)'(t) = Th(t)\frac{h(t)}{t}\cdot$$

Hence,

$$h(t) = \frac{t\,(Th)'(t)}{Th(t)},$$

and so

$$0 < \alpha_{Th} \leq \beta_{Th} < 1.$$

Clearly,

$$(T \circ T^{-1})(g)(t) = g(t) \text{ and } (T^{-1} \circ T)(h)(t) = h(t). \quad \blacksquare$$

The next propositions give a list of properties that the classes B_K and B_Ψ satisfy. As we shall see later, they turn out to be of vital importance for the development of our theory on Lorentz spaces.

PROPOSITION 1.1.8. ([**GU**])

Suppose $f \in B_K$. Then:

(6) $\underline{f}(s)\overline{f}(\frac{1}{s}) = 1$, *where* $\underline{f}(s) = \inf_{t>0} \frac{f(st)}{f(t)}\cdot$

(7) $0 < \underline{f}(s)f(t) \leq f(st) \leq \overline{f}(s)f(t)$, ($f$ *is quasi-homogeneous*)$\cdot$

(8) $\underline{f}$ *and* $\overline{f}$ *are non-decreasing and* $\underline{f}(1) = \overline{f}(1) = 1\cdot$

(9) $\overline{f}(st) \leq \overline{f}(s)\overline{f}(t)$, ($\overline{f}$ *is submultiplicative*)$\cdot$

(10)

$$\overline{f}(s) = o(\max(1, s)),\ s \longrightarrow 0 \text{ or } \infty\cdot$$

(11)

For sufficiently small $\varepsilon > 0$, $\overline{f}(s) = o(\max(s^{\varepsilon}, s^{1-\varepsilon}))$, $s \longrightarrow 0$ *or* $\infty\cdot$

(12)

$$\int_0^\infty \left(\min(1, 1/t)\overline{f}(t)\right)^p \frac{dt}{t} < \infty, \text{ for any } p > 0\cdot$$

(13)
$$\frac{1}{p^{1/p}} \le \frac{f(s)}{s}\left(\int_0^s \left(\frac{t}{f(t)}\right)^p \frac{dt}{t}\right)^{1/p} \le \left(\int_1^\infty \left(\frac{\overline{f}(t)}{t}\right)^p \frac{dt}{t}\right)^{1/p}$$
for any $p > 0$; *i.e.,*
$$\int_0^s \left(\frac{t}{f(t)}\right)^p \frac{dt}{t} \approx \left(\frac{s}{f(s)}\right)^p.$$

(14)
$$\left(\int_1^\infty \left(\frac{1}{\overline{f}(t)}\right)^p \frac{dt}{t}\right)^{1/p} \le f(s)\left(\int_s^\infty \left(\frac{1}{f(t)}\right)^p \frac{dt}{t}\right)^{1/p}$$
$$\le \left(\int_0^1 (\overline{f}(t))^p \frac{dt}{t}\right)^{1/p}, \quad \textit{for any } p > 0; \; \textit{i.e.,}$$
$$\int_s^\infty \left(\frac{1}{f(t)}\right)^p \frac{dt}{t} \approx \frac{1}{f^p(s)}.$$

If $g \in B_\Psi$, property (11) can be improved using Proposition 1.1.7:

PROPOSITION 1.1.9. *Let* $g \in B_\Psi$. *Then*

$$\min(t^\alpha, t^\beta) \le g(t) \le \overline{g}(t) \le \max(t^\alpha, t^\beta).$$

PROOF: By Proposition 1.1.7,

$$g(t) = \exp\left(\int_1^t \frac{h(s)}{s}\, ds\right)$$

where $\alpha \le h(s) \le \beta$, all $s > 0$. Hence,

$$g(t) \ge \begin{cases} t^\alpha & \text{if } t \ge 1 \\ t^\beta & \text{if } t \le 1. \end{cases}$$

Thus, $\min(t^\alpha, t^\beta) \le g(t)$.

Similarly, since
$$\frac{g(ts)}{g(s)} = \exp\left(\int_s^{st} \frac{h(u)}{u}\, du\right),$$
then
$$\frac{g(ts)}{g(s)} \le \begin{cases} t^\beta & \text{if } t \ge 1 \\ t^\alpha & \text{if } t \le 1. \end{cases}$$
Hence, $\overline{g}(t) \le \max(t^\alpha, t^\beta)$. ∎

We now use the following result to improve (13) and (14) above.

PROPOSITION 1.1.10. ([**SA**])

Let m be a positive function. Then, for all r, $0 < r < \infty$,

$$\int_0^r m(s)\frac{ds}{s} \approx m(r)$$

if and only if for all r, $0 < r < \infty$,

$$\int_r^\infty \frac{1}{m(s)}\frac{ds}{s} \approx \frac{1}{m(r)}\cdot$$

COROLLARY 1.1.11. *Let $f \in B_K$. Then*

$$\int_s^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} \approx \left(\frac{f(s)}{s}\right)^p, \ p > 0\cdot \tag{15}$$

$$\int_0^s (f(t))^p \frac{dt}{t} \approx (f(s))^p, \ p > 0\cdot \tag{16}$$

As was mentioned at the beginning of this section, the class B_Ψ was introduced to generalize the real method of interpolation of Peetre and Lions. It is then natural that to obtain the reiteration theorem or the duality theorem (see [**BE-LO**]), we show that B_Ψ is closed under certain types of operations.

PROPOSITION 1.1.12. *Let $g \in B_\Psi$ and $\beta = \beta_g$ as in (4).*

(17) *If $a, b \in \mathbf{R}$, then*

$$g^a(t^b) \in B_\Psi \qquad \text{if and only if} \qquad 0 < ab < \beta^{-1}\cdot$$

(18) *If $0 < b \leq a$, with $a\beta < 1$, then*

$$\frac{1}{2}(g^a(t) + g^b(t)) \in B_\Psi.$$

(19) *If $0 < b \leq a,\ d > 0,\ c \geq 0$ and $ca + d\beta < 1$, then*

$$\frac{1}{2^c}(t^a + t^b)^c g^d(t) \in B_\Psi.$$

(20) *The functions $tg(1/t)$ and $t/g(t)$ belong to B_Ψ.*

(21) *Suppose $f \in B_\Psi$ and if we set $\tau(t) = f(t)/g(t)$ then τ satisfies that*

$$\left|\frac{t\tau'(t)}{\tau(t)}\right| \geq \varepsilon > 0, \ \text{for some } \varepsilon > 0.$$

Then for all $h \in B_\Psi$, the function

$$\varphi(t) = g(t)\ h\left(\frac{f(t)}{g(t)}\right)$$

belongs to B_Ψ. In particular, $f^\theta(t) g^{1-\theta}(t) \in B_\Psi$, $0 < \theta < 1$.

PROOF: The proofs are elementary. For further references see [**CO**], [**ME 1**].

(§2) Weighted Hardy's inequalities.

Hardy's inequalities can be considered as a fundamental tool in the study of a wide variety of function spaces in Harmonic Analysis and, in particular, the Lorentz spaces. Our goal in this section is to provide some results dealing with these inequalities (in particular the work of B. Muckenhoupt [**MU**]), and establish more properties of functions in B_Ψ related to Hardy's inequalities.

DEFINITION 1.2.1. *We define the Hardy operator as*

$$Sf(t) = \frac{1}{t}\int_0^t f(s)\, ds,\ \ f \geq 0.$$

We also define the formal adjoint of S by

$$S^* f(t) = \int_t^\infty f(s)\, \frac{ds}{s},\ \ f \geq 0.$$

REMARK 1.2.2: S^* is in fact the adjoint of S:

$$\begin{aligned}\langle Sf, g\rangle = \int_0^\infty Sf(t)\ g(t)\ dt &= \int_0^\infty \left(\frac{1}{t}\int_0^t f(s)\ ds\right) g(t)\ dt \\ &= \int_0^\infty f(s) \left(\int_s^\infty g(t)\ \frac{dt}{t}\right)\ ds = \langle f, S^* g\rangle.\end{aligned}$$

We can now formulate Hardy's inequalities:

THEOREM 1.2.3. *If $q \geq 1,\ r > 0$, and $f \geq 0$, then*

$$\left(\int_0^\infty \left(\int_0^t f(s)\ ds\right)^q t^{-r-1}\ dt\right)^{1/q} \leq \frac{q}{r}\left(\int_0^\infty (sf(s))^q s^{-r-1}\ ds\right)^{1/q}. \tag{22}$$

$$\left(\int_0^\infty \left(\int_t^\infty f(s)\ ds\right)^q t^{r-1}\ dt\right)^{1/q} \leq \frac{q}{r}\left(\int_0^\infty (sf(s))^q s^{r-1}\ ds\right)^{1/q}. \tag{23}$$

For a proof of the theorem see [**HA-LI-PO**] Chapter IX or [**ST**].

Using the notation introduced above, this theorem can be restated as:

(24) S is a bounded operator on $L^q(s^{-r/q-1/q+1})$

and

(25) S^* is a bounded operator on $L^q(s^{r/q-1/q})$

where for a non-negative function v defined on $\mathbf{R}^+$, we set, for $1 \leq q \leq \infty$,

$$L^q(v) = \left\{f :\ \left(\int_0^\infty (|f(t)|v(t))^q\ dt\right)^{1/q} < \infty\right\}.$$

Hence, (22) and (23) are weighted inequalities for the operators S and S^*. A natural problem is to characterize all pairs of weights (u, v) for which S or S^* is bounded from $L^q(u)$ to $L^q(v)$, $1 \leq q \leq \infty$. A complete solution was given by B. Muckenhoupt in [**MU**].

THEOREM 1.2.4. *If (u, v) is a pair of non-negative weights on $\mathbf{R}^+$ and $1 \leq q \leq \infty$, then S is a bounded operator from $L^q(u)$ to $L^q(v)$ if and only if*

$$\sup_{r>0}\left(\int_r^\infty s^{-q}(v(s))^q\ ds\right)^{1/q}\left(\int_0^r (u(s))^{-q'}\ ds\right)^{1/q'} < \infty. \tag{26}$$

Similarly we have a result for S^*.

THEOREM 1.2.5. *If (u, v) is a pair of non-negative weights on $\mathbf{R}^+$ and $1 \leq q \leq \infty$, then S^* is a bounded operator from $L^q(u)$ to $L^q(v)$, if and only if*

$$\sup_{r>0}\left(\int_0^r (v(s))^q\ ds\right)^{1/q}\left(\int_r^\infty s^{-q'}(u(s))^{-q'}\ ds\right)^{1/q'} < \infty. \tag{27}$$

DEFINITION 1.2.6. *Suppose* $1 \le q \le \infty$. *Let*

$$W_q(S) = \big\{u: \ u \text{ weight and } (u,u) \text{ satisfies } (26)\big\}, \tag{28}$$

$$W_q(S^*) = \big\{u: \ u \text{ weight and } (u,u) \text{ satisfies } (27)\big\}\cdot \tag{29}$$

We can now extend the list of properties given in (§1) for functions in B_Ψ. We will use the following result very often.

THEOREM 1.2.7. ([**HE-SO**])

Suppose $f \in B_K$, $1 \le q \le \infty$ *and* $w(t) = t^{1-1/q}/f(t)$. *Then*

$$w \in W_q(S) \cap W_q(S^*).$$

PROOF: Assume first that $1 \le q < \infty$.

To show that $w \in W_q(S)$ we need to check (26) for $u = v = w$.

$$\begin{aligned}
&\sup_{r>0}\left(\int_r^\infty s^{-q}(w(s))^q\, ds\right)^{1/q}\left(\int_0^r (w(s))^{-q'}\, ds\right)^{1/q'}\\
=&\sup_{r>0}\left(\int_r^\infty \left(\frac{1}{f(s)}\right)^q \frac{ds}{s}\right)^{1/q}\left(\int_0^r (f(s))^{q'}\,\frac{ds}{s}\right)^{1/q'} \quad \text{(by (14) and (16))}\\
=&\sup_{r>0} C\frac{1}{f(r)}f(r) = C < \infty\cdot
\end{aligned}$$

Similarly, to show that $w \in W_q(S^*)$ we have to check (27) for $u = v = w$.

$$\begin{aligned}
&\sup_{r>0}\left(\int_0^r (w(s))^q\, ds\right)^{1/q}\left(\int_r^\infty s^{-q'}(w(s))^{-q'}\, ds\right)^{1/q'}\\
=&\sup_{r>0}\left(\int_0^r \left(\frac{s}{f(s)}\right)^q \frac{ds}{s}\right)^{1/q}\left(\int_r^\infty \left(\frac{f(s)}{s}\right)^{q'}\frac{ds}{s}\right)^{1/q'} \quad \text{(by (13) and (15))}\\
=&\sup_{r>0} C\frac{r}{f(r)}\frac{f(r)}{r} = C < \infty\cdot
\end{aligned}$$

For the case $q = \infty$, conditions (26) and (27) read as follows:

$$\sup_{r>0}\left(\sup_{r<s}\frac{v(s)}{s}\right)\left(\int_0^r \frac{1}{u(s)}\, ds\right) < \infty$$

and

$$\sup_{r>0} \, (\sup_{0<s<r} v(s)) \left(\int_r^\infty \frac{1}{su(s)} \, ds \right) < \infty \cdot$$

By Proposition 1.1.6 we may assume that $f \in B_\Psi$.

Now, with $u = v = w(t) = t/f(t)$:

$$\sup_{r>0} \left(\sup_{r<s} \frac{1}{f(s)} \right) \left(\int_0^r f(s) \, \frac{ds}{s} \right)$$

(f is an increasing function and (16))

$$\leq \sup_{r>0} C \frac{1}{f(r)} f(r) = C < \infty \cdot$$

Hence, $w \in W_\infty(S)$.

To show that $w \in W_\infty(S^*)$ we first observe that w is an increasing function. In fact,

$$w'(t) = \frac{1}{f(t)} \left(1 - \frac{tf'(t)}{f(t)} \right) > 0$$

since $\beta_f < 1$.

Hence,

$$\sup_{r>0} \, (\sup_{0<s<r} w(s)) \left(\int_r^\infty \frac{f(s)}{s} \, \frac{ds}{s} \right) \leq \sup_{r>0} Cw(r) \frac{f(r)}{r} = C < \infty,$$

by the observation just made and (15). ∎

As a consequence of this theorem and the characterization of B_Ψ given in Proposition 1.1.7 we can construct, in a very simple way, a large class of weights satisfying Hardy's inequalities.

COROLLARY 1.2.8. *Let $h : \mathbf{R}^+ \longrightarrow [0,1]$ be a continuous function satisfying*

$$\|h\|_\infty < 1 \text{ and } \|1/h\|_\infty < \infty$$

and let $1 \leq q \leq \infty$. Then

$$\left(\int_0^\infty \left(\exp\left(- \int_1^t h(s) \, \frac{ds}{s} \right) \int_0^t f(s) \, ds \right)^q \, \frac{dt}{t} \right)^{1/q}$$

$$\leq C \left(\int_0^\infty \left(t \exp\left(- \int_1^t h(s) \, \frac{ds}{s} \right) f(t) \right)^q \, \frac{dt}{t} \right)^{1/q}$$

and

$$\left(\int_0^\infty \left(t\exp\left(-\int_1^t h(s)\ \frac{ds}{s}\right)\int_t^\infty f(s)\ \frac{ds}{s}\right)^q\ \frac{dt}{t}\right)^{1/q}$$

$$\leq C\left(\int_0^\infty \left(t\exp\left(-\int_1^t\ h(s)\ \frac{ds}{s}\right)f(t)\right)^q\ \frac{dt}{t}\right)^{1/q}$$

for all $f \geq 0$.

Chapter II
Weighted Lorentz spaces

(§1) Definitions and properties.

Lorentz spaces were first studied by G. G. Lorentz ([**LO**]). They are rearrangement invariant spaces and appear, in a natural way, as interpolation spaces between L^1 and L^∞ (see [**ST-WE**]).

To define these spaces, we need to introduce some notation.

DEFINITION 2.1.1. *Suppose (X, Σ, μ) is a σ-finite measurable space and f is a function on X. We define the distribution function of f, with respect to the measure μ as*

$$\lambda_f(t) = \lambda_f(t, d\mu) = \mu(\{x \in X : |f(x)| > t\}), \text{ for all } t > 0\cdot$$

DEFINITION 2.1.2. *We define the non-increasing rearrangement of f,*

$$f^*(t) = \inf_{s>0}\{\lambda_f(s) \leq t\}\cdot$$

For further information about the properties of the functions λ_f and f^*, we refer to [**BE-SH**], Chapter II and [**KR-PE-TE**].

DEFINITION 2.1.3. *Let w be a non-negative function on $\mathbf{R}^+$ and let $0 < q \leq \infty$. We define the weighted Lorentz space:*

$$\Lambda^q(w) = \left\{ f : \|f\|_{\Lambda^q(w)} = \left(\int_0^\infty (w(t) f^*(t))^q \, dt \right)^{1/q} < \infty \right\}\cdot$$

For $q = \infty$, this is understood as:

$$\|f\|_{\Lambda^\infty(w)} = \sup_{0<t<\infty} w(t) f^*(t)\cdot$$

EXAMPLES:

(a) With $w(t) = t^{1/p-1/q}$ we get that $\Lambda^q(w) = L^{p,q}$, the classical (p,q)-Lorentz space.

(b) With $w(t) = t^{1/p-1/q}(1+|\log t|)^\gamma$ then $\Lambda^q(w) = L^{p,q}(\log L)^\gamma$ (see [**ME 1**]).

For now, we only assume that w is nonnegative, but for applications we shall consider later, w will be of the form:

$$w(t) = \frac{t^{1-1/q}}{\varphi(t)}, \qquad \varphi \in B_\Psi\cdot$$

As in the case of $L^{p,q}$ spaces (see [**NE**]), there exists an equivalent "quasi-norm" on $\Lambda^q(w)$ in terms of the distribution function. The next two lemmas give additional insight about functions in B_Ψ and generalize some properties proved in Proposition 1.1.8.

LEMMA 2.1.4. *Let $\varphi \in B_\Psi$, $0 < q < \infty$ and $0 \le a \le b < \infty$. Then*

$$\frac{1}{q}\frac{b^q - a^q}{\varphi^q(b)} \le \int_a^b \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} \le \frac{1}{q(1-\beta)}\frac{b^q - a^q}{\varphi^q(b)}$$

where $\beta = \beta_\varphi$.

PROOF: Set $s = t/b$. Then

$$\begin{aligned}
&\int_a^b \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} = \int_{a/b}^1 \left(\frac{bs}{\varphi(bs)}\right)^q \frac{ds}{s} \\
&= \left(\frac{b}{\varphi(b)}\right)^q \int_{a/b}^1 \left(\frac{s\varphi(b)}{\varphi(bs)}\right)^q \frac{ds}{s} \qquad \left(\frac{\varphi(bs)}{\varphi(b)} \le \overline{\varphi}(s)\right) \\
&\ge \left(\frac{b}{\varphi(b)}\right)^q \int_{a/b}^1 \left(\frac{s}{\overline{\varphi}(s)}\right)^q \frac{ds}{s} \qquad (\overline{\varphi} \text{ increases and } \overline{\varphi}(1) = 1) \\
&\ge \left(\frac{b}{\varphi(b)}\right)^q \int_{a/b}^1 s^{q-1}\, ds = \frac{1}{q}\frac{b^q - a^q}{\varphi^q(b)}\cdot
\end{aligned}$$

On the other hand,

$$\int_a^b \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} = \left(\frac{b}{\varphi(b)}\right)^q \int_{a/b}^1 \left(\frac{s\varphi(b)}{\varphi(bs)}\right)^q \frac{ds}{s}$$

$$\leq \left(\frac{b}{\varphi(b)}\right)^q \int_{a/b}^1 (s\overline{\varphi}(1/s))^q \frac{ds}{s} = \left(\frac{b}{\varphi(b)}\right)^q \int_1^{b/a} \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t}$$

(by Proposition 1.1.9)

$$\leq \left(\frac{b}{\varphi(b)}\right)^q \int_1^{b/a} t^{q(\beta-1)} \frac{dt}{t} = \frac{1}{q(1-\beta)} \left(\frac{b}{\varphi(b)}\right)^q \left(1 - \left(\frac{a}{b}\right)^{q(1-\beta)}\right)$$

$$\leq \frac{1}{q(1-\beta)} \frac{b^q - a^q}{\varphi^q(b)}$$

since

$$\frac{1 - r^{q(1-\beta)}}{1 - r^q} \leq 1, \text{ if } \ 0 < r < 1 \cdot \quad \blacksquare$$

LEMMA 2.1.5. *Let* $\varphi \in B_\Psi$ *and* $0 < y < x$. *Then*

$$\frac{x\varphi(y) - y\varphi(x)}{(x-y)\varphi(y)} \geq 1 - \beta > 0$$

where $\beta = \beta_\varphi$.

PROOF: By Proposition 1.1.7, we have that

$$\frac{\varphi(x)}{\varphi(y)} = \frac{\exp\left(\int_1^x \ h(s) \ \frac{ds}{s}\right)}{\exp\left(\int_1^y \ h(s) \ \frac{ds}{s}\right)} \qquad (\alpha_\varphi \leq h(t) \leq \beta_\varphi)$$

$$= \exp\left(\int_y^x h(s) \ \frac{ds}{s}\right) \leq \exp(\beta \log \frac{x}{y}) = \left(\frac{x}{y}\right)^\beta \cdot$$

Hence,

$$\frac{x\varphi(y) - y\varphi(x)}{(x-y)\varphi(y)} = \frac{x}{x-y} - \frac{y}{x-y}\frac{\varphi(x)}{\varphi(y)}$$

$$\geq \frac{x}{x-y} - \frac{y}{x-y}\left(\frac{x}{y}\right)^\beta = \frac{1}{1-(y/x)} - \frac{1}{(x/y)-1}\left(\frac{x}{y}\right)^\beta.$$

Set $r = y/x$. Then, we want to show that

$$\frac{1}{1-r} - \frac{1}{(1/r)-1}\left(\frac{1}{r}\right)^\beta \geq 1-\beta, \ 0 < r < 1 \cdot$$

Now,

$$\frac{1}{1-r} - \frac{r^{1-\beta}}{1-r} = \frac{1-r^{1-\beta}}{1-r} \geq 1-\beta \iff \beta \geq r^{1-\beta} + (\beta-1)r \cdot$$

But if we set $f(r) = r^{1-\beta} + (\beta-1)r$ then $f'(r) = (1-\beta)r^{-\beta} - 1 + \beta$ which is positive on the interval $(0,1)$ and $f(0) = 0, \ f(1) = \beta$. $\blacksquare$

THEOREM 2.1.6. *Let $\varphi \in B_\Psi$, $0 < q < \infty$ and $w(t) = t^{1-1/q}/\varphi(t)$. Suppose $f \in \Lambda^q(w)$, then*

$$\|f\|_{\Lambda^q(w)} \approx \left(\int_0^\infty (tw(\lambda_f(t)))^q \,\lambda_f(t)\, \frac{dt}{t}\right)^{1/q}.$$

REMARKS:

(i) Since $\lambda_f(t) = 0$ if and only if $t > \|f\|_\infty$ then the right hand side above equals

$$\left(\int_0^{\|f\|_\infty} \left(tw(\lambda_f(t))\right)^q \lambda_f(t)\, \frac{dt}{t}\right)^{1/q}$$

so that w is never evaluated at zero.

(ii) If $\Lambda^q(w) = L^{p,q}$ $(w(t) = t^{1/p-1/q})$ then we get

$$\|f\|_{L^{p,q}} \approx \left(\int_0^\infty \left(t\lambda_f^{1/p-1/q}(t)\right)^q \lambda_f(t)\, \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty \left(t\lambda_f^{1/p}(t)\right)^q \frac{dt}{t}\right)^{1/q}.$$

PROOF OF THE THEOREM: By a density argument, it suffices to show the equivalence of the quasi-norms for simple functions; i.e., we may assume that

$$f(x) = \sum_{j=1}^n a_j \chi_{E_j}(x)$$

where the sets E_j are pairwise disjoint with finite measure. Since we have that $\|f\|_{\Lambda^q(w)} = \|\,|f|\,\|_{\Lambda^q(w)}$ we may also assume that $a_1 > a_2 > ... > a_n > 0$.

It is easy to show (see **[BE-SH]**) that

$$\lambda_f(t) = \sum_{j=1}^n m_j \chi_{[a_{j+1}, a_j)}(t) \quad \text{and} \quad f^*(t) = \sum_{j=1}^n a_j \chi_{[m_{j-1}, m_j)}(t)$$

where $m_j = \displaystyle\sum_{i=1}^j |E_i|$, $m_0 = 0$ and $a_{n+1} = 0$.

Hence,

$$\|f\|^q_{\Lambda^q(w)} = \int_0^\infty (w(t)f^*(t))^q\, dt = \int_0^\infty \left(w(t)\sum_{j=1}^n a_j\chi_{[m_{j-1},m_j)}(t)\right)^q dt$$

$$= \sum_{j=1}^n \left(\int_{m_{j-1}}^{m_j} w^q(t)\, dt\right) a_j^q = \sum_{j=1}^n \left(\int_{m_{j-1}}^{m_j} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t}\right) a_j^q \quad \text{(by Lemma 2.1.4)}$$

$$\approx \sum_{j=1}^n \frac{m_j^q - m_{j-1}^q}{\varphi^q(m_j)} a_j^q.$$

On the other hand,

$$
\begin{aligned}
&\int_0^\infty \big(tw(\lambda_f(t))\big)^q \lambda_f(t)\,\frac{dt}{t}\\
&=\int_0^{a_1}\left(tw\left(\sum_{j=1}^n m_j\chi_{[a_{j+1},a_j)}(t)\right)\right)^q \sum_{j=1}^n m_j\chi_{[a_{j+1},a_j)}(t)\,\frac{dt}{t}\\
&=\sum_{j=1}^n\int_{a_{j+1}}^{a_j}(tw(m_j))^q m_j\,\frac{dt}{t}=\sum_{j=1}^n w^q(m_j)m_j\int_{a_{j+1}}^{a_j}t^{q-1}\,dt\\
&\approx\sum_{j=1}^n\frac{a_j^q-a_{j+1}^q}{\varphi^q(m_j)}m_j^q.
\end{aligned}
$$

Therefore, we need to show

$$\sum_{j=1}^n\frac{m_j^q-m_{j-1}^q}{\varphi^q(m_j)}a_j^q\approx\sum_{j=1}^n\frac{a_j^q-a_{j+1}^q}{\varphi^q(m_j)}m_j^q.$$

Set $x_j=m_j^q$, $y_j=a_j^q$ and $\Psi(t)=\varphi^q(t^{1/q})$. Then $\Psi\in B_\Psi$ (in fact $\alpha_\Psi=\alpha_\varphi$ and $\beta_\Psi=\beta_\varphi$), and the above expression gives

$$\sum_{j=1}^n\frac{x_j-x_{j-1}}{\Psi(x_j)}y_j\approx\sum_{j=1}^n\frac{y_j-y_{j+1}}{\Psi(x_j)}x_j.$$

Summing by parts we obtain

$$
\begin{aligned}
\sum_{j=1}^n\frac{y_j-y_{j+1}}{\Psi(x_j)}x_j&=\frac{x_n}{\Psi(x_n)}y_{n+1}-\sum_{j=1}^n\left(\frac{x_{j-1}}{\Psi(x_{j-1})}-\frac{x_j}{\Psi(x_j)}\right)y_j\\
&=\sum_{j=1}^n\frac{y_j}{\Psi(x_j)}\left(x_j-\frac{\Psi(x_j)}{\Psi(x_{j-1})}x_{j-1}\right).
\end{aligned}
$$

We still have to show that

$$\sum_{j=1}^n\frac{y_j}{\Psi(x_j)}(x_j-x_{j-1})\approx\sum_{j=1}^n\frac{y_j}{\Psi(x_j)}\left(x_j-\frac{\Psi(x_j)}{\Psi(x_{j-1})}x_{j-1}\right).$$

Since $x_{j-1}<x_j$ and Ψ increases, $\dfrac{\Psi(x_j)}{\Psi(x_{j-1})}>1$ and so

$$\sum_{j=1}^n\frac{y_j}{\Psi(x_j)}(x_j-x_{j-1})\geq\sum_{j=1}^n\frac{y_j}{\Psi(x_j)}\left(x_j-\frac{\Psi(x_j)}{\Psi(x_{j-1})}x_{j-1}\right).$$

Using Lemma 2.1.5 we have

$$\frac{x_j - \frac{\Psi(x_j)}{\Psi(x_{j-1})}x_{j-1}}{x_j - x_{j-1}} = \frac{x_j\Psi(x_{j-1}) - x_{j-1}\Psi(x_j)}{\Psi(x_{j-1})(x_j - x_{j-1})} \geq 1 - \beta\cdot$$

Therefore,

$$\sum_{j=1}^{n} \frac{y_j}{\Psi(x_j)}(x_j - x_{j-1}) \leq \frac{1}{1-\beta}\sum_{j=1}^{n} \frac{y_j}{\Psi(x_j)}\left(x_j - \frac{\Psi(x_j)}{\Psi(x_{j-1})}x_{j-1}\right)\cdot \quad \blacksquare$$

The following deep result, due to E. Sawyer, gives us some very useful information about $\Lambda^q(w)$ and its dual space.

THEOREM 2.1.7. **[SAW]**

(i) Suppose $1 < q < \infty$ and w is a non-negative weight. The following are equivalent statements:

(a) $\Lambda^q(w)$ *is a Banach space.*

(b) $\Lambda^q(w) = \Gamma^q(w) = \left\{ f : \|f\|_{\Gamma^q(w)} = \left(\int_0^\infty (f^{**}(t)w(t))^q\, dt\right)^{1/q} < \infty \right\}$ *with* $\|f\|_{\Gamma^q(w)} \approx \|f\|_{\Lambda^q(w)}$ *where* $f^{**}(t) = S(f^*)(t) = \dfrac{1}{t}\displaystyle\int_0^t f^*(s)\, ds$. *In this case,* $\|\cdot\|_{\Gamma^q(w)}$ *is a norm in* $\Lambda^q(w)$.

(c) $\left(\int_0^r w^q(t)\,\frac{dt}{t}\right)^{1/q}\left(\int_0^r \left(\frac{1}{t}\int_0^t w^q(s)\,ds\right)^{1-q'}\frac{dt}{t}\right)^{1/q'} \leq Cr$, *for all* $r > 0$.

(d) $\left(\int_r^\infty t^{-q}w^q(t)\,dt\right)^{1/q} \leq \frac{C}{r}\left(\int_0^r w^q(t)\,dt\right)^{1/q}$, *for all* $r > 0$.

(ii) If $\int_0^\infty w^q(t)\,dt = \infty$ *then the dual of* $\Lambda^q(w)$ *can be identified with* $\Gamma^{q'}(\tilde{w})$, *where* $\tilde{w}(t) = \left(\frac{1}{t}\int_0^t w^q(s)\,ds\right)^{-1}(w(t))^{q/q'}$, *under the pairing* $\langle f, g\rangle = \int_{\mathbf{R}^n} f(y)g(y)\,dy$, $f \in \Lambda^q(w)$ *and* $g \in \Gamma^{q'}(\tilde{w})$.

(iii)

(a) *For* $1 < p \leq q < \infty$, $\Lambda^p(v) \subset \Lambda^q(w)$, *if and only if*

$$(1) \qquad \left(\int_0^r w^q(t)\,dt\right)^{1/q} \leq C\left(\int_0^r v^p(t)\,dt\right)^{1/p} \qquad \textit{for all } r > 0.$$

(b) *For $1 < q < p < \infty$, $\Lambda^p(v) \subset \Lambda^q(w)$, if and only if*

$$\left(\int_0^\infty \left(\left(\int_0^t w^q(s)\,ds\right)^{1/p}\left(\int_0^t v^p(s)\,ds\right)^{-1/p}\right)^r w^q(t)\,dt\right)^{1/r} < \infty \tag{2}$$

where $1/r = 1/q - 1/p$.

We now examine what this theorem says in the case where the weights are constructed from functions in B_Ψ. From the three parts of the theorem, we get the following three corollaries:

COROLLARY 2.1.8. *Suppose $\varphi \in B_\Psi$ and $1 < q < \infty$. Let $w(t) = t^{1-1/q}/\varphi(t)$. Then $\Lambda^q(w)$ is a Banach space and*

$$\|f\|_{\Lambda^q(w)} \approx \left(\int_0^\infty (f^{**}(t)w(t))^q\,dt\right)^{1/q}. \tag{3}$$

PROOF: Using (i)-(b) of Theorem 2.1.7, it suffices to show (3). Since

$$f^*(t) \le \frac{1}{t}\int_0^t f^*(s)\,ds = f^{**}(t)$$

it is clear that

$$\|f\|_{\Lambda^q(w)} \le \left(\int_0^\infty (f^{**}(t)w(t))^q\,dt\right)^{1/q}.$$

On the other hand, by Theorem 1.2.7, we know that $w \in W_q(S)$; i.e.,

$$\left(\int_0^\infty (w(t)S(g)(t))^q\,dt\right)^{1/q} \le C\left(\int_0^\infty (w(t)g(t))^q\,dt\right)^{1/q}.$$

Hence, with $g \equiv f^*$,

$$\left(\int_0^\infty (w(t)f^{**}(t))^q\,dt\right)^{1/q} \le C\|f\|_{\Lambda^q(w)}. \quad \blacksquare$$

COROLLARY 2.1.9. *Suppose $\varphi \in B_\Psi$, $1 < q < \infty$ and $w(t) = t^{1-1/q}/\varphi(t)$. Then the dual of $\Lambda^q(w)$ is $\Lambda^{q'}(1/w)$. Moreover, $1/w$ is also of the form $t^{1-1/q'}/\psi(t)$, with $\psi \in B_\Psi$.*

PROOF: By (ii) of Theorem 2.1.7, we need to show that $\int_0^\infty w^q(t)\,dt = \infty$. But by Proposition 1.1.8 (13),

$$\int_0^t w^q(s)\,ds = \int_0^t \left(\frac{s}{\varphi(s)}\right)^q \frac{ds}{s} \approx \left(\frac{t}{\varphi(t)}\right)^q$$

and by Proposition 1.1.9, $\varphi(t) \le t^\beta$, for $t \ge 1$, where $\beta = \beta_\varphi < 1$. Hence

$$\left(\frac{t}{\varphi(t)}\right)^q \ge t^{q(1-\beta)}, \qquad t \ge 1$$

and therefore

$$\int_0^\infty w^q(s)\, ds = \lim_{t\to\infty} \int_0^t w^q(s)\, ds \ge C \lim_{t\to\infty} t^{q(1-\beta)} = \infty.$$

We still need to show that $\tilde{w}(t) \approx \dfrac{1}{w(t)}\cdot$ But

$$\begin{aligned}
\tilde{w}(t) &= \left(\frac{1}{t}\int_0^t w^q(s)\, ds\right)^{-1} (w(t))^{q/q'} \\
&= \left(\frac{1}{t}\int_0^t \left(\frac{s}{\varphi(s)}\right)^q \frac{ds}{s}\right)^{-1} (w(t))^{q/q'} \quad \text{(by Proposition 1.1.8- (13))} \\
&\approx \left(\frac{1}{t}\left(\frac{t}{\varphi(t)}\right)^q\right)^{-1} (w(t))^{q/q'} = \frac{\varphi^q(t)}{t^{q-1}}(w(t))^{q/q'} = (w(t))^{-q}(w(t))^{q-1} = \frac{1}{w(t)}\cdot
\end{aligned}$$

Now, $1/w(t) = t^{1-1/q'}/\psi(t)$, where $\psi(t) = t/\varphi(t)$ and by Proposition 1.1.12-(20), we have that $\psi \in B_\Psi$.

Thus, by the last corollary,

$$(\Lambda^q(w))^* = \Gamma^{q'}(1/w) = \Lambda^{q'}(1/w)\cdot \qquad \blacksquare$$

COROLLARY 2.1.10. *Suppose $1 < p, q < \infty$, φ, $\psi \in B_\Psi$ and consider the weights $w(t) = t^{1-1/q}/\varphi(t)$, $v(t) = t^{1-1/p}/\psi(t)$.*

(a) *For $1 < p \le q < \infty$, $\Lambda^p(v) \subset \Lambda^q(w)$ if and only if $\psi(t) \le C\varphi(t)$, $t > 0$.*

(b) *For $1 < q < p < \infty$, $\Lambda^p(v) \subset \Lambda^q(w)$ if and only if*

$$\int_0^\infty \left(\frac{\psi(t)}{\varphi(t)}\right)^r \frac{dt}{t} < \infty,$$

where $1/r = 1/q - 1/p$.

PROOF: To prove (a) we use (iii)-(a) of the theorem:

$$\begin{aligned}
\Lambda^p(v) \subset \Lambda^q(w) &\iff \left(\int_0^r \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t}\right)^{1/q} \le C\left(\int_0^r \left(\frac{t}{\psi(t)}\right)^p \frac{dt}{t}\right)^{1/p} \\
&\iff \frac{r}{\varphi(r)} \le C\frac{r}{\psi(r)},\ r > 0 \quad \text{by Proposition 1.1.8-(13))} \\
&\iff \psi(r) \le C\varphi(r), \quad \text{for all } r > 0\cdot
\end{aligned}$$

Similarly, to prove (b) we use (iii)-(b):

$$\Lambda^p(v) \subset \Lambda^q(w)$$
$$\iff \int_0^\infty \left(\left(\left(\int_0^t \left(\frac{s}{\varphi(s)} \right)^q \frac{ds}{s} \right)^{\frac{1}{p}} \left(\int_0^t \left(\frac{s}{\psi(s)} \right)^p \frac{ds}{s} \right)^{\frac{-1}{p}} \right)^r \left(\frac{t}{\varphi(t)} \right)^q \right) \frac{dt}{t} < \infty$$
$$\iff \int_0^\infty \left(\left(\left(\frac{t}{\varphi(t)} \right)^{q/p} \left(\frac{t}{\psi(t)} \right)^{-1} \right)^r \left(\frac{t}{\varphi(t)} \right)^q \right) \frac{dt}{t} < \infty$$
$$\iff \int_0^\infty \left(\left(\frac{t}{\varphi(t)} \right)^{\frac{qr}{p}+q} \left(\frac{\psi(t)}{t} \right)^r \right) \frac{dt}{t} < \infty \quad (\frac{qr}{p} + q = r)$$
$$\iff \int_0^\infty \left(\frac{\psi(t)}{\varphi(t)} \right)^r \frac{dt}{t} < \infty \cdot \qquad \blacksquare$$

REMARK 2.1.11:

(i) Since $L^{p,q} = \Lambda^q(w)$, with $w(t) = t^{1-1/q}/\varphi_p(t)$ and $\varphi_p(t) = t^{1-1/p} \in B_\Psi$ and $\varphi_{p_0}(t) \leq C\varphi_{p_1}(t) \iff p_0 = p_1$ and

$$\int_0^\infty \left(\frac{\varphi_{p_0}(t)}{\varphi_{p_1}(t)} \right)^r \frac{dt}{t} = \infty, \quad \text{for all choices of } \ 1 < p_0, p_1 < \infty$$

then we conclude that

$$L^{p_0,q_0} \subset L^{p_1,q_1} \iff p_0 = p_1 \text{ and } q_0 \leq q_1 \cdot$$

(ii) By Proposition 1.1.9, it is very easy to see that a necessary condition for part (b) of the last corollary to hold is that $\alpha_\varphi < \beta_\psi$ and $\alpha_\psi < \beta_\varphi$.

A priori, it is not clear that we can find two functions φ, $\psi \in B_\Psi$ satisfying condition (b). The following proposition shows that even more is true.

PROPOSITION 2.1.12. *Suppose* $1 < q < p < \infty$ *and* $\varphi \in B_\Psi$. *Then we can find* $\psi \in B_\Psi$ *such that if* $w(t) = t^{1-1/q}/\varphi(t)$ *and* $v(t) = t^{1-1/p}/\psi(t)$ *then* $\Lambda^p(v) \subset \Lambda^q(w)$.

PROOF: We know that it suffices to find $\psi \in B_\Psi$ such that

$$\int_0^\infty \left(\frac{\psi(t)}{\varphi(t)} \right)^r \frac{dt}{t} < \infty, \qquad 1/r = 1/q - 1/p$$

By Proposition 1.1.12-(17) we can assume that $r = 1$. Also, by Proposition 1.1.6 we only need to find $g \in B_K$ satisfying the above condition.
By Proposition 1.1.8-(11), there exists $0 < \varepsilon < 1/2$ so that

$$\overline{\varphi}(t) \leq \begin{cases} Ct^{\varepsilon}, & \text{if } 0 < t \leq 1 \\ Ct^{1-\varepsilon}, & \text{if } 1 \leq t < \infty \cdot \end{cases}$$

Pick $0 < \delta < \min(\alpha_\varphi, \varepsilon)$ and set

$$g(t) = \begin{cases} t^{\delta}\varphi(t) & 0 < t \leq 1 \\ t^{-\delta}\varphi(t) & 1 < t < \infty \ , \end{cases}$$

g is a continuous function, $g(1) = 1$ and g is non-decreasing. In fact, if $0 < t \leq 1$, then g is the product of two increasing functions, and if $1 < t < \infty$ then we have that $g'(t) = t^{-\delta}\varphi'(t) - \delta t^{-\delta-1}\varphi(t)$. Hence $g'(t) > 0 \Longleftrightarrow \dfrac{t\varphi'(t)}{\varphi(t)} > \delta$. But $\dfrac{t\varphi'(t)}{\varphi(t)} \geq \alpha_\varphi > \delta$.

A simple calculation shows that

$$\overline{g}(t) \leq \begin{cases} t^{-\delta}\overline{\varphi}(t) & 0 < t \leq 1 \\ t^{\delta}\overline{\varphi}(t) & 1 \leq t < \infty \cdot \end{cases}$$

Therefore,

$$\begin{aligned} &\int_0^\infty \min(1, \frac{1}{t}) \ \overline{g}(t) \, \frac{dt}{t} = \int_0^1 \overline{g}(t) \, \frac{dt}{t} + \int_1^\infty t^{-1} \ \overline{g}(t) \, \frac{dt}{t} \\ \leq &\int_0^1 t^{-\delta-1} \ \overline{\varphi}(t) \, dt + \int_1^\infty t^{-2+\delta} \ \overline{\varphi}(t) \, dt \\ \leq &C\left(\int_0^1 t^{\varepsilon-\delta-1} \, dt + \int_1^\infty t^{-2+\delta+1-\varepsilon} \, dt\right) < \infty, \quad \text{since } \ \delta < \varepsilon \cdot \ \blacksquare \end{aligned}$$

(§2) Atomic decomposition and discrete characterization of $\Lambda^q(w)$.

We now introduce a family of sequence spaces that allows us to give a discrete version of the $\Lambda^q(w)$ spaces.

DEFINITION 2.2.1. *Let $w = \{w_k\}_k$ be a sequence of non-negative numbers and let $0 < q \le \infty$. We define the weighted Lorentz space of sequences,*

$$\lambda^q(w) = \left\{ \{a_k\}_k : \|\{a_k\}_k\|_{\lambda^q(w)} = \left(\sum_{k=-\infty}^{\infty} 2^k (w_k a_k^*)^q \right)^{1/q} < \infty \right\}$$

where $\{a_k^*\}_k$ is the non-increasing rearrangement of $\{a_k\}_k$, as in Definition 2.1.2.

PROPOSITION 2.2.2. *Let w be a non-negative weight and let $0 < q < \infty$.*

(i) *Assume there exists $C > 0$ such that*

$$\int_{2^k}^{2^{k+1}} w^q(t)\,dt \le C 2^k w^q(2^k), \qquad \text{all } \ k \in \mathbf{Z}. \tag{4}$$

Then,

$$\|f\|_{\Lambda^q(w)} \le C \|\{f^*(2^k)\}_k\|_{\lambda^q(w)}.$$

(ii) *Assume there exists $C > 0$ such that*

$$\int_{2^k}^{2^{k+1}} w^q(t)\,dt \ge C 2^k w^q(2^{k+1}), \qquad \text{all } \ k \in \mathbf{Z}. \tag{5}$$

Then,

$$\|f\|_{\Lambda^q(w)} \ge C \|\{f^*(2^k)\}_k\|_{\lambda^q(w)}.$$

(iii) *If $w(t) = \dfrac{t^{1-1/q}}{\varphi(t)}$, with $\varphi \in B_\Psi$, then*

$$\|f\|_{\Lambda^q(w)} \approx \|\{f^*(2^k)\}_k\|_{\lambda^q(w)}.$$

PROOF: To prove (i) and (ii) we use the fact that f^* is a non-decreasing function:

$$\begin{aligned} \|f\|^q_{\Lambda^q(w)} &= \int_0^\infty (w(t) f^*(t))^q\,dt \\ &= \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} (w(t) f^*(t))^q\,dt \le C \sum_{k=-\infty}^{\infty} 2^k (w(2^k) f^*(2^k))^q \end{aligned}$$

and similarly,

$$\|f\|^q_{\Lambda^q(w)} \ge C \sum_{k=-\infty}^{\infty} 2^k (w(2^{k+1}) f^*(2^{k+1}))^q.$$

To prove (iii) we want to show that there exists $C > 1$ such that

$$\frac{1}{C}\left(\frac{2^{k+1}}{\varphi(2^{k+1})}\right)^q \le \int_{2^k}^{2^{k+1}} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} \le C\left(\frac{2^k}{\varphi(2^k)}\right)^q, \text{ all } k \in \mathbf{Z}\cdot$$

But

$$\begin{aligned}\int_{2^k}^{2^{k+1}} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} &\le \int_0^{2^{k+1}} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} \\ &\le C\left(\frac{2^{k+1}}{\varphi(2^{k+1})}\right)^q \le C2^q\left(\overline{\varphi}(\frac{1}{2})\right)^q\left(\frac{2^k}{\varphi(2^k)}\right)^q.\end{aligned}$$

For the other estimate, we must carefully keep track of the constants in Proposition 1.1.8-(13):

$$\begin{aligned}&\int_{2^k}^{2^{k+1}} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} = \int_0^{2^{k+1}} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} - \int_0^{2^k} \left(\frac{t}{\varphi(t)}\right)^q \frac{dt}{t} \\ &\ge \left(\int_1^\infty \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t}\right)\left(\frac{2^{k+1}}{\varphi(2^{k+1})}\right)^q - \frac{1}{q}\left(\frac{2^k}{\varphi(2^k)}\right)^q \\ &\ge \left(\int_1^\infty \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t}\right)\left(\frac{2^{k+1}}{\varphi(2^{k+1})}\right)^q - \frac{1}{q}2^{-q}(\overline{\varphi}(2))^q\left(\frac{2^{k+1}}{\varphi(2^{k+1})}\right)^q \\ &= \left(\left(\int_1^\infty \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t}\right) - \frac{1}{q}2^{-q}(\overline{\varphi}(2))^q\right)\left(\frac{2^{k+1}}{\varphi(2^{k+1})}\right)^q.\end{aligned}$$

Thus, we only need to show that

$$\int_1^\infty \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t} > \frac{1}{q}2^{-q}(\overline{\varphi}(2))^q\cdot$$

But, by Proposition 1.1.8-(8):

$$\int_1^\infty \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t} \ge \int_2^\infty \left(\frac{\overline{\varphi}(t)}{t}\right)^q \frac{dt}{t} > (\overline{\varphi}(2))^q \int_2^\infty t^{-q-1}\,dt = \frac{1}{q}2^{-q}(\overline{\varphi}(2))^q\cdot \quad \blacksquare$$

Before proving the atomic decomposition for $\Lambda^q(w)$ we need the following observation:

LEMMA 2.2.3. *Let $0 < q < \infty$. Then*

(6) $$\|f\|_{\Lambda^q(w)} = \|f^q\|_{\Lambda^1(w^q)}^{1/q}$$

and

(7) $$\|\{\alpha_k\}_k\|_{\lambda^q(w)} = \|\{\alpha_k^q\}_k\|_{\lambda^1(w^q)}^{1/q}\cdot$$

PROOF: This follows easily from the fact that $\lambda_{f^q}(t) = \lambda_f(t^{1/q})$ and, as a consequence $(f^q)^*(t) = (f^*)^q(t)\cdot$ $\blacksquare$

THEOREM 2.2.4. *Suppose* $0 < q < \infty$ *and* w *is a non-negative weight satisfying* (4), (5) *and*

$$\int_0^{2t} w^q(s)\,ds \approx t w^q(t), \qquad \textit{for all } t > 0\cdot \tag{8}$$

Then, the following are equivalent statements:

(i) $f \in \Lambda^q(w)$.

(ii) *There exist* $\{\alpha_k\}_k \in \lambda^q(w)$ *and* $\{g_k\}_k \subset L^\infty$ *such that*

(a) $\|g_k\|_\infty \leq 1$.

(b) $|\text{supp } g_k| \leq 2^{k+1}$.

(c) $|\text{supp } g_k \cap \text{supp } g_j| = 0$, *if* $j \neq k$.

(d) $f \equiv \sum\limits_k \alpha_k g_k$.

In this case, $\|f\|_{\Lambda^q(w)} \approx \inf \|\{\alpha_k\}_k\|_{\lambda^q(w)}$, *where the infimum is taken over all sequences satisfying (d).*

PROOF: Suppose $f \in \Lambda^q(w)$. Define

$$g_k(x) = \begin{cases} \dfrac{f(x)}{f^*(2^k)} & f^*(2^{k+1}) \leq |f(x)| < f^*(2^k) \\ 0 & \text{otherwise} \end{cases}$$

and $\alpha_k = f^*(2^k)$. Then (a) is obvious,

$$|\text{supp } g_k| \leq |\{x : |f(x)| \geq f^*(2^{k+1})\}| = \lambda_f(f^*(2^{k+1})) \leq 2^{k+1}$$

and (c) and (d) are immediate. By Proposition 2.2.2, we get that $\{\alpha_k\}_k \in \lambda^q(w)$.

Now suppose that (ii) holds. We can assume that $\{\alpha_k\}_k = \{\alpha_k^*\}_k$. Suppose first that $q = 1$. To estimate $\|f\|_{\Lambda^1(w)}$ we calculate $\|g_k\|_{\Lambda^1(w)}$ for each g_k:

$$\begin{aligned} \|g_k\|_{\Lambda^1(w)} &= \int_0^\infty w(t) g_k^*(t)\,dt = \int_0^{|\text{supp } g_k|} w(t) g_k^*(t)\,dt \\ &\leq \|g_k^*\|\infty \int_0^{2^{k+1}} w(t)\,dt \leq C 2^k w(2^k), \text{ (by (8) and (a))}\cdot \end{aligned}$$

Thus,

$$\|f\|_{\Lambda^1(w)} \leq C \sum_k |\alpha_k| \, \|g_k\|_{\Lambda^1(w)} \leq C \sum_k \alpha_k^* 2^k w(2^k) = C \|\{\alpha_k\}_k\|_{\lambda^1(w)}\cdot$$

Hence, $\|f\|_{\Lambda^1(w)} \leq C \inf \|\{\alpha_k\}_k\|_{\lambda^1(w)}$. Proposition 2.2.2 gives the desired equivalence.

Now suppose that $q > 1$, and assume that (ii) holds; i.e.,

$$f \equiv \sum_k \alpha_k g_k, \qquad \{\alpha_k\}_k \in \lambda^q(w),\ \|g_k\|_\infty \leq 1,\ |\text{supp } g_k| \leq 2^{k+1}$$

and $|\text{supp } g_j \cap \text{supp } g_k| = 0,\ j \neq k$.

Set $v \equiv w^q,\ h \equiv f^q,\ \beta_k = \alpha_k^q$ and $u_k \equiv g_k^q$. Then,

$$h \equiv \sum_k \beta_k u_k, \qquad \{\beta_k\}_k \in \lambda^1(v),\ \{u_k\}_k \subset \lambda^\infty,\ \|u_k\|_\infty \leq 1,\ |\text{supp } u_k| \leq 2^{k+1}$$

and $|\text{supp } u_j \cap \text{supp } u_k| = 0,\ j \neq k$.

Moreover, v satisfies (4), (5) and (8), with $q = 1$. Hence, by the first part we conclude that $h \in \Lambda^1(v)$ and $\|h\|_{\Lambda^1(v)} \approx \inf \|\{\beta_k\}_k\|_{\lambda^1(v)}$. Therefore, by Lemma 2.2.3:

$$\|f\|_{\Lambda^q(w)} = \|h\|_{\Lambda^1(v)}^{1/q} \approx \inf \|\{\beta_k\}_k\|_{\lambda^1(v)}^{1/q} \approx \inf \|\{\alpha_k\}_k\|_{\lambda^q(v)} \cdot \quad \blacksquare$$

COROLLARY 2.2.5. *Suppose* $0 < q < \infty$ *and* $w(t) = t^{1-1/q}/\varphi(t)$ *where* $\varphi \in B_\Psi$. *Then* w *satisfies conditions* (4), (5) *and* (8) *and hence the above atomic decomposition holds true for* $\Lambda^q(w)$.

PROOF: (4) and (5) are proved in Proposition 2.2.2-(iii) and (8) is a consequence of Proposition 1.1.8-(13) and the fact that $\varphi(t) \approx \varphi(2t)$. $\blacksquare$

REMARK 2.2.6: For the case of $L^{p,q}$-spaces (ii) of Theorem 2.2.4 can be found in [**COL**]. A similar result, for $L^{p,1}$ is also proved in [**BO-JO**].

(§3) Interpolation results. Main theorem.

Interpolation Theory has developed considerably since the classical results of Riesz-Thorin and Marcinkiewicz were announced (see [**BE-LO**] Chapter 1). The proof of the former contains the idea behind the complex interpolation method of Calderón (see [**CA**]) and from the latter one can see the main ingredients of the real interpolation method of Lions and Peetre (see [**LI-PE**] and [**PE 1**]). Both methods of interpolation have been studied and generalized to more general settings. In particular, we want to mention the work of Coifman, Cwikel, Rochberg, Sagher and Weiss for a theory of complex interpolation for families of Banach spaces, indexed by the boundary of the unit disc on $\mathbf{C}$ (see [**CCRSW 1-2**]). With regard to real interpolation, we have already mentioned the method of interpolation with a function parameter (see [**KA**] and [**GU**]).

The results in this section deal with both methods simultaneously. We find a reiteration theorem that relates the interpolation of a couple of Banach spaces with a family of function parameters, which can then be interpolated using the method of CCRSW. As a consequence of this theorem, we can identify the interpolation spaces for $\Lambda^q(w)$. We will see another application of the reiteration theorem for Tent spaces in the next chapter.

Before stating and proving our results, we give a brief description of the complex and real interpolation methods.

The complex method of interpolation.

We describe the complex method of interpolation for families of Banach spaces as given in [**CCRSW 1**]. Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. To simplify notation we shall write $\theta \in \mathbf{T}$ instead of $e^{i\theta} \in \mathbf{T}$. Let $\{B(\theta)\}_{\theta\in\mathbf{T}}$ be a family of Banach spaces. We say that this family is an interpolation family of Banach spaces (or interpolation family, for short) if each $B(\theta)$ is continuously embedded in a Banach space $(U, \|\cdot\|_U)$, the function $\theta \longrightarrow \|b\|_{B(\theta)}$ is measurable for each $b \in \cap_{\theta\in\mathbf{T}} B(\theta)$, and if we define

$$\mathcal{B} = \left\{ b \in \cap_{\theta\in\mathbf{T}} B(\theta) : \int_0^{2\pi} \log^+ \|b\|_{B(\theta)}\, d\theta < \infty \right\}$$

then $\|b\|_U \leq k(\theta)\|b\|_{B(\theta)}$ for all $b \in \mathcal{B}$, with $\log^+ k(\theta) \in L^1(\mathbf{T})$. The space $\mathcal{B}$ is called the log-intersection space of the given family and U is called the containing space.

Let us denote by $N^+(\mathcal{B})$ the space of all $\mathcal{B}$-valued analytic functions of the form $g(z) = \sum_{j=1}^{m} \mathcal{X}_j(z)b_j$ for which $\|g\|_\infty = \sup_\theta \|g(\theta)\|_{B(\theta)} < \infty$, where $\mathcal{X}_j \in N^+$ and $b_j \in \mathcal{B},\ j = 1, ..., m$. ($N^+$ denotes the positive Nevalinna class for the unit disc $D = \{z \in \mathbf{C} : |z| \leq 1\}$.) The completion of the space $N^+(\mathcal{B})$ with respect to $\|\cdot\|_\infty$ is denoted by $\mathcal{F}(\mathcal{B})$. For $z \in D$, the space $[B(\theta)]_z$ will consist of all elements of the form $f(z)$ for $f \in \mathcal{F}(\mathcal{B})$. A Banach space norm is defined on $[B(\theta)]_z$ by

$$\|v\|_z = \inf\big\{\|f\|_\infty : f \in \mathcal{F}(\mathcal{B}),\ f(z) = v\big\}, \quad v \in [B(\theta)]_z,\ z \in D\cdot$$

These spaces are also called in the literature the St. Louis spaces. It can be proved that $([B(\theta)]_z, \|\cdot\|_z)$ is a Banach space and $\mathcal{B}$ is dense in each $[B(\theta)]_z$. Other properties of these spaces, such as the interpolation property and reiteration are discussed in [**CCRSW 1-2**]. The only one we shall need in order to prove our main theorem is the following subharmonicity property, which is contained in Proposition (2.4) of [**CCRSW 1**]:

PROPOSITION 2.3.1. *For each $f \in \mathcal{F}(\mathcal{B})$ and each $z \in D$,*

$$\|f(z)\|_z \leq \exp \int_{\mathbf{T}} \big(\log \|f(\theta)\|_{B(\theta)}\big)\, P_z(\theta)\, d\theta$$

where $P_z(\theta) = \Re\left(\frac{1}{2\pi}\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right)$ *is the Poisson Kernel for D.*

For further references see also [**HE**].

The real method of interpolation with a function parameter.

Let $A_0,\ A_1$ be two Banach spaces. We say that A_0 and A_1 are compatible if there is a Hausdorff topological vector space $\mathcal{U}$ such that A_0 and A_1 are subspaces of $\mathcal{U}$. We then can form their sum and their intersection. We set $\overline{A} = (A_0, A_1)$ and $\Sigma(\overline{A}) = A_0 + A_1$ and define the Peetre K-functional by

$$K(t,a) = K(t,a;\overline{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \qquad \text{all} \ \ t > 0, \ a \in \Sigma(\overline{A})\cdot$$

It is easy to see that for each t, $K(t,a)$ is an equivalent norm on $\Sigma(\overline{A})$. More precisely, we have ([**BE-LO**] Lemma 3.1.1):

(9) $K(t,a) \le \max(1, t/s)K(s,a)$.

We also need to introduce the J-functional. Set $\Delta(\overline{A}) = A_0 \cap A_1$, and for each $a \in \Delta(\overline{A})$ and $t > 0$, define

$$J(t,a) = J(t,a;\overline{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1})\cdot$$

Clearly $J(t,a)$ is an equivalent norm on $\Delta(\overline{A})$ for a given $t > 0$. More precisely, we have (see [**BE-LO**]):

(10) $J(t,a) \le \max(1, t/s)J(s,a)$

(11) $K(t,a) \le \min(1, t/s)J(s,a)$, all $a \in \Delta(\overline{A})$.

Given a couple $\overline{A} = (A_0, A_1)$ of compatible Banach spaces, a function parameter $\varphi \in B_\Psi$ and $0 < q \le \infty$ we define

$$\overline{A}_{\varphi,q,K} = (A_0, A_1)_{\varphi,q,K} = \left\{ a \in \Sigma(\overline{A}) : \|a\|^q_{\varphi,q,K} = \int_0^\infty \left(\frac{1}{\varphi(t)} K(t,a) \right)^q \frac{dt}{t} < \infty \right\}\cdot$$

Also

$$\overline{A}_{\varphi,q,J} = \left\{ a \in \Sigma(\overline{A}) : \ a = \int_0^\infty u(t) \, \frac{dt}{t} \right\},$$

where the integral converges in $\Sigma(\overline{A})$ and $u(t)$ is measurable with values in $\Delta(\overline{A})$ and

$$\int_0^\infty \left(\frac{1}{\varphi(t)} J(t, u(t)) \right)^q \frac{dt}{t} < \infty\cdot$$

In this case, we have

$$\|a\|_{\overline{A}_{\varphi,q,J}} = \inf_u \left(\int_0^\infty \left(\frac{1}{\varphi(t)} J(t, u(t)) \right)^q \frac{dt}{t} \right)^{1/q}\cdot$$

In the classical case $\varphi(t)=t^\theta$, $0<\theta<1$, we have the equivalence theorem (see [**BE-LO**] Theorem 3.3.1):

$$\overline{A}_{\varphi,q,K}=\overline{A}_{\varphi,q,J}, \tag{12}$$

with equivalent norms.

The results needed to prove (12) are based on a discretization of the corresponding quasi-norms and an extension of the fundamental lemma of interpolation theory (see [**BE-LO**] Lemma 3.3.2).

PROPOSITION 2.3.2. *If $\varphi\in B_\Psi$, $0<q\le\infty$, then*

$$\|a\|_{\overline{A}_{\varphi,q,K}}\approx\left(\sum_{n=-\infty}^{\infty}\left(\frac{1}{\varphi(2^n)}K(2^n,a)\right)^q\right)^{1/q}. \tag{13}$$

$$\|a\|_{\overline{A}_{\varphi,q,J}}\approx\inf\left\{\left(\sum_{n=-\infty}^{\infty}\left(\frac{1}{\varphi(2^n)}J(2^n,u_n)\right)^q\right)^{1/q}\right\} \tag{14}$$

where $a=\sum_n u_n$, with convergence in $\Sigma(\overline{A})$ and $u_n\in\Delta(\overline{A})$. In particular,

$$\|a\|_{\overline{A}_{\varphi,q,J}}\le C\left(\int_0^\infty\overline{\varphi}(s)\min(1,\frac{1}{s})\frac{ds}{s}\right)\left(\sum_{n=-\infty}^{\infty}\left(\frac{1}{\varphi(2^n)}J(2^n,u_n)\right)^q\right)^{1/q}.$$

PROOF: Similar to the case $\varphi(t)=t^\theta$ (see [**BE-LO**] Lemmas 3.1.3 and 3.2.3). ∎

PROPOSITION 2.3.3. *If $\varphi\in B_\Psi$, $0<q\le\infty$, then,*

$$K(t,a)\le C\varphi(t)\|a\|_{\overline{A}_{\varphi,q,K}}.$$

In particular, $\|a\|_{A_0+A_1}\equiv K(1,a)\le C\|a\|_{\overline{A}_{\varphi,q,K}}$.

PROOF: By (9), $\min(1,s/t)K(t,a)\le K(s,a)$. Hence,

$$\begin{aligned}\|a\|_{\overline{A}_{\varphi,q,K}}=&\left(\int_0^\infty\left(\frac{1}{\varphi(s)}K(s,a)\right)^q\frac{ds}{s}\right)^{1/q}\\ \ge&K(t,a)\left(\int_0^\infty\left(\frac{1}{\varphi(s)}\min(1,s/t)\right)^q\frac{ds}{s}\right)^{1/q}.\end{aligned}$$

Changing variables and using the definition of $\underline{\varphi}$ (see (6) Chapter I):

$$\begin{aligned}
\|a\|_{\overline{A}_{\varphi,q,K}} &\geq K(t,a)\left(\int_0^\infty \left(\frac{1}{\varphi(t/u)}\min(1,1/u)\right)^q \frac{du}{u}\right)^{1/q} \\
&\geq \frac{K(t,a)}{\varphi(t)}\left(\int_0^\infty \left(\underline{\varphi}(u)\min(1,1/u)\right)^q \frac{du}{u}\right)^{1/q} \qquad \text{(by (6) Chapter I)} \\
&= \frac{K(t,a)}{\varphi(t)}\left(\int_0^\infty \left(\frac{1}{\overline{\varphi}(1/u)}\min(1,1/u)\right)^q \frac{du}{u}\right)^{1/q} \\
&= \frac{K(t,a)}{\varphi(t)}\left(\int_0^\infty \left(\frac{1}{\overline{\varphi}(u)}\min(1,u)\right)^q \frac{du}{u}\right)^{1/q}
\end{aligned}$$

and the integral is finite by Proposition 1.1.5 and Proposition 1.1.8-(12). ∎

PROPOSITION 2.3.4. (Fundamental lemma of interpolation theory)

If $\varphi \in B_\Psi$ and $0 < q \leq \infty$, then for each $a \in \overline{A}_{\varphi,q,K}$ and for all $\varepsilon > 0$ there exists a sequence $\{u_n\}_n \subset \Delta(\overline{A})$ such that $a = \sum_{n=-\infty}^{\infty} u_n$, with convergence in $\Sigma(\overline{A})$ and

$$J(2^n, u_n) \leq 3(1+\varepsilon)K(2^n, a),\ \ n \in \mathbf{Z}\cdot$$

PROOF: Since, by Proposition 1.1.9 we have that

$$\varphi(t) \leq \max(t^{\alpha_\varphi}, t^{\beta_\varphi}) \qquad \text{and} \qquad K(t,a) \leq C\varphi(t)\|a\|_{\overline{A}_{\varphi,q,K}}$$

then

$$\min(1,1/t)K(t,a) \longrightarrow 0 \quad \text{as} \ \ t \longrightarrow 0 \text{ or } \infty\cdot$$

Now apply Lemma 3.3.2 in **[BE-LO]** to obtain the desired sequence $\{u_n\}_n$. ∎

NOTE 2.3.5:

(a) Because of (12) we will denote by $\overline{A}_{\varphi,q}$ either of the spaces $\overline{A}_{\varphi,q;K}$ or $\overline{A}_{\varphi,q;J}$.

(b) We can now give the result that is equivalent to Proposition 2.3.3 for the $J-$ functional; i.e.,

$$\|a\|_{\overline{A}_{\varphi,q}} \leq C\frac{J(t,a)}{\varphi(t)}, \qquad a \in \Delta(\overline{A})\cdot$$

In fact,

$$
\begin{aligned}
\|a\|_{\overline{A}_{\varphi,q}} &= \left(\int_0^\infty \left(\frac{1}{\varphi(s)} K(s,a)\right)^q \frac{ds}{s}\right)^{1/q} \qquad \text{(by (11))}\\
&\le J(t,a)\left(\int_0^\infty \left(\frac{1}{\varphi(s)} \min(1,\frac{s}{t})\right)^q \frac{ds}{s}\right)^{1/q}\\
&= J(t,a)\left(\int_0^\infty \left(\frac{1}{\varphi(t/u)} \min(1,\frac{1}{u})\right)^q \frac{du}{u}\right)^{1/q}\\
&\le \frac{J(t,a)}{\varphi(t)}\left(\int_0^\infty \left(\overline{\varphi}(u) \min(1,\frac{1}{u})\right)^q \frac{du}{u}\right)^{1/q}
\end{aligned}
$$

and the integral is finite, since $\varphi \in B_K$.

We are now ready to prove the reiteration theorem mentioned above. We will now be dealing with functions $F : \mathbf{T} \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ such that $F_\theta(t) = F(\theta, t)$ is measurable on θ for every $t > 0$ and $F_\theta \in B_\Psi$ for every $\theta \in \mathbf{T}$. Given $q : \mathbf{T} \longrightarrow \mathbf{R}^+$ measurable, the basic assumption on F is the following:

DEFINITION 2.3.6. *Let F and q be as above. We say that F satisfies the condition (A), with respect to q, if:*

$$
\int_0^{2\pi} \log^+\left(\int_0^\infty \left(\overline{F}_\theta(t)\min(1,1/t)\right)^{q(\theta)} \frac{dt}{t}\right)^{1/q(\theta)} d\theta < \infty \cdot \tag{15}
$$

$$
\int_0^{2\pi} \log\left(\int_0^\infty \left(\overline{F}_\theta(t)\min(1,1/t)\right) \frac{dt}{t}\right) d\theta < \infty \cdot \tag{16}
$$

$$
\int_0^{2\pi} \log^+\left(\int_0^\infty \left(\left(\overline{F}_\theta(t)\right)^{-1}\min(1,t)\right)^{q(\theta)} \frac{dt}{t}\right)^{-1/q(\theta)} d\theta < \infty \cdot \tag{17}
$$

As we shall see later in the proof of the theorem, condition (A) is the appropriate one for the complex method of interpolation of families.

THEOREM 2.3.7. *Suppose $\overline{A} = (A_0, A_1)$ is a pair of compatible Banach spaces, and F satisfies condition (A) with respect to q. Set $A(\theta) = (A_0, A_1)_{F_\theta, q(\theta)}$. Then $\{A(\theta)\}_{\theta \in \mathbf{T}}$ is an interpolation family of Banach spaces and*

$$
[A(\theta)]_z = (A_0, A_1)_{F_z, q(z)},
$$

with equivalent norms, where

$$F_z(t) = \exp\left(\int_{\mathbf{T}} (\log F(\theta,t)) P_z(\theta)\, d\theta\right) \quad \textit{and} \quad \frac{1}{q(z)} = \int_{\mathbf{T}} \frac{1}{q(\theta)} P_z(\theta)\, d\theta\cdot$$

PROOF: We begin by showing that $F_z \in B_\Psi$ for all $z \in D$, so that it makes sense to write $(A_0, A_1)_{F_z, q(z)}$. For $F_\theta \in B_\Psi$ we set

$$G_\theta(t) = \frac{tF'_\theta(t)}{F_\theta(t)}\cdot$$

As in Proposition 1.1.7 we get

$$F_\theta(t) = \exp\left(\int_1^t \frac{G_\theta(s)}{s}\, ds\right)\cdot \tag{18}$$

Thus,

$$F_z(t) = \exp\left(\int_{\mathbf{T}} \left(\int_1^t \frac{G_\theta(s)}{s}\, ds\right) P_z(\theta)\, d\theta\right) = \exp\left(\int_1^t \frac{G_z(s)}{s}\, ds\right)$$

where $G_z(s) = G(z,s)$ is the Poisson integral of $G(\cdot, s)$. Hence,

$$G_z(t) = \frac{tF'_z(t)}{F_z(t)}\cdot$$

The Maximum Principle and the continuity of $F(\theta, \cdot)$ show that

$$\sup_{t>0}(G_z(t)) \leq \sup_{t>0}\sup_{\theta\in\mathbf{T}}(G_\theta(t)) < 1$$

and similarly

$$\inf_{t>0}(G_z(t)) > 0\cdot$$

Therefore, $F_z \in B_\Psi$ as we wanted.

Our next step is to prove that $\{A(\theta)\}_{\theta\in\mathbf{T}}$ is an interpolation family. To see this, observe that $A(\theta) \hookrightarrow A_0 + A_1$ and by Proposition 2.3.3 we have the estimate $\|a\|_{A_0+A_1} \leq k(\theta)\|a\|_{A(\theta)}$, with

$$k(\theta) = \left(\int_0^\infty \left(\left(\overline{F}_\theta(t)\right)^{-1} \min(1,t)\right)^{q(\theta)} \frac{dt}{t}\right)^{-1/q(\theta)}$$

so that by (17), $\log^+ k(\theta) \in L^1(\mathbf{T})$, which is all we need to show.

We now come to the main part of the proof. We first prove the inclusion

$$(19) \qquad [A(\theta)]_z \subset (A_0, A_1)_{F_z, q(z)}$$

and the corresponding norm inequality.

Let $a \in [A(\theta)]_z$ and take $\varepsilon > 0$. We can find $f \in \mathcal{F}(A)$ with $f(z) = a$ such that

$$(20) \qquad \|f\|_\infty \leq \|a\|_z(1+\varepsilon)\cdot$$

By the subharmonicity of $\log K(t, f(z))$ (see Lemma 4.1 of [**HE 2**]) and the definition of $F_z(t)$ we obtain

$$\begin{aligned} \|a\|_{F_z, q(z)} &= \left(\int_0^\infty \left(\frac{1}{F_z(t)} K(t, f(z)) \right)^{q(z)} \frac{dt}{t} \right)^{1/q(z)} \\ &\leq \left(\int_0^\infty \left(\frac{1}{F_z(t)} \exp\left(\int_{\mathbf{T}} (\log K(t, f(\theta))) P_z(\theta)\, d\theta \right) \right)^{q(z)} \frac{dt}{t} \right)^{1/q(z)} \\ &= \left(\int_0^\infty \left(\exp\left(\int_{\mathbf{T}} \log\left(\frac{1}{F_\theta(t)} K(t, f(\theta)) \right) P_z(\theta)\, d\theta \right) \right)^{q(z)} \frac{dt}{t} \right)^{1/q(z)} . \end{aligned}$$

We need the following result:

LEMMA 2.3.8. (Fundamental inequality, Proposition 3.1 in [**HE 1**])

Let $p : \overline{D} \longrightarrow [1, \infty)$ be a function such that $1/p(z)$ is harmonic in D and $G : \mathbf{T} \times \mathbf{R}^+ \longrightarrow \mathbf{C}$ be a function for which

$$\int_{\mathbf{T}} \|G(\theta, \cdot)\|_{L^{p(\theta)}} P_z(\theta)\, d\theta < \infty,$$

for some $z \in D$, (G is called p-admissible). Then

$$\log \|u_G(z, \cdot)\|_{L^{p(z)}} \leq \int_{\mathbf{T}} \log \|G(\theta, \cdot)\|_{L^{p(\theta)}} P_z(\theta)\, d\theta,$$

where

$$u_G(z, x) = \exp\left(\int_{\mathbf{T}} \log |G(\theta, x)|\ H_z(\theta)\, d\theta \right)$$

and

$$P_z(\theta) = \Re(H_z(\theta)) = \Re\left(\frac{1}{2\pi}\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right),$$

for all $z \in D\cdot$

Using the lemma and (20):

$$\begin{aligned}\|a\|_{F_z,q(z)} &\leq \exp\left(\int_{\mathbf{T}} \log\left(\left(\int_0^\infty \left(\frac{1}{F_\theta(t)}K(t,f(\theta))\right)^{q(\theta)}\frac{dt}{t}\right)^{1/q(\theta)}\right) P_z(\theta)\,d\theta\right)\\ &\leq \exp\left(\int_{\mathbf{T}} (\log\|f(\theta)\|_{F_\theta,q(\theta)})P_z(\theta)\,d\theta\right) \leq \|f\|_\infty \leq \|a\|_z(1+\varepsilon)\cdot\end{aligned}$$

The inclusion (19) now follows with norm less than or equal to 1, upon letting $\varepsilon \longrightarrow 0$.

To prove the inclusion

$$[A(\theta)]_z \supset (A_0, A_1)_{F_z,q(z)} \tag{21}$$

and the corresponding norm inequality, we need the following lemma:

LEMMA 2.3.9. *Under condition (A),* $A_0 \cap A_1 \subset \mathcal{A}$*, where* $\mathcal{A}$ *denotes the log-intersection space of the family* $\{A(\theta)\}_{\theta\in\mathbf{T}}$*.*

PROOF: Since $K(t,a) \leq \min(1,t)\|a\|_{A_0\cap A_1}$, we deduce

$$\|a\|^{q(\theta)}_{F_\theta,q(\theta)} \leq \|a\|^{q(\theta)}_{A_0\cap A_1}\left(\int_0^1\left(\frac{t}{F_\theta(t)}\right)^{q(\theta)}\frac{dt}{t} + \int_1^\infty\left(\frac{1}{F_\theta(t)}\right)^{q(\theta)}\frac{dt}{t}\right).$$

Using Proposition 1.1.8-(13) and (14), with $s = 1$, we obtain

$$\|a\|^{q(\theta)}_{F_\theta,q(\theta)} \leq \|a\|^{q(\theta)}_{A_0\cap A_1}\left(\int_0^\infty \left(\overline{F_\theta}(t)\min(1,1/t)\right)^{q(\theta)}\frac{dt}{t}\right).$$

The desired result now follows from condition (15). ∎

To prove (21) let $a \in (A_0, A_1)_{F_z,q(z)}$ and $\varepsilon > 0$. By Proposition 2.3.4, there is a representation of a of the form $a = \sum_n u_n$ (convergence in $A_0 + A_1$) with $u_n \in A_0 \cap A_1$ and such that

$$J(2^n, u_n) \leq 3(1+\varepsilon)K(2^n, a), \qquad \text{for all } n \in \mathbf{Z}\cdot \tag{22}$$

Fix $t > 0$ and let $\widetilde{G}(\xi, t),\ \xi \in D$ be the harmonic conjugate of $G(\cdot, t)$ normalized by $\widetilde{G}(z, t) = 0$. Similarly, let $(1/q)^{\sim}$ be the harmonic conjugate of $1/q$ such that $(1/q)^{\sim}(z) = 0$. Set $W(\xi, t) = G(\xi, t) + i\widetilde{G}(\xi, t)$, and

$$\frac{1}{s(\xi)} = \frac{1}{q(\xi)} + i\left(\frac{1}{q}\right)^{\sim}(\xi), \qquad \xi \in D\cdot$$

Let $H(\xi, t)$ be so that $W(\xi, t) = \dfrac{tH'(\xi, t)}{H(\xi, t)}$; that is

$$H(\xi, t) = \exp\left(\int_1^t \frac{W(\xi, s)}{s}\, ds\right)\cdot$$

Define

$$A_n(\xi) = \frac{H(\xi, 2^n)}{F_z(2^n)}\left(\frac{J(2^n, u_n)}{F_z(2^n)}\right)^{-1+\frac{q(z)}{s(\xi)}}, \quad n \in \mathbf{Z}\cdot$$

We now show that A_n is bounded for every n. In fact,

$$|H(\xi, 2^n)| = \exp\left(\int_1^{2^n} \frac{G(\xi, 2^n)}{s}\, ds\right) \le 2^n,$$

since $G(\xi, s) \le 1$ and

$$\begin{aligned}\left|\left(\frac{J(2^n, u_n)}{F_z(2^n)}\right)^{-1+\frac{q(z)}{s(\xi)}}\right| &= \left(\frac{J(2^n, u_n)}{F_z(2^n)}\right)^{-1+\frac{q(z)}{q(\xi)}} \\ &\le \frac{F_z(2^n)}{J(2^n, u_n)} \max\left(1, \left(\frac{J(2^n, u_n)}{F_z(2^n)}\right)^{q(z)}\right).\end{aligned}$$

These two estimates give the desired result.

Set $g_N(\xi) = \sum_{n=-N}^{N} f_n(\xi),\ \xi \in D$ where $f_n(\xi) = u_n A_n(\xi) \in A_0 \cap A_1$, so that by Lemma 2.3.9 and the boundedness of A_n we have $g_N \in N^+(A)$, for all positive integers N. Let

$$C(\theta) = C\int_0^\infty \overline{F_\theta}(s) \min(1, 1/s)\, \frac{ds}{s},$$

be the constant coming from the equivalence of norms in Proposition 2.3.2-(6).

Using this proposition and the definition of A_n, we obtain

$$\|g_N(\theta)\|_{F_\theta,q(\theta)} \le C(\theta)\left(\sum_{n=-N}^{N}\left(\frac{1}{F_\theta(2^n)}J(2^n,f_n(\theta))\right)^{q(\theta)}\right)^{1/q(\theta)}$$

$$= C(\theta)\left(\sum_{n=-N}^{N}\left(\frac{1}{F_\theta(2^n)}\left|\frac{H(\theta,2^n)}{F_z(2^n)}\left(\frac{J(2^n,u_n)}{F_z(2^n)}\right)^{-1+\frac{q(z)}{s(\theta)}}\right|J(2^n,u_n)\right)^{q(\theta)}\right)^{1/q(\theta)}$$

$$= C(\theta)\left(\sum_{n=-N}^{N}\left(\frac{1}{F_z(2^n)}J(2^n,u_n)\right)^{q(z)}\right)^{1/q(\theta)}.$$

Using (22), we obtain

$$\|g_N(\theta)\|_{F_\theta,q(\theta)} \le 3(1+\varepsilon)C(\theta)\left(\sum_{n=-N}^{N}\left(\frac{1}{F_z(2^n)}K(2^n,a)\right)^{q(z)}\right)^{1/q(\theta)}.$$

Proposition 2.3.1 now implies

$$\|g_N(z)\|_z \le 3(1+\varepsilon)\exp\left(\int_{\mathbf{T}}(\log C(\theta))P_z(\theta)\,d\theta\right)\left(\sum_{n=-N}^{N}\left(\frac{K(2^n,a)}{F_z(2^n)}\right)^{q(z)}\right)^{\frac{1}{q(z)}}$$

where $C(z) = \exp\left(\int_{\mathbf{T}}(\log C(\theta))P_z(\theta)\,d\theta\right)$ is finite due to (16) of condition (A).

Notice that $\lim_{N\to\infty} g_N(z)$ coincides formally with $\sum_{n=-\infty}^{\infty} u_n = a$, convergence in $A_0 + A_1$, so that a density argument will give $\|a\|_z \le k(1+\varepsilon)C(z)\|a\|_{F_z,q(z)}$, after using Proposition 2.3.2-(5).

The details of this density argument are similar to the ones given on page 89 of [**HE 2**] and therefore, omitted. The inclusion (21) follows upon letting $\varepsilon \longrightarrow 0$ and hence Theorem 2.3.7 is proved. ▮

Before using this theorem to get the interpolation results for the $\Lambda^q(w)$ spaces, we want to show, as a consequence of it, two particular cases that were already known in the literature. The first one deals with the classical real interpolation method, and it is due to E. Hernández ([**HE 2**]) and the second result involves the classical complex interpolation of Calderón, and it is due to Karadžov ([**KAR**]) and Berg (see [**BE-LO**]).

COROLLARY 2.3.10. **[HE 2]**

Suppose $0 < \alpha(\theta) < 1$, $1 \le q(\theta) \le \infty$ *are two measurable functions defined on* $\mathbf{T}$ *and* $A(\theta) = (A_0, A_1)_{\alpha(\theta),q(\theta)}$, *where* $A_0,\ A_1$ *is a pair of compatible Banach spaces. Then* $\{A(\theta)\}_{\theta\in\mathbf{T}}$ *is an interpolation family of Banach spaces. If, in addition, we suppose*

$$(A')\quad \int_0^{2\pi} \frac{1}{q(\theta)}\, d\theta > 0,\quad \int_0^{2\pi} \log\alpha(\theta)\, d\theta > -\infty,\ \text{and} \int_0^{2\pi} \log(1-\alpha(\theta))\, d\theta > -\infty$$

the spaces $[A(\theta)]_z$ *and* $(A_0, A_1)_{\alpha(z),q(z)}$ *coincide and their norms are equivalent, where* $z \in D$ *and* $\alpha(z)$, $1/q(z)$ *are the harmonic functions on* D *whose boundary values are* $\alpha(\theta)$ *and* $1/q(\theta)$ *respectively.*

PROOF: Set $F(\theta, t) = t^{\alpha(\theta)}$, $\theta \in \mathbf{T}$, $t > 0$. Then condition (A') implies condition (A) of Theorem 2.3.7 and since

$$F_z(t) = \exp\left(\int_{\mathbf{T}} (\log t^{\alpha(\theta)}) P_z(\theta)\, d\theta\right) = t^{\left(\int_{\mathbf{T}} \alpha(\theta) P_z(\theta)\, d\theta\right)} = t^{\alpha(z)}$$

we obtain the corollary. ∎

We need the following proposition to prove our next result.

PROPOSITION 2.3.11. (Corollary 5.1 in **[CCRSW 1]**)

Suppose $\mathbf{T}_0$, $\mathbf{T}_1$ *are two disjoint measurable subsets of* $\mathbf{T}$, *whose union is* $\mathbf{T}$, *and* $\overline{A} = (A_0, A_1)$ *is a pair of compatible Banach spaces. If* $\{A(\theta)\}_{\theta\in\mathbf{T}}$ *is such that* $A(\theta) = A_0$ *for all* $\theta \in \mathbf{T}_0$ *and* $A(\theta) = A_1$ *for all* $\theta \in \mathbf{T}_1$, *then* $[A(\theta)]_z = [A_0, A_1]_{\theta(z)}$, *where*

$$\theta(z) = \int_{\mathbf{T}_1} dP_z(\gamma)\cdot$$

COROLLARY 2.3.12. *Suppose* $\overline{A} = (A_0, A_1)$ *is a pair of compatible Banach spaces and* $\varphi_0,\ \varphi_1 \in B_\Psi$. *Then, if* $0 < \theta < 1$, $1 \le q_0, q_1 < \infty$ *:*

$$[\overline{A}_{\varphi_0,q_0}, \overline{A}_{\varphi_1,q_1}]_\theta = \overline{A}_{\varphi,q}$$

where

$$\varphi(s) = (\varphi_0(s))^{1-\theta}(\varphi_1(s))^\theta \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\cdot$$

PROOF: Let $\mathbf{T}_1 = \{e^{i\gamma} \in \mathbf{T} \ : \ 0 \leq \gamma \leq 2\pi\theta\}$, $\mathbf{T}_0 = \mathbf{T} \setminus \mathbf{T}_1$, $A(\gamma) = \overline{A}_{\varphi_0,q_0}$ if $\gamma \in \mathbf{T}_0$ and $A(\gamma) = \overline{A}_{\varphi_1,q_1}$ if $\gamma \in \mathbf{T}_1$. Put

$$F(\gamma,t) = \begin{cases} \varphi_0(t) & \gamma \in \mathbf{T}_0 \\ \varphi_1(t) & \gamma \in \mathbf{T}_1 \end{cases} \qquad \text{and} \qquad q(\gamma) = \begin{cases} q_0 & \gamma \in \mathbf{T}_0 \\ q_1 & \gamma \in \mathbf{T}_1 \end{cases}.$$

We are then under the hypotheses of Theorem 2.3.7 and Proposition 2.3.11; hence, $[A(\gamma)]_{z=0} = (A_0, A_1)_{F_0,q(0)}$, where

$$F_0(t) = \exp\left(\int_{\mathbf{T}_0} \log \varphi_0(t)\, d\gamma + \int_{\mathbf{T}_1} \log \varphi_1(t)\, d\gamma \right) = (\varphi_0(t))^{1-\theta} (\varphi_1(t))^{\theta}$$

and

$$\frac{1}{q(0)} = \int_{\mathbf{T}} \frac{1}{q(\gamma)}\, dP_0(\gamma) = \int_{\mathbf{T}_0} \frac{1}{q_0}\, d\gamma + \int_{\mathbf{T}_1} \frac{1}{q_1}\, d\gamma = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Thus $[A(\gamma)]_{z=0} = (A_0, A_1)_{\varphi,q} = \overline{A}_{\varphi,q}$. Now, by Proposition 2.3.11, we know that $[A(\gamma)]_{z=0} = [\overline{A}_{\varphi_0,q_0}, \overline{A}_{\varphi_1,q_1}]_\theta$. Hence,

$$[\overline{A}_{\varphi_0,q_0}, \overline{A}_{\varphi_1,q_1}]_\theta = \overline{A}_{\varphi,q}\,. \quad \blacksquare$$

COROLLARY 2.3.13. (See Theorem 4.7.2 in **[BE-LO]**)

If $0 < \theta_0 < \theta_1 < 1$, $\theta = (1-\eta)\theta_0 + \eta\theta_1$, $1 \leq p_i < \infty$, $i = 0, 1$, *and if we set* $1/p = (1-\eta)/p_0 + \eta/p_1$, *then* $[\overline{A}_{\theta_0,p_0}, \overline{A}_{\theta_1,p_1}]_\eta = \overline{A}_{\theta,p}$.

PROOF: In the last corollary, take $\varphi_0(t) = t^{\theta_0}$, $\varphi_1(t) = t^{\theta_1}$ and $\theta = \eta$. Then $\overline{A}_{\varphi_i,p_i} = \overline{A}_{\theta_i,p_i}$ and $[\overline{A}_{\theta_0,p_0}, \overline{A}_{\theta_1,p_1}]_\eta = \overline{A}_{\theta,p}$, because $t^\theta = (\varphi_0(t))^{1-\theta}(\varphi_1(t))^\theta$. $\blacksquare$

We now want to show the interpolation property for the weighted Lorentz spaces, when using the complex method for families of Banach spaces. We do this by showing that this is a consequence of our main result, Theorem 2.3.7. The corresponding result, for the real method of interpolation has been proved by Gustavsson in **[GU]**: first one shows that $\Lambda^q(w)$ is an intermediate space between L^1 and L^∞, and then, using the reiteration theorem it is easy to identify the intermediate spaces of the spaces $\Lambda^q(w)$. (See **[ME 2]** for more details). We shall need Gustavsson's theorem to prove our result:

THEOREM 2.3.14. ([**GU**])

If $\varphi \in B_\Psi$, $1 \le q \le \infty$ *and* $w(t) = t^{1-1/q}/\varphi(t)$, *then* $(L^1, L^\infty)_{\varphi,q} = \Lambda^q(w)$, *with equivalent norms.*

The proof of the previous theorem is a simple consequence of the fact that $K(t, f; L^1, L^\infty) = \int_0^t f^*(s)\, ds$ and our Theorem 1.2.7. Notice, that using Proposition 2.3.2, this theorem also gives an equivalent discrete norm on $\Lambda^q(w)$, as in Theorem 2.2.4.

We now prove the interpolation result:

THEOREM 2.3.15. *Suppose* $q : \mathbf{T} \longrightarrow [1, \infty)$ *and* $w : \mathbf{T} \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *are measurable functions on* $\mathbf{T}$ *such that* $\varphi_\theta(t) = t^{1-1/q(\theta)}/w(\theta, t)$ *belongs to* B_Ψ, *for every* $\theta \in \mathbf{T}$ *and satisfies condition (A) of Definition 2.3.6. Then* $\{\Lambda^{q(\theta)}(w(\theta, \cdot))\}_{\theta \in \mathbf{T}}$ *is an interpolation family of Banach spaces and*

$$[\Lambda^{q(\theta)}(w(\theta, \cdot))]_z = \Lambda^{q(z)}(w(z, \cdot)),$$

with equivalent norms, where

$$\frac{1}{q(z)} = \int_{\mathbf{T}} \frac{1}{q(\theta)} P_z(\theta)\, d\theta$$

and

$$w(z, s) = \exp\left(\int_{\mathbf{T}} (\log w(\theta, s)) P_z(\theta)\, d\theta \right).$$

PROOF: By Theorem 2.3.14

$$\Lambda^{q(\theta)}(w(\theta, \cdot)) = (L^1, L^\infty)_{\varphi(\theta), q(\theta)}.$$

By Theorem 2.3.7, $\{\Lambda^{q(\theta)}(w(\theta, \cdot))\}_{\theta \in \mathbf{T}}$ is an interpolation family, and

$$[\Lambda^{q(\theta)}(w(\theta, \cdot))]_z = (L^1, L^\infty)_{\varphi_z, q(z)} = \Lambda^{q(z)}(w(z, \cdot)) \cdot \quad \blacksquare$$

COROLLARY 2.3.16. *Suppose* $\varphi_0,\ \varphi_1 \in B_\Psi,\ 1 \leq q_0, q_1 < \infty$ *and consider the weigths* $w_i(t) = t^{1-1/q_i}/\varphi_i(t),\ i = 0, 1$. *Then,*

$$[\Lambda^{q_0}(w_o), \Lambda^{q_1}(w_1)]_\theta = \Lambda^q(w)$$

where $w(s) = [w_0(s)]^{1-\theta}[w_1(s)]^\theta$ *and* $1/q = (1-\theta)/q_0 + \theta/q_1\cdot$

PROOF: Use Corollary 2.3.12, and argue as above. ∎

(§4) Reiteration.

As we saw in the previous section, the weighted Lorentz spaces $\Lambda^q(w)$ show up naturally as the intermediate spaces between L^1 and L^∞. In this section, we will show that this correspondence also holds for other endpoint spaces such as the Hardy space H^1, the space of bounded mean oscillation functions BMO, the space of continuous functions that tend to zero at infinity C_0, the space of finite Borel measures M, etc. In particular, for the case of H^1 and C_0, or similarly for M and BMO, this gives an answer to a question that Yoram Sagher asked the author.

The idea of the proof is to reduce the interpolation with a function parameter to the classical case, by means of a reiteration theorem, (see **[GU]** and **[PER]**).

THEOREM 2.4.1. *Suppose* $\overline{A} = (A_0, A_1)$ *is a pair of compatible Banach spaces,* $\varphi_0, \varphi_1, \Psi \in B_\Psi$, *and* $1 \leq q_0, q_1, q \leq \infty$. *Let* $\overline{A}_{\varphi_i, q_i} = (A_0, A_1)_{\varphi_i, q_i}$, $i = 0, 1$. *Assume also that if we let* $\varphi(t) = \varphi_1(t)/\varphi_0(t)$ *then* $|t\varphi'(t)/\varphi(t)| \geq \varepsilon > 0$. *Then*

$$(\overline{A}_{\varphi_0, q_0}, \overline{A}_{\varphi_1, q_1})_{\Psi, q} = (A_0, A_1)_{\xi, q},$$

where

$$\xi(t) = \varphi_0(t)\Psi\left(\frac{\varphi_1(t)}{\varphi_0(t)}\right).$$

PROOF: We only need to show the following two estimates:

(23) $K(t, a; A_0, A_1) \leq C\varphi_i(t)\|a\|_{\overline{A}_{\varphi_i, q_i}}$, $a \in \Sigma(\overline{A})$, $i = 0, 1$; i.e., $\overline{A}_{\varphi_i, q_i}$ is of class $C_K(\varphi_i, \overline{A})$.

(24) $\|a\|_{\overline{A}_{\varphi_i,q_i}} \leq \dfrac{C}{\varphi_i(t)} J(t,a;A_0,A_1)$, $a \in \Delta(\overline{A})$, $i = 0,1$; i.e., $\overline{A}_{\varphi_i,q_i}$ is of class $C_J(\varphi_i, \overline{A})$.

(23) is proved in Proposition 2.3.3 and (24) is Note 2.3.5-(b). Thus, Theorem 2.3 in **[GU]** finishes the proof. ∎

LEMMA 2.4.2. *Let $\varphi \in B_\Psi$, $0 < p_0 < 1/(1-\alpha_\varphi)$ and $1/(1-\beta_\varphi) < p_1$. Let $a = 1 - 1/p_0$ and $b = 1/p_0 - 1/p_1$. Set $\Psi(t) = t^{-a/b}\varphi(t^{1/b})$. Then $\Psi \in B_\Psi$.*

PROOF: Since

$$\Psi'(t) = \frac{-a}{b} t^{-a/b-1}\varphi(t^{1/b}) + \frac{t^{-a/b+1/b-1}}{b}\varphi'(t^{1/b}),$$

then

$$\frac{t\Psi'(t)}{\Psi(t)} = \frac{-a}{b} + \frac{1}{b}\frac{t^{1/b}\varphi'(t^{1/b})}{\varphi(t^{1/b})}\cdot$$

Hence,

$$\alpha_\Psi = \frac{-a}{b} + \frac{\alpha_\varphi}{b}, \qquad \beta_\Psi = \frac{-a}{b} + \frac{\beta_\varphi}{b}\cdot$$

That is,

$$\alpha_\Psi = \frac{1}{b}(\frac{1}{p_0} - 1 + \alpha_\varphi) > 0, \qquad \beta_\Psi < 1 \Longleftrightarrow p_1 > \frac{1}{1-\beta_\varphi}\cdot$$

Therefore $0 < \alpha_\Psi \leq \beta_\Psi < 1$ and so $\Psi \in B_\Psi$. ∎

THEOREM 2.4.3. *Suppose $\varphi \in B_\Psi$ and $1 \leq q \leq \infty$. Set $w(t) = t^{1-1/q}/\varphi(t)$. Then*

(a) $(H^1, C_0)_{\varphi,q} = \Lambda^q(w)$,

(b) $(M, BMO)_{\varphi,q} = \Lambda^q(w)$,

(c) $(L^1, BMO)_{\varphi,q} = \Lambda^q(w)$,

(d) $(H^1, BMO)_{\varphi,q} = \Lambda^q(w)$.

PROOF: We only need to give the proof of (a), since the others are similar.

If $\varphi(t) = t^\theta$, $0 < \theta < 1$, (a) follows from Theorem (4) of **[RI-SA]** and the fact that $\Lambda^q(t^{1-1/q-\theta}) = L^{p,q}$ where $1/p = 1-\theta$.

Choose $1 \leq p_0 < 1/(1-\alpha_\varphi)$ and $p_1 > 1/(1-\beta_\varphi)$. Set $\theta_i = 1 - 1/p_i$, $i = 0,1$, and define $\Psi(t) = t^{\theta_0/(\theta_0-\theta_1)}\varphi(t^{1/(\theta_1-\theta_0)})$. Then, by Lemma 2.4.2, $\Psi \in B_\Psi$. Also,

$(L^{p_0}, L^{p_1})_{\Psi,q} = \Lambda^q(w)$. In fact, since $L^{p_i} = (L^1, L^\infty)_{\theta_i,p_i},\ i = 0, 1$ and we have that $\varphi(t) = t^{\theta_0}\Psi(t^{\theta_1-\theta_0})$, then by Theorem 2.4.1 and Theorem 2.3.14 we get that

$$(L^{p_0}, L^{p_1})_{\Psi,q} = (L^1, L^\infty)_{\varphi,q} = \Lambda^q(w)\cdot$$

Again, by Theorem 2.4.1 and the result of [**RI-SA**],

$$(H^1, C_0)_{\varphi,q} = ((H^1, C_0)_{\theta_0,p_0},\ (H^1, C_0)_{\theta_1,p_1})_{\Psi,q} = (L^{p_0}, L^{p_1})_{\Psi,q} = \Lambda^q(w)\cdot \quad \blacksquare$$

REMARK 2.4.4:
(i) One can also get the theorem for the case of quasi-Banach spaces; i.e., when $0 < q < 1$.
(ii) It is also possible to give a direct proof of some of the previous results. For example, to prove (a), if we denote by $M_r(f)(x) = (M(|f|^r)(x))^{1/r},\ x \in \mathbf{R}^n$, and $1 < r < \infty$, where M is the Hardy-Littlewood maximal operator and

$$f_r(t) = \left(S\left(((M_r f)^*)^r\right)(t)\right)^{1/r},\ t > 0$$

then it is proved in [**RI-SA**] that

$$K(t, f; H^1, C_0) \le Ctf_r(t),\ t > 0$$

and $f \in C_0\cdot$ One shows that if $\varphi \in B_\Psi,\ 1 \le q \le \infty$ and if $1 < r < 1/(1-\alpha_\varphi)$ then

$$M_r : \Lambda^q(w) \longrightarrow \Lambda^q(w),$$

where $w(t) = t^{1-1/q}/\varphi(t)\cdot$ Then using the weighted Hardy's inequalities of our

Theorem 1.2.7, we conclude that

$$\begin{aligned}
\|f\|_{(H^1,C_0)_{\varphi,q}} &= \left(\int_0^\infty \left(\frac{1}{\varphi(t)}K(t,f;H^1,C_0)\right)^q \frac{dt}{t}\right)^{1/q} \\
&\leq C\left(\int_0^\infty \left(\frac{1}{\varphi(t)}tf_r(t)\right)^q \frac{dt}{t}\right)^{1/q} \\
&= C\left(\int_0^\infty \left(\frac{1}{\varphi(t)}t\left(\frac{1}{t}\int_0^t ((M_rf)^*(s))^r\,ds\right)^{1/r}\right)^q \frac{dt}{t}\right)^{1/q} \\
&= C\left(\int_0^\infty \left(w(t)\left(S\left((M_rf)^*\right)^r(t)\right)^{1/r}\right)^q dt\right)^{1/q} \\
&= C\left(\int_0^\infty \left(w^r(t)S\left((M_rf)^*\right)^r(t)\right)^{q/r} dt\right)^{1/q} \qquad (w^r \in W_{q/r}(S)) \\
&\leq C\left(\int_0^\infty \left(w^r(t)\left((M_rf)^*(t)\right)^r\right)^{q/r} dt\right)^{1/q} \\
&\leq C\left(\int_0^\infty (w(t)(M_rf)^*(t))^q\,dt\right)^{1/q} \\
&= C\|M_rf\|_{\Lambda^q(w)} \leq C\|f\|_{\Lambda^q(w)},
\end{aligned}$$

for all $f \in C_0$, and hence, by a density argument $\Lambda^q(w) \subset (H^1,C_0)_{\varphi,q}$.

The other inclusion follows easily, since

$$(H^1,C_0)_{\varphi,q} \subset (L^1,L^\infty)_{\varphi,q} = \Lambda^q(w)\cdot$$

(iii) As in (ii) one can also give a direct proof of (c) by the same argument that shows the result for the case $\varphi(t) = t^\theta,\ 0<\theta<1$, (see Theorem 8.11 **[BE-SH]**): the condition needed there to get $(L^1,BMO)_{\varphi,q} \subset \Lambda^q(w)$ also holds now; i.e., if $f \in (L^1,BMO)_{\varphi,q}$ then

$$\int_1^\infty \left(f^\#\right)^*(s)\,\frac{ds}{s} < \infty, \tag{25}$$

where

$$f^\#(t) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy, \qquad\qquad f_Q = \frac{1}{|Q|}\int_Q f(y)\,dy\cdot$$

To show (25) one uses the equivalence

$$K(t, f; L^1, BMO) \approx t \left(f^{\#}\right)^* (t)$$

and the estimate

$$\int_1^\infty \left(f^{\#}\right)^* (s) \, \frac{ds}{s} < \infty \leq \left(\int_1^\infty \left(\frac{\varphi(t)}{t} \right)^{q'} \frac{dt}{t} \right)^{1/q'} \|f\|_{(L^1, BMO)_{\varphi,q}}.$$

The integral is finite by Proposition 1.1.11-(15) and finally, Theorem 1.2.7 gives the desired norm estimate.

Chapter III
Tent spaces

(§1) Introduction.

The theory of Tent spaces was first developed by Coifman, Meyer and Stein (see [**CO-ME-ST 1**]), as a tool to get a simpler proof of the boundedness of commutator integrals of Calderón, that are related to the Cauchy Integral operator (see [**JOU**]). Later, in [**CO-ME-ST 2**], they studied in detail properties of Tent spaces, as (quasi) Banach spaces of measurable functions on the upper half space. The main results that they obtain deal with the equivalence of norms on these spaces in terms of several functionals (balayages), the atomic decomposition, from which one can obtain a simple and elegant proof of the atomic decomposition of the Hardy spaces H^p, interpolation theorems for both the real and complex method (results that can be used to get the similar theorems for Hardy spaces), and finally a very interesting conection, by duality, with Carleson measures.

As is written in [**CO-ME-ST 2**], "... these spaces lead to unifications and simplifications of some basic techniques in harmonic analysis." Besides the ones already mentioned, one can also point out the recent works of Alvarez and Milman (see [**AL-MI 1**], [**AL-MI 2**]) and Bonami and Johnson (see [**BO-JO**]), where applications to maximal operators, square functions, weighted norm inequalities, as well as other applications are given. These authors have also extended the class of tent spaces by considering not only L^p- but also $L^{p,q}$-norms.

Our goal in this chapter is to further enhance the range of the tent spaces to include the more general weighted Lorentz spaces $\Lambda^q(w)$. One way to justify this comes from the fact that the spaces that one gets, are precisely the intermediate spaces obtained when one applies the real method of interpolation with a function parameter to the Tent spaces of Coifman, Meyer and Stein. Another feature is that the duality theorems now give a richer class of Carleson-type measures. As we shall see later, this can be used to get more general weighted inequalities for maximal operators.

(§2) Definitions and first properties.

DEFINITION 3.2.1.

(i) *Let $x \in \mathbf{R^n}$. We shall denote by $\Gamma(x)$ the standard cone of aperture 1, whose vertex is $x \in \mathbf{R^n}$. That is,*

$$\Gamma(x) = \left\{(y,t) \in \mathbf{R}_+^{\mathbf{n}+1} \ : \ |x-y| < t\right\}\cdot$$

(ii) *For any set $F \subset \mathbf{R^n}$, $\mathcal{R}F = \cup_{x\in F}\Gamma(x)$ is the union of the cones with vertices in F.*

(iii) *Let Ω be the set which is the complement of F, $\Omega = {}^cF$. The tent over Ω is denoted*

$$T(\Omega) = \widehat{\Omega} = {}^c\mathcal{R}F = \left\{(x,t) \in \mathbf{R}_+^{\mathbf{n}+1} \ : \ B(x,t) \subset \Omega\right\}\cdot$$

Notice that $T(\Omega)$ coincides with the set lying below the Lipschitz graph

$$\left\{(x,t) \in \mathbf{R}_+^{\mathbf{n}+1} \ : \ \mathrm{dist}(x,F) = t\right\}\cdot$$

DEFINITION 3.2.2. *Let $0 < q \leq \infty$. The A_q functionals mapping functions on $\mathbf{R}_+^{\mathbf{n}+1}$*

into functions on $\mathbf{R^n}$ are defined by

$$A_q(f)(x) = \left(\int\limits_{\Gamma(x)} |f(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{1/q} \qquad \text{if } q < \infty$$

$$A_\infty(f)(x) = \sup_{\Gamma(x)} |f(y,t)|\cdot$$

DEFINITION 3.2.3. *Let w be a weight on $\mathbf{R}^+$, $0 < q \leq \infty,\ 0 < p < \infty$. Assume that w^p is a locally integrable function on $\mathbf{R}^+$; i.e., $\int_a^b w^p(t)\ dt < \infty,\ 0 \leq a < b < \infty$. The tent space $T_q(\Lambda^p(w))$ is defined as the space of functions f on $\mathbf{R}_+^{\mathbf{n}+1}$ such that $A_q(f) \in \Lambda^p(w)\cdot$*

$$T_q(\Lambda^p(w)) = \left\{ f \ : \ \|f\|_{T_q(\Lambda^p(w))} = \|A_q(f)\|_{\Lambda^p(w)} < \infty \right\}$$

The case $q = \infty$, requires a natural modification: $T_\infty(\Lambda^p(w))$ will denote the class of all f which are continuous in $\mathbf{R}_+^{\mathbf{n}+1}$, for which $A_\infty(f) \in \Lambda^p(w)$ and such that $\|f_\varepsilon - f\|_{T_\infty(\Lambda^p(w))} \longrightarrow 0$, as $\varepsilon \longrightarrow 0^+$, where $f_\varepsilon(x,t) = f(x, t+\varepsilon)$.

REMARK 3.2.4:

(i) If $w(t) = t^{1/r-1/p}$ then we obtain the spaces $T_q^{r,p} = T_q(\Lambda^p(w))$ of **[AL-MI 1,2]** and **[BO-JO]**. In particular, if $p = r$ then $T_q^p = T_q(\Lambda^p(w))$ as in **[CO-ME-ST 2]**.

(ii) It is clear that $\|\cdot\|_{T_q(\Lambda^p(w))}$ is a quasi-norm. Moreover, if $1 \leq p,\ 1 \leq q$ and $w(t) = t^{1-1/p}/\varphi(t)$, where $\varphi \in B_\Psi$, then $\|\cdot\|_{T_q(\Lambda^p(w))}$ is equivalent to a norm, by Corollary 2.1.8.

(iii) The completeness of $T_q(\Lambda^p(w))$ follows as in **[CO-ME-ST 2]** and **[BO-JO]**.

For $q = \infty$ it is easy to show

$$|f(x,t)| \leq \inf_{y \in B(x,t)} A_\infty(f)(y),$$

which implies that

$$|f(x,t)|\chi_{B(x,t)}(y) \leq A_\infty(f)(y)\ ,$$

for all $y \in \mathbf{R^n}\cdot$ Taking the $\Lambda^p(w)$ (quasi) norm on both sides,

$$|f(x,t)|\ \|\chi_{B(x,t)}\|_{\Lambda^p(w)} \leq C\|f\|_{T_\infty(\Lambda^p(w))}\cdot$$

Therefore, since by the hypothesis on w, $0 < \|\chi_{B(x,t)}\|_{\Lambda^p(w)} < \infty$, we get

$$|f(x,t)| \leq \frac{C}{\|\chi_{B(x,t)}\|_{\Lambda^p(w)}} \|f\|_{T_\infty(\Lambda^p(w))}. \tag{1}$$

If $\{f_k\}_k$ is a Cauchy sequence in $T_\infty(\Lambda^p(w))$, (1) shows that there exists a function f such that $\lim_{k\to\infty} f_k(x,t) = f(x,t)$ for all $(x,t) \in \mathbf{R}^{\mathbf{n}+1}_+$. Thus

$$|f_k(x,t) - f(x,t)| = \lim_{j\to\infty} |f_k(x,t) - f_j(x,t)| \leq \lim_{j\to\infty} A_\infty(f_k - f_j)(y)$$

for all $y \in \mathbf{R}^{\mathbf{n}}$ and $(x,t) \in \Gamma(y)$. Thus,

$$A_\infty(f_k - f)(y) \leq \lim_{j\to\infty} A_\infty(f_k - f_j)(y).$$

Now taking "$\Lambda^p(w)$ norms" we obtain

$$\|f_k - f\|_{T_\infty(\Lambda^p(w))} \leq \| \lim_{j\to\infty} A_\infty(f_k - f_j)\|_{\Lambda^p(w)} \leq \lim_{j\to\infty} \|f_k - f_j\|_{T_\infty(\Lambda^p(w))}$$

by the Fatou property for rearrangement invariant spaces (see [**BE-SH**]).

Finally,

$$\lim_{k\to\infty} \|f_k - f\|_{T_\infty(\Lambda^p(w))} \leq \lim_{k\to\infty} \lim_{j\to\infty} \|f_k - f_j\|_{T_\infty(\Lambda^p(w))} = 0.$$

The case $q < \infty$ is handled similarly, except for using the following inequality: for any compact set $K \subset \mathbf{R}^{\mathbf{n}+1}_+$, there exists $x_0 \in \mathbf{R}^{\mathbf{n}}$, and a constant $C(K,q)$ such that

$$\left(\int_K |f(x,t)|^q \, dxdt\right)^{1/q} \leq C(K,q) A_q(f)(x_0) < \infty$$

for all $f \in T_q(\Lambda^p(w))$, (see [**BO-JO**]).

(§3) Atomic decompositions.

As in Chapter II, where we considered the atomic decomposition of the $\Lambda^q(w)$ spaces, we now want to obtain a suitable "discrete norm" in terms of a distribution function. This will be the right tool to get the needed estimates for the atoms introduced below. We achieve this using Theorem 2.1.6 and a rather technical result which has independent interest of its own.

DEFINITION 3.3.1. *Let $0 \leq x < y < \infty$. We define*

$$K(x,y) = \{k \in \mathbf{Z} \ : \ x \leq 2^k < y\}$$

and also

$$C(x,y) = \sum_{k \in K(x,y)} 2^k.$$

If $K(x,y) = \emptyset$ then $C(x,y)$ is understood to be zero. The following result shows the close connection between the function $C(x,y)$ and the linear measure of the interval (x,y), though as we will see later, they are far from being equivalent. The difficulties arise when $\sharp K(x,y) = 0$ or 1. $\sharp E$ denotes the cardinality of the set E.

LEMMA 3.3.2. *Let $0 \leq x < y < \infty$.*

(i) *If $\sharp K(x,y) \geq 1$ then $C(x,y) \geq \frac{2}{3}(y-x)$.*

(ii) *If $\sharp K(x,y) \neq 1$ then $C(x,y) \leq 3(y-x)$.*

(iii) *Suppose that there exists an $N \in \mathbf{Z}$ such that $\dfrac{2^{N+1}}{3} \leq y \leq 2^N$, then $C(x,y) \leq 3(y-x)$.*

PROOF:

(i) If $x = 0$ the inequality is trivial. Now, if $x \neq 0$ and $K(x,y) \geq 1$ then there exist $k \in \mathbf{Z}$ and $l = 0, 1, 2, \cdots$ such that

$$2^{k-1} < x \leq 2^k < \cdots < 2^{k+l} \leq y \leq 2^{k+l+1}.$$

Hence $C(x,y) = \sum_{j=k}^{k+l} 2^j = 2^{k+l+1} - 2^k = 2^k(2^{l+1}-1)$. Also $y - x \leq 2^{k+l+1} - 2^{k-1} = 2^k(2^{l+1} - \frac{1}{2})$. Thus, we need to show that $2^{l+1} - \frac{1}{2} \leq \frac{3}{2}(2^{l+1}-1)$, $l = 0, 1, \cdots$; i.e.,

$$\frac{2^x - \frac{1}{2}}{2^x - 1} \leq \frac{3}{2}, \qquad x \geq 1.$$

But

$$\frac{2^x - \frac{1}{2}}{2^x - 1} = \frac{2^x - 1}{2^x - 1} + \frac{\frac{1}{2}}{2^x - 1} \leq 1 + \frac{1}{2} = \frac{3}{2}.$$

(ii) If $\sharp K(x,y)=0$ then the inequality is trivially true.

Observe also that $\sharp K(x,y)=\infty$ if and only if $x=0$. Hence if $2^k<y\le 2^{k+1}$, some $k\in\mathbf{Z}$, we have

$$C(0,y)=\sum_{j=-\infty}^{k}2^j=2^{k+1}\le 2y\le 3y.$$

If $2\le\sharp K(x,y)<\infty$ then there are $k\in\mathbf{Z}$ and $l=1,2,\cdots$ so that

$$2^{k-1}<x\le 2^k<\cdots<2^{k+l}<y\le 2^{k+l+1}.$$

Hence,

$$C(x,y)=\sum_{j=k}^{k+l}2^j=2^k(2^{l+1}-1)$$

and $y-x\ge 2^{k+l}-2^k=2^k(2^l-1)$. So we need to show

$$\frac{2^x-1}{2^{x+1}-1}\ge\frac{1}{3},\qquad x\ge 1.$$

But

$$\frac{2^x-1}{2^{x+1}-1}=\frac{1}{2}\left(1-\frac{1}{2^{x+1}-1}\right)\ge\frac{1}{2}\left(1-\frac{1}{4-1}\right)=\frac{1}{3}.$$

(iii) Suppose that $\dfrac{2^{N+1}}{3}\le y\le 2^N$, some $N\in\mathbf{Z}$ .By (ii) we may assume that $\sharp K(x,y)=1$; i.e.,

$$2^{N-2}<x\le 2^{N-1}<\frac{2^{N+1}}{3}\le y\le 2^N.$$

Hence, $C(x,y)=2^{N-1}$ and

$$y-x\ge\frac{2^{N+1}}{3}-2^{N-1}=\frac{C(x,y)}{3}\cdot \quad\blacksquare$$

Since the proof of Theorem 3.3.4 is rather involved, we want to present first the simple case $n=2$, to make things clearer.

LEMMA 3.3.3. *If $0=x_0<x_1<x_2$ and $r_1>r_2>0$, then*

$$\frac{1}{2}\sum_{j=1}^{2}r_jC(x_{j-1},x_j)\le\sum_{j=1}^{2}r_j(x_j-x_{j-1})\le\frac{3}{2}\sum_{j=1}^{2}r_jC(x_{j-1},x_j).$$

PROOF: To prove the inequality on the left we may assume

$$0 = x_0 < 2^k < x_1 \leq 2^{k+1} < x_2 \leq 2^{k+2}$$

(any other possibility is already taken care of, by (ii) of the previous lemma). Then

$$\frac{1}{2}\sum_{j=1}^{2} r_j C(x_{j-1}, x_j) = \frac{1}{2}(r_1 2^{k+1} + r_2 2^{k+1}) = 2^k(r_1 + r_2) = 2^k r_1 (1 + \frac{r_2}{r_1}).$$

Thus, we want to show: $2^k(1 + \frac{r_2}{r_1}) \leq x_1 + \frac{r_2}{r_1}(x_2 - x_1)$. But

$$x_1 + \frac{r_2}{r_1}(x_2 - x_1) \geq x_1(1 - \frac{r_2}{r_1}) + \frac{r_2}{r_1}2^{k+1} \geq 2^k(1 - \frac{r_2}{r_1} + 2\frac{r_2}{r_1}) = 2^k(1 + \frac{r_2}{r_1})\cdot$$

To show the inequality on the right, we may also assume that $2^k < x_1 < x_2 \leq 2^{k+1}$, for some $k \in \mathbf{Z}$ (by (i) of the previous lemma). Thus, we have to show

$$x_1 + \frac{r_2}{r_1}(x_2 - x_1) \leq \frac{3}{2}2^{k+1}.$$

But $x_1 + r(x_2 - x_1) \leq x_1 + x_2 - x_1 = x_2 \leq 2^{k+1} < \frac{3}{2}2^{k+1}$. ∎

We now prove the result for a general n.

THEOREM 3.3.4. *Without loss of generality, we may assume $n > 2$. If we choose two sequences satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$, and $r_1 > \cdots > r_n > 0$, then,*

$$\frac{1}{2}\sum_{j=1}^{n} r_j C(x_{j-1}, x_j) \leq \sum_{j=1}^{n} r_j(x_j - x_{j-1}) \leq \frac{3}{2}\sum_{j=1}^{n} r_j C(x_{j-1}, x_j). \tag{2}$$

PROOF: Let $k \in \mathbf{Z}$ be so that $2^k < x_1 \leq 2^{k+1}$. Let $0 = l_0 \leq l_1 \leq \cdots \leq l_{(n-1)}$, $l_j \in \mathbf{Z}$, so that

$$2^{k+l_{(j-1)}} < x_j \leq 2^{k+l_{(j-1)}+1}, \qquad j = 1, \cdots, n. \tag{3}$$

Suppose first that $l_0 = l_1 = \cdots = l_{(n-1)} = 0$. Then, $C(0, x_1) = 2^{k+1}$ and $C(x_{j-1}, x_j) = 0$, $j = 2, \cdots, n$.

Hence,

$$\frac{1}{2}\sum_{j=1}^{n} r_j C(x_{j-1}, x_j) = \frac{r_1}{2} 2^{k+1} < r_1 x_1 < \sum_{j=1}^{n} r_j (x_j - x_{j-1}).$$

Let us now assume that there exists $1 \leq j_0 \leq n$ such that $l_{j_0} > 0$, (choose the least j_0 satisfying this property). Also choose l_{j_p} so that

$$l_{j_p} > l_{(j_p-1)}, \qquad \text{i.e. } l_{j_p} \geq l_{(j_p-1)} + 1, \qquad p = 0, \cdots, m. \tag{4}$$

Notice that this is equivalent to $C(x_{j_p}, x_{j_p+1}) > 0$.

Then, $1 \leq j_0 \leq \cdots \leq j_m \leq n-1$ and

$$\begin{aligned} 0 = l_1 = \cdots = l_{(j_0-1)} < l_{j_0} = l_{(j_0+1)} = \cdots = l_{(j_1-1)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ < l_{j_{(m-1)}} = l_{j_{(m-1)}+1} = \cdots = l_{(j_m-1)} < l_{j_m} = l_{j_m+1} = \cdots = l_{(n-1)}, \end{aligned} \tag{5}$$

in particular $l_{(j_p-1)} = l_{j_{(p-1)}}$). Now we calculate the infimum of

$$\sum_{j=1}^{n} r_j (x_j - x_{j-1}), \qquad 2^{k+l_{(j-1)}} < x_j \leq 2^{k+l_{(j-1)}+1}.$$

Claim: Setting $l_{j_{(-1)}} = 0$, then

$$\sum_{j=1}^{n} r_j (x_j - x_{j-1}) \geq r_1 2^k + \sum_{p=0}^{m} r_{j_p+1}(2^{k+l_{j_p}} - 2^{k+l_{j_{(p-1)}}}).$$

In fact,

$$\begin{aligned} &\sum_{j=1}^{n} r_j (x_j - x_{j-1}) = r_1 x_1 + \sum_{j=2}^{n} r_j (x_j - x_{j-1}) \\ \geq & r_1 2^k + r_1(x_1 - 2^k) + \sum_{j=2}^{j_0+1} r_j (x_j - x_{j-1}) + \cdots \\ &+ \sum_{j=j_{(m-2)}+2}^{j_{(m-1)}+1} r_j (x_j - x_{j-1}) + \sum_{j=j_{(m-1)}+2}^{j_m+1} r_j (x_j - x_{j-1}) \\ \geq & r_1 2^k + r_{j_0+1}(x_1 - 2^k) + r_{j_0+1}(x_{j_0+1} - x_1) + \cdots \end{aligned}$$

$$
\begin{aligned}
&\cdots + r_{j_{(m-1)}+1}(x_{j_{(m-1)}+1} - x_{j_{(m-2)}+1}) + r_{j_m+1}(x_{j_m+1} - x_{j_{(m-1)}+1})\\
\geq & r_1 2^k + r_{j_0+1}(x_{j_0+1} - 2^k) + \cdots + r_{j_{(m-1)}+1}(x_{j_{(m-1)}+1} - 2^{k+l_{j_{(m-1)}}}\\
&+ 2^{k+l_{j_{(m-1)}}} - x_{j_{(m-2)}+1}) + r_{j_m+1}(x_{j_m+1} - x_{j_{(m-1)}+1})\\
\geq & r_1 2^k + r_{j_0+1}(x_{j_0+1} - 2^k) + \cdots + r_{j_m+1}(x_{j_{(m-1)}+1} - 2^{k+l_{j_{(m-1)}}})\\
&+ r_{j_m+1}(x_{j_m+1} - x_{j_{(m-1)}+1})\\
\geq & r_1 2^k + \sum_{p=0}^{m} r_{j_p+1}(2^{k+l_{j_p}} - 2^{k+l_{j_{(p-1)}}}).
\end{aligned}
$$

On the other hand:

$$
\frac{1}{2}\sum_{j=1}^{n} r_j C(x_{j-1}, x_j) = \frac{1}{2}\left(r_1 2^{k+1} + \sum_{j=2}^{n} r_j 2^{k+1}(2^{l_{(j-1)}} - 2^{l_{(j-2)}})\right),
$$

since $C(x_{j-1}, x_j) = 2^{k+l_{(j-2)}+1} + \cdots + 2^{k+l_{(j-1)}}$. But

$$
\begin{aligned}
&\sum_{j=2}^{n} r_j(2^{l_{(j-1)}} - 2^{l_{(j-2)}}) =\\
&\sum_{j=2}^{j_0} r_j(2^{l_{(j-1)}} - 2^{l_{(j-2)}}) + \sum_{j=j_0+1}^{j_1} r_j(2^{l_{(j-1)}} - 2^{l_{(j-2)}}) + \cdots\\
&\cdots + \sum_{j=j_{(m-1)+1}}^{j_m} r_j(2^{l_{(j-1)}} - 2^{l_{(j-2)}}) + \sum_{j=j_m+1}^{n} r_j(2^{l_{(j-1)}} - 2^{l_{(j-2)}}) \qquad \text{(using (5))}\\
&= 0 + r_{j_0+1}(2^{l_{j_0}} - 2^{l_{(j_0-1)}}) + \cdots + r_{j_{(m-1)+1}}(2^{l_{j_{(m-1)}}} - 2^{l_{j_{(m-2)}}})\\
&+ r_{j_m+1}(2^{l_{j_m}} - 2^{l_{j_{(m-1)}}}) = \sum_{p=0}^{m} r_{j_p+1}(2^{l_{j_p}} - 2^{l_{j_{(p-1)}}}).
\end{aligned}
$$

Hence,

$$
\frac{1}{2}\sum_{j=1}^{n} r_j C(x_{j-1}, x_j) = \frac{1}{2}(r_1 2^{k+1} + \sum_{p=0}^{m} r_{j_p+1} 2^{k+1}(2^{l_{j_p}} - 2^{l_{j_{(p-1)}}})) \leq \sum_{j=1}^{n} r_j(x_j - x_{j-1}).
$$

Now we show the inequality on the right. As before, suppose

$$
2^{k+l_{(j-1)}} < x_j \leq 2^{k+l_{(j-1)}+1}, \qquad j = 1, \cdots, n.
$$

If $l_0 = l_1 = \cdots = l_{(n-1)} = 0$ then $C(0, x_1) = 2^{k+1}$ and $C(x_{j-1}, x_j) = 0$, for all $j = 2, \cdots, n$. Thus

$$\sum_{j=1}^{n} r_j (x_j - x_{j-1}) \leq r_1 x_n \leq r_1 2^{k+1} = \sum_{j=1}^{n} r_j C(x_{j-1}, x_j) \cdot$$

If there exists $1 \leq j_0 \leq n$ such that $l_{j_0} > 0$ then choose l_{j_p} as in (4) and (5). Then, by a similar argument to the one used in the first part, we get

$$\begin{aligned}
&\sum_{j=1}^{n} r_j (x_j - x_{j-1}) \\
&\leq r_1 2^{k+1} + r_{j_0+1}(2^{k+1+l_{(j_1-1)}} - 2^{k+1}) + \cdots + r_{j_m+1}(2^{k+1+l_{(n-1)}} - 2^{k+l_{(j_m-1)}+1}) \\
&\quad = r_1 2^{k+1} + r_{j_0+1}(2^{k+1+l_{j_0}} - 2^{k+1}) + \cdots + r_{j_m+1}(2^{k+1+l_{j_m}} - 2^{k+l_{(j_m-1)}+1}) \\
&\qquad = r_1 2^{k+1} + \sum_{p=0}^{m} r_{j_p+1} 2^{k+1} (2^{l_{j_p}} - 2^{l_{j_{(p-1)}}}) = \sum_{j=1}^{n} r_j C(x_{j-1}, x_j) \cdot \quad \blacksquare
\end{aligned}$$

REMARK 3.3.5: The theorem also holds for any $a > 1$, namely

$$\sum_{j=1}^{n} r_j \left(\sum_{\substack{k \in \mathbf{Z} \\ x_{j-1} \leq a^k < x_j}} a^k \right) \approx \sum_{j=1}^{n} r_j (x_j - x_{j-1}) \cdot$$

We now give the discrete version of Theorem 2.1.6.

THEOREM 3.3.6. *Suppose $\varphi \in B_\Psi$, $0 < q < \infty$ and set $w(t) = t^{1-1/q}/\varphi(t)$. Then*

$$\|f\|_{\Lambda^q(w)} \approx \left(\sum_{k \in \mathbf{Z}} (2^k w(\lambda_f(2^k)))^q \lambda_f(2^k) \right)^{1/q} \cdot \tag{6}$$

PROOF: As in the proof of Theorem 2.1.6 it suffices to show (6) for simple functions of the form $f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$, where the sets E_j are pairwise disjoint with finite measure and $a_1 > a_2 > \cdots > a_n > 0$. As the proof of the theorem shows

$$\|f\|^q_{\Lambda^q(w)} \approx \sum_{j=1}^{n} \left(\frac{m_j}{\varphi(m_j)} \right)^q (a_j^q - a_{j+1}^q)$$

where $m_j = \sum_{i=1}^{j} |E_i|$. Now

$$\sum_{k\in\mathbf{Z}} (2^k w(\lambda_f(2^k)))^q \lambda_f(2^k) = \sum_{k\in\mathbf{Z}} \left(2^{kq} \left(\frac{\lambda_f(2^k)}{\varphi(\lambda_f(2^k))} \right)^q \right)$$
$$= \sum_{j=1}^{n} \left(\sum_{\substack{k\in\mathbf{Z} \\ a_{j+1}\le 2^k < a_j}} \left(2^{kq} \left(\frac{m_j}{\varphi(m_j)} \right)^q \right) \right).$$

Hence, we only need to show

$$\sum_{j=1}^{n} \left(\frac{m_j}{\varphi(m_j)} \right)^q \left(\sum_{\substack{k\in\mathbf{Z} \\ a_{j+1}\le 2^k < a_j}} 2^{kq} \right) \approx \sum_{j=1}^{n} \left(\frac{m_j}{\varphi(m_j)} \right)^q (a_j^q - a_{j+1}^q). \tag{7}$$

Set

$$r_1 = \left(\frac{m_n}{\varphi(m_n)} \right)^q, \; r_2 = \left(\frac{m_{n-1}}{\varphi(m_{n-1})} \right)^q, \; \cdots \;, \; r_n = \left(\frac{m_1}{\varphi(m_1)} \right)^q,$$

$x_0 = 0,\ x_1 = a_n^q,\ \cdots\ ,\ x_n = a_1^q$ and $a = 2^q$. Then (7) is equivalent to showing that

$$\sum_{j=1}^{n} r_j \left(\sum_{\substack{k\in\mathbf{Z} \\ x_{j-1}\le a^k < x_j}} a^k \right) \approx \sum_{j=1}^{n} r_j (x_j - x_{j-1}). \tag{8}$$

Since m_j is an increasing sequence, then $\left(\frac{m_j}{\varphi(m_j)} \right)^q$ increases also and thus we get $r_1 > r_2 > \cdots > r_n > 0$. Also, $x_n > x_{n-1} > \cdots > x_1 > 0$ and $a > 1$. Hence we can now use Remark 3.3.5 to show that (8) holds true and this completes the proof. ∎

REMARK 3.3.7: Theorem 3.3.6 is trivially true if either w is an increasing function or a power function, since then it suffices to write the integral as a sum of integrals over disjoint dyadic intervals of the form $[2^k, 2^{k+1})$ and use basic estimates.

Our goal now is to find an atomic decomposition for functions in a suitable class of weighted Tent spaces. As in **[BO-JO]** we shall be most interested in the case of those spaces based on the Lorentz space $\Lambda(w)$. We first give the definition of atoms:

DEFINITION 3.3.8. *Suppose* $0 < p < q \le \infty$ *and* w *is a positive weight in* $\mathbf{R}^+$ *such that*

$$(9) \qquad 0 < \left(\int_0^t (w(s))^{pq/(q-p)}\, dt \right)^{(q-p)/pq} < \infty, \qquad \textit{for all } t > 0\cdot$$

We say that a measurable function $a : \mathbf{R}_+^{\mathbf{n}+1} \longrightarrow \mathbf{C}$ *is a* (w, q, p)*-atom if:*
(10) *There exists an open set* $\Omega \subset \mathbf{R}^{\mathbf{n}}$ *such that* $\operatorname{supp} a \subset T(\Omega)\cdot$

$$(11) \qquad \left(\int\limits_{T(\Omega)} |a(x,t)|^q \frac{dxdt}{t} \right)^{1/q} \le \left(\int_0^{|\Omega|} (w(t))^{pq/(q-p)} dt \right)^{(p-q)/pq}.$$

If $q = \infty$, (11) *has to be understood as*

$$\sup_{(x,t)\in T(\Omega)} |a(x,t)| \le \left(\int_0^{|\Omega|} (w(t))^p \; dt \right)^{-1/p}.$$

REMARK 3.3.9: Suppose $\varphi \in B_\Psi$, $1/(1-\beta_\varphi) < q$ and $w(t) = t^{1-1/p}/\varphi(t)$. Then the condition (11) is independent of p; i.e., a is a (w, q, p)-atom if and only if a is a $(w, q, 1)$-atom, up to a multiplicative constant. In fact:

$$\begin{aligned} \left(\int_0^r (w(t))^{pq/(q-p)} dt \right)^{(q-p)/pq} &= \left(\int_0^r \frac{t^{q(p-1)/(q-p)}}{\varphi^{pq/(q-p)}(t)}\, dt \right)^{(q-p)/pq} \\ &= \left(\int_0^r \left(\frac{t}{\varphi_q(t)} \right)^s \frac{dt}{t} \right)^{(q-p)/pq} \end{aligned}$$

where $s = p(q-1)/(q-p)$ and $\varphi_q(t) = \varphi^{q/(q-1)}(t)$. Since $1/(1-\beta_\varphi) < q$ then $\varphi_q \in B_\Psi$ and, hence, by Theorem 1.1.8-(13),

$$\left(\int_0^r \left(\frac{t}{\varphi_q(t)} \right)^s \frac{dt}{t} \right)^{(q-p)/pq} \approx r^{1/(q-1)} \varphi(r)$$

which is independent of $p\cdot$ In this case, we shall say that a is a (w, q)-atom, with no reference to the p parameter.

DEFINITION 3.3.10. (See **[CO-ME-ST 2]**). *Let* Ω *be an open set with finite measure,* $F = {}^c\Omega$ *and* $0 < \gamma < 1$. *We say that* $x \in \mathbf{R^n}$ *is a point of global* γ*-density with respect to* F *if*

$$\frac{|F \cap B(x)|}{|B(x)|} \geq \gamma$$

for all balls $B(x)$ *centered at* x.

We define F^* *to be the set of all points of global* γ*-density with respect to* F *and* $\Omega^* = {}^cF^*$.

REMARK 3.3.11: Notice that $\Omega^* = \{x : \; M(\chi_\Omega)(x) > 1 - \gamma\}$. In fact

$$x \in \Omega^* \iff \text{ there exist a ball } B(x) \text{ such that } \frac{|F \cap B(x)|}{|B(x)|} < \gamma$$
$$\iff \frac{|\Omega \cap B(x)|}{|B(x)|} > 1 - \gamma \iff M(\chi_\Omega)(x) > 1 - \gamma \cdot$$

Also, since F is closed $F^* \subset F$, hence $\Omega \subset \Omega^*$ and

$$|\Omega^*| \leq \frac{c}{1-\gamma} \|\chi_\Omega\|_1 = C_\gamma |\Omega| \cdot$$

Clearly, Ω^* is open, and thus, F^* is a closed set.

We shall also need the following estimate.

LEMMA 3.3.12. (See Lemma 2, **[CO-ME-ST 2]**). *Suppose* F *is a closed set whose complement has finite measure and* ϕ *is a non-negative function. Then there exists a* γ, $0 < \gamma < 1$ *sufficiently close to* 1 *such that*

$$\int_{\mathcal{R}(F^*)} \phi(y,t) t^n \, dydt \leq C_\gamma \int_F \left(\int_{\Gamma(x)} \phi(y,t) \, dydt \right) \, dx \tag{12}$$

where F^* *is the set of points of global* γ*-density with respect to* F.

We now give the atomic decomposition for the spaces $T_q(\Lambda^p(w))$.

THEOREM 3.3.13. *Suppose* $\varphi \in B_\Psi$, $0 < p \leq 1 < q < \infty$, *and* $q > 1/(1 - \beta_\varphi) \cdot$ *Set* $w(t) = t^{1-1/p}/\varphi(t) \cdot$ *Then*

$$T_q(\Lambda^p(w)) = \big\{ f : \; f \equiv \sum_j \; r_j a_j \big\}, \tag{13}$$

where a_j is a $(w,q)-$atom and $\sum_j |r_j|^p < \infty\cdot$ Moreover,

$$\|f\|_{T_q(\Lambda^p(w))} \approx \inf(\sum_j |r_j|^p)^{1/p},$$

where the infimum is taken over all sequences $\{r_j\}$ satisfying (13).

If $q=\infty$ then for all $f\in T_\infty(\Lambda^p(w))$ there exist a sequence $\{a_j\}_j$ of (w,∞)-atoms and a sequence $\{r_j\}_j$ such that

$$f \equiv \sum_j r_j a_j \qquad \text{and} \qquad (\sum_j |r_j|^p)^{1/p} \leq C\|f\|_{T_\infty(\Lambda^p(w))}\cdot$$

PROOF: Let a be a $(w,q)-$ atom, that is, there exists an open set $\Omega\subset\mathbf{R^n}$ such that supp $a\subset T(\Omega)$ and

$$\left(\int_{T(\Omega)} |a(y,t)|^q \frac{dydt}{t}\right)^{1/q} \leq \left(\int_0^{|\Omega|} (w(t))^{q'}dt\right)^{-1/q'}.$$

Since supp $A_q(a)\subset\Omega$ and using Hölder's inequality (with exponent $q/p>1$):

$$\begin{aligned}
\|a\|^p_{T_q(\Lambda^p(w))} &= \int_0^\infty ((A_q(a))^*(t)w(t))^p dt = \int_0^{|\Omega|} ((A_q(a))^*(t)w(t))^p dt \\
&\leq \left(\int_0^{|\Omega|} ((A_q(a))^*(t))^q dt\right)^{p/q} \left(\int_0^{|\Omega|} (w(t))^{pq/(q-p)} dt\right)^{(q-p)/q} \\
&= C\left(\int_\Omega ((A_q(a)(x))^q dx\right)^{p/q} \left(\int_0^{|\Omega|} (w(t))^{pq/(q-p)} dt\right)^{(q-p)/q} \\
&\leq C\left(\int_\Omega ((A_q(a)(x))^q dx\right)^{p/q} \left(\int_0^{|\Omega|} (w(t))^{q'} dt\right)^{p/q'}
\end{aligned}$$

by Remark 3.3.9. But

$$\begin{aligned}
&\int_\Omega ((A_q(a)(x))^q dx = \int_\Omega \left(\int_{\Gamma(x)} |a(y,t)|^q \frac{dydt}{t^{n+1}}\right) dx \\
&= \int_{\mathcal{R}(\Omega)} \left(\int_{B(y,t)\cap\Omega} dx\right) |a(y,t)|^q \frac{dydt}{t^{n+1}} = \int_{T(\Omega)} |B(y,t)\cap\Omega|\; |a(y,t)|^q \frac{dydt}{t^{n+1}} \\
&= C\int_{T(\Omega)} |a(y,t)|^q \frac{dydt}{t} \leq C\left(\int_0^{|\Omega|} (w(t))^{q'} dt\right)^{q/q'}.
\end{aligned}$$

Therefore, $\|a\|_{T_q(\Lambda^p(w))} \leq C$ and so, if $f \equiv \sum_j r_j a_j$ then

$$\|f\|_{T_q(\Lambda^p(w))} \leq C \sum_j |r_j| \leq C(\sum_j |r_j|^p)^{1/p}.$$

Conversely, let $f \in T_q(\Lambda^p(w))$. We want to find an atomic decomposition in terms of tents over open sets.

Let $k \in \mathbf{Z}$ and set

$$O_k = \left\{x \in \mathbf{R^n} : \ A_q(f)(x) > 2^k\right\}\cdot$$

Fix $0 < \gamma < 1$ and consider the set O_k^* as in Definition 3.3.10. It is clear that

$$\text{supp } f \subset \cup_k T(O_k^*), \qquad\qquad |O_k^*| \leq C_\gamma |O_k|\cdot$$

Define

$$r_k = c^{1/q} \frac{|O_k^*|}{\varphi(|O_k^*|)} 2^{k+1},$$

c will be chosen later, and

$$a_k(y,t) = r_k^{-1} \chi_{T(O_k^*)\setminus T(O_{k+1}^*)}(y,t) f(y,t)\cdot$$

Clearly, $f \equiv \sum_k r_k a_k$, and also supp $a_k \subset T(O_k^*)$. Thus it only remains to show (10) and the equivalence of norms:

Set $F = {}^cO_{k+1}$, $F^* = {}^cO_{k+1}^*$, so that $\mathcal{R}(F^*) = {}^cT(O_{k+1}^*)$ and set

$$\phi(y,t) = |f(y,t)|^q \chi_{T(O_k^*)}(y,t) t^{-1-n}.$$

Using Lemma 3.3.12, we get

$$\int_{T(O_k^*)} |a_k(y,t)|^q \frac{dydt}{t} = r_k^{-q} \int_{T(O_k^*)\setminus T(O_{k+1}^*)} |f(y,t)|^q \frac{dydt}{t}$$

$$= r_k^{-q} \int_{\mathcal{R}(F^*)} \phi(y,t) t^n dydt$$

$$\leq c_\gamma r_k^{-q} \int_{{}^cO_{k+1}} \left(\int_{\Gamma(x)} |f(y,t)|^q \chi_{T(O_k^*)}(y,t) \frac{dydt}{t^{n+1}} \right) dx$$

$$
\begin{aligned}
&\leq \frac{c_\gamma \varphi^q(|O_k^*|)}{c2^{q(k+1)}|O_k^*|^q} \int_{O_k^* \cap\ {}^c O_{k+1}} \left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dydt}{t^{n+1}} \right) dx \\
&= \frac{c_\gamma \varphi^q(|O_k^*|)}{c2^{q(k+1)}|O_k^*|^q} \int_{O_k^* \cap\ {}^c O_{k+1}} (A_q(f)(x))^q \, dx \\
&\leq \frac{c_\gamma \varphi^q(|O_k^*|)}{c2^{q(k+1)}|O_k^*|^q} |O_k^*| 2^{q(k+1)} = \frac{c_\gamma}{c} |O_k^*|^{1-q} \varphi^q(|O_k^*|).
\end{aligned}
$$

As we proved in Remark 3.3.9, we have

$$
\int_0^{|O_k^*|} (w(t))^{q'} \, dt \approx \frac{|O_k^*|}{\varphi^{q'}(|O_k^*|)}. \tag{14}
$$

Thus

$$
\begin{aligned}
\left(\int_{T(O_k^*)} |a_k(y,t)|^q \frac{dydt}{t} \right)^{1/q} &\leq \left(\frac{c_\gamma}{c}\right)^{1/q} \left(\frac{|O_k^*|^{1-1/q}}{\varphi(|O_k^*|)} \right)^{-1} \\
&\leq \left(\frac{A_\gamma}{c}\right)^{1/q} \left(\int_0^{|O_k^*|} (w(t))^{q'} \, dt \right)^{-1/q'}.
\end{aligned}
$$

Hence, if we choose $c = A_\gamma$, then a satisfies (11); i.e., a is a (w,q)-atom. Finally,

$$
\begin{aligned}
\sum_k |r_k|^p &= C \sum_k \left(2^{k+1} \frac{|O_k^*|}{\varphi(|O_k^*|)} \right)^p \leq C \sum_k \left(2^k \frac{\lambda_{A_q(f)}(2^k)}{\varphi(\lambda_{A_q(f)}(2^k))} \right)^p \\
&= C \sum_k \left(2^k w(\lambda_{A_q(f)}(2^k)) \right)^p \lambda_{A_q(f)}(2^k) \quad \text{(by Theorem 3.3.6)} \\
&\leq C \|A_q(f)\|^p_{\Lambda^p(w)} = C \|f\|^p_{T_q(\Lambda^p(w))}.
\end{aligned}
$$

For the case $q = \infty$ we define r_k and a_k as above, with

$$
O_k = \left\{ x \in \mathbf{R^n} : \ A_\infty(f)(x) > 2^k \right\}.
$$

Now we can easily get the norm estimate for a_k since

$$
\sup_{(x,t) \in T(O_k^*)} |a_k(x,t)| \leq r_k^{-1} \sup_{(x,t) \notin T(O_{k+1})} |f(x,t)| \leq \frac{\varphi(|O_k^*|)}{|O_k^*|} \approx \left(\int_0^{|O_k|} w(t) \, dt \right)^{-1}
$$

and hence a_k is a (w,∞)-atom. To get the norm estimate for $\{r_j\}_j$ one now argues as before. ∎

REMARK 3.3.14:

(i) We see from the proof of the theorem that for any (w, q, p)-atom a, where w only satisfies condition (10) and any $0 < p \leq 1 < q$ we get

$$\|a\|_{T_q(\Lambda^p(w))} \leq C$$

uniformly on $a\cdot$

(ii) From the proof of the second part we can also obtain an atomic decomposition of the space $T_q(\Lambda^p(w))$ for all $0 < p < q\cdot$

(iii) If $p = 1$, $r \geq 1$ and $w(t) = t^{1/r-1}$ then $T_q(\Lambda^p(w)) = T_q^{r,1}$ and the theorem gives the atomic decomposition proved in [**BO-JO**].

(§4) Duality and Carleson measures.

The duality results for the Tent spaces T_q^p, $1 \leq p < \infty$, $1 < q \leq \infty$ and $T_q^{p,1}$, $1 \leq p < q < \infty$, have been proved in [**CO-ME-ST 2**], [**AL-MI 1**] and also in [**BO - JO**]. The most interesting case is given when $q = \infty$, since then we get a family of measures that extend the definition of a Carleson measure given in [**CAR**]. This turns out to be extremely useful to obtain different pointwise and norm estimates for certain operators.

We find now the dual spaces of the weighted Tent spaces $T_q(\Lambda^p(w))$ and for the case $q = \infty$ we introduce a new class of Carleson-type measures.

THEOREM 3.4.1. *Suppose $\varphi \in B_\Psi$, $1 \leq p, q < \infty$, and $w(t) = t^{1-1/p}/\varphi(t)$. Then the pairing*

$$\langle f, g\rangle = \int_{\mathbf{R}_+^{\mathbf{n}+1}} f(x,t)g(x,t)\frac{dxdt}{t}, \quad f \in T_q(\Lambda^p(w)),\ g \in T_{q'}(\Lambda^{p'}(1/w))$$

shows $T_{q'}(\Lambda^{p'}(1/w))$ to be equivalent to the Banach space dual of $T_q(\Lambda^p(w))$.

PROOF: First, by Corollary 2.1.9 implies $(\Lambda^p(w))^* = \Lambda^{p'}(1/w)$. Let $f \in T_q(\Lambda^p(w))$ and $g \in T_{q'}(\Lambda^{p'}(1/w))$. Then

$$\begin{aligned}
&\int_{\mathbf{R}^{\mathbf{n}+1}_+} |f(y,t)g(y,t)| \frac{dydt}{t} \\
&= C \int_{\mathbf{R}^{\mathbf{n}}} \left(\int_{|x-y|<t} |f(y,t)g(y,t)| \frac{dydt}{t^{n+1}} \right) dx \qquad \text{(Hölder's)} \\
&\leq C \int_{\mathbf{R}^{\mathbf{n}}} A_q(f)(x) A_{q'}(g)(x)\, dx \qquad \text{(Theorem 2.1.7)} \\
&\leq C \|A_q(f)\|_{\Lambda^p(w)} \|A_{q'}(g)\|_{\Lambda^{p'}(1/w)} = C \|f\|_{T_q(\Lambda^p(w))} \|g\|_{T_{q'}(\Lambda^{p'}(1/w))}.
\end{aligned}$$

For the converse, we only give the proof for the cases where p and $1/(1-\beta_\varphi)$ are less than q. The other cases follow by showing that $T_q(\Lambda^p(w))$ is a reflexive space (see **[CO-ME-ST 2]** for more details).

Let l be a bounded linear functional on $T_q(\Lambda^p(w))$, K a compact subset of $\mathbf{R}^{\mathbf{n}+1}_+$, and let f be a function with supp $f \subset K$. Then there exists $\tilde{K} \subset \mathbf{R}^{\mathbf{n}}$, ($\tilde{K}$ depends only on K), compact, such that supp $A_q(f) \subset \tilde{K}$. Hence if $r_\varepsilon = 1+\varepsilon$ and $r_\varepsilon{}'$ is its conjugate exponent, we have

$$\begin{aligned}
\|f\|^p_{T_q(\Lambda^p(w))} = \|A_q(f)\|^p_{\Lambda^p(w)} &\leq \int_0^{|\tilde{K}|} \left((A_q(f))^*(t) w(t) \right)^p dt \\
&\leq \left(\int_0^{|\tilde{K}|} (w(t))^{pr_\varepsilon} dt \right)^{1/r_\varepsilon} \left(\int_0^{|\tilde{K}|} \left((A_q(f))^*(t) \right)^{pr_\varepsilon{}'} dt \right)^{1/r_\varepsilon{}'}.
\end{aligned}$$

Now choose ε so that:

(15) $\dfrac{pr_\varepsilon}{pr_\varepsilon - \varepsilon}\beta_\varphi < 1$, (notice that $\dfrac{pr_\varepsilon}{pr_\varepsilon - \varepsilon} \longrightarrow 1$ as $\varepsilon \longrightarrow 0$) and

(16) $r_\varepsilon{}' \dfrac{p}{q} > 1$, ($r_\varepsilon{}' \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$).

Hence,

$$\left(\int_0^{|\tilde{K}|} (w(t))^{pr_\varepsilon} dt \right)^{1/r_\varepsilon} = \left(\int_0^{|\tilde{K}|} \left(\frac{t}{\varphi^{pr_\varepsilon/(pr_\varepsilon-\varepsilon)}(t)} \right)^{pr_\varepsilon - \varepsilon} \frac{dt}{t} \right)^{1/r_\varepsilon} = C_K$$

since $\varphi^{pr_\varepsilon/(pr_\varepsilon-\varepsilon)} \in B_\Psi$ by (15) and using Proposition 1.1.8-(13).

On the other hand

$$\int_0^{|\tilde{K}|} \left((A_q(f))^*(t)\right)^{pr_\varepsilon'} dt = \int_{\tilde{K}} (A_q(f)(x))^{pr_\varepsilon'} dx$$

$$= \int_{\tilde{K}} \left(\int_{\Gamma(x)\cap K} |f(y,t)|^q \frac{dydt}{t^{n+1}} \right)^{pr_\varepsilon'/q} dx$$

(by (16) and Jensen's inequality)

$$\leq C_K \int_{\tilde{K}} \int_{\Gamma(x)\cap K} |f(y,t)|^{pr_\varepsilon'} \frac{dydt}{t^{n+1}} \, dx$$

$$\leq C_K \int_K \left(\int_{B(y,t)} dx \right) |f(y,t)|^{pr_\varepsilon'} \frac{dydt}{t^{n+1}}$$

$$\leq C_K \int_K |f(y,t)|^{pr_\varepsilon'} \frac{dydt}{t} = C_K \|f\|^{pr_\varepsilon'}_{L^{pr_\varepsilon'}(K)}.$$

Therefore,

$$\|f\|_{T_q(\Lambda^p(w))} \leq C_K \|f\|_{L^{pr_\varepsilon'}(K)}.$$

Thus, if we set $a = pr_\varepsilon'$, then l induces a bounded linear functional on $L^a(K)$ and is thus representable by a $g = g_K \in L^{a'}(K)$. Taking an increasing family of such K which exhaust $\mathbf{R}^{\mathbf{n}+1}_+$ produces a function g in $\mathbf{R}^{\mathbf{n}+1}_+$ which is locally in $L^{a'}$ and such that

$$l(f) = \int_{\mathbf{R}^{\mathbf{n}+1}_+} f(x,t) g(x,t) \, \frac{dxdt}{t},$$

whenever $f \in T_q(\Lambda^p(w))$ and f has compact support.

For K an arbitrary compact subset of $\mathbf{R}^{\mathbf{n}+1}_+$ set $g_K = g\chi_K$. Then it suffices to show that

$$\text{(17)} \qquad \|g_K\|_{T_{q'}(\Lambda^{p'}(1/w))} = \|A_{q'}(g_K)\|_{\Lambda^{p'}(1/w)} \leq C\|l\|,$$

where C is independent of K. Set $r = p(q-1)/(q-p)$ and $u = w^{q'}$. Without loss of generality let us assume that g is non-negative. Then

$$\|A_{q'}(g_K)\|^{q'}_{\Lambda^{p'}(1/w)} = \sup_\phi \int_{\mathbf{R}^\mathbf{n}} A^{q'}_{q'}(g_K)(x)\phi(x) \, dx$$

where the supremum is taken over all functions satisfying $\|\phi\|_{\Lambda^r(u)} \leq 1$, (this holds since $(\Lambda^r(u))^* = \Lambda^{p'/q'}(1/w^{q'}))$. Set

$$M_t(\phi)(x) = \sup_{|s|\leq t} \frac{1}{s^n} \int_{|y|\leq s} \phi(x-y)\,dy\cdot$$

Then

$$\int_{\mathbf{R^n}} A_{q'}^{q'}(g_K)(x)\phi(x)\,dx = \int_{\mathbf{R^n}} \int_{|x-y|<t} |g_K(y,t)|^{q'} \frac{dydt}{t^{n+1}} \phi(x)\,dx$$

$$= \int_{\mathbf{R}_+^{\mathbf{n+1}}} |g_K(y,t)|^{q'} M_t(\phi)(y) \frac{dydt}{t} = \langle f_\phi, g\rangle,$$

where

$$f_\phi(y,t) = |g_K(y,t)|^{q'-1} M_t(\phi)(y)\cdot$$

Therefore,

$$\|A_{q'}(g_K)\|_{\Lambda^{p'}(1/w)}^{q'} \leq \sup_\phi \langle f_\phi, g\rangle \leq \sup_\phi C\|f_\phi\|_{T_q(\Lambda^p(w))}\|l\|$$

with f_ϕ as above. But,

$$A_q(f_\phi)(x) = \left(\int_{\Gamma(x)} |g_K(y,t)|^{q(q'-1)} \left(M_t(\phi)(y)\right)^q \frac{dydt}{t^{n+1}}\right)^{1/q}$$

$$\leq CM(\phi)(x)\left(A_{q'}(g_K)(x)\right)^{q'/q},$$

where M is the Hardy-Littlewood maximal operator.

We now recall the following result (see **[KR-PE-SE]** p. 67): Let F, G be two measurable functions. Then

$$(FG)^*(t_1+t_2) \leq F^*(t_1)G^*(t_2), \qquad t_1,\ t_2 > 0\cdot$$

In particular, if $t_1 = t_2 = \dfrac{t}{2}$ then

(18) $$(FG)^*(t) \leq F^*(t/2)G^*(t/2)\cdot$$

We also recall that w satisfies the Δ_2-condition; that is, there exists a $C>1$ such that

(19) $$\frac{1}{C}w(t) \leq w(2t) \leq Cw(t), \qquad \text{for all } \ t>0\cdot$$

Now

$$\|f_\phi\|_{T_q(\Lambda^p(w))} = \|A_q(f_\phi)\|_{\Lambda^q(w)}$$

$$\leq C\|M(\phi)\,(A_{q'}(g_K))^{q'/q}\|_{\Lambda^p(w)}$$

$$= C\left(\int_0^\infty \left(\left(M(\phi)\,(A_{q'}(g_K))^{q'/q}\right)^*(t)w(t)\right)^p dt\right)^{1/p} \quad \text{(by (18))}$$

$$\leq C\left(\int_0^\infty \left((M(\phi))^*(t/2)w^{q'}(t)\right)^p \left(\left((A_{q'}(g_K))^{q'/q}\right)^*(t/2)w^{1-q'}(t)\right)^p dt\right)^{1/p}$$

(Hölder's with index $\frac{q-1}{q-p}$ and (19))

$$\leq C\|M(\phi)\|_{\Lambda^r(u)}\|A_{q'}(g_k)\|^{q'-1}_{\Lambda^{p'}(1/w)}.$$

Since $q > 1/(1-\beta_\varphi)$ then $q'\beta_\varphi < 1$ and hence M is a bounded operator on $\Lambda^r(u)$ (see [**AR-MU**]). Therefore,

$$\|f_\phi\|_{T_q(\Lambda^p(w))} \leq C\|\phi\|_{\Lambda^r(u)}\|A_{q'}(g_K)\|^{q'-1}_{\Lambda^{p'}(1/w)} \leq C\|A_{q'}(g_K)\|^{q'-1}_{\Lambda^{p'}(1/w)}.$$

Thus,

$$\|A_{q'}(g_k)\|_{\Lambda^{p'}(1/w)} \leq C\|l\| \cdot \quad \blacksquare$$

We now consider the duality of the Tent spaces when $q = \infty$. As in the case of L^∞, the dual spaces consist of measures defined on measurable subsets of $\mathbf{R}^{\mathbf{n}+1}_+$.

DEFINITION 3.4.2. *Suppose w is a weight on $\mathbf{R}^+$ that is locally in $L(\mathbf{R}^+)$, and μ is a Borel measure on $\mathbf{R}^{\mathbf{n}+1}_+$. We say that μ is a w-Carleson measure, and write $\mu \in V^w$, if*

$$\|\mu\|_{V^w} = \sup_\Omega \frac{|\mu|(\widehat{\Omega})}{\displaystyle\int_0^{|\Omega|} w(t)\,dt} < \infty, \tag{20}$$

where the supremum is taken over all open and bounded $\Omega \subset \mathbf{R}^\mathbf{n}$ and $\widehat{\Omega}$ is the tent over Ω.

EXAMPLES 3.4.3:

(i) If $w(t) = t^{\alpha-1}$, $\alpha > 0$, condition (20) gives

$$|\mu|(\widehat{\Omega}) \leq C|\Omega|^\alpha$$

which coincides with the definition of a Carleson measure of order α given, for example, in [**AL - MI 1**]. In particular for $\alpha = 1$, these are the measures defined originally by Carleson (see [**CAR**]). In this case we will denote V^α by $V^{t^{\alpha-1}}$.

(ii) If $\delta_{(x_0,t_0)}$ is the Dirac mass at $(x_0,t_0) \in \mathbf{R}^{\mathbf{n}+1}_+$ then for any $\Omega \subset \mathbf{R}^{\mathbf{n}}$ we have

$$\delta_{(x_0,t_0)}(\widehat{\Omega}) = \begin{cases} 1 & \text{if } B(x_0,t_0) \subset \Omega \\ 0 & \text{otherwise} \end{cases}.$$

Hence $|\Omega| \geq c_n t_0^n$ if $B(x_0,t_0) \subset \Omega$, and so

$$1 \leq \left(\int_0^{c_n t_0^n} w(t)\, dt \right)^{-1} \left(\int_0^{|\Omega|} w(t)\, dt \right) \cdot$$

Therefore,

$$\delta_{(x_0,t_0)}(\widehat{\Omega}) \leq \left(\int_0^{c_n t_0^n} w(t)\, dt \right)^{-1} \left(\int_0^{|\Omega|} w(t)\, dt \right)$$

that is

$$\|\delta_{(x_0,t_0)}\|_{V^w} \leq \left(\int_0^{c_n t_0^n} w(t)\ dt \right)^{-1} \cdot$$

(iii) Let $W \in L^p(\mathbf{R}^{\mathbf{n}})$, $W \geq 0$, $1 \leq p \leq \infty$ and σ a positive finite measure in $\mathbf{R}^+$. Let $w(t) = W^*(t)$ be the non-increasing rearrangement of W and set

$$d\mu(x,t) = W(x)\, dx d\sigma(t) \cdot$$

Then $\mu \in V^w$ and $\|\mu\|_{V^w} \leq \|\sigma\|$. In fact, since $W \in L^p(\mathbf{R}^{\mathbf{n}})$, then $w \in L^p(\mathbf{R}^+)$ and, hence, $w \in L^1_{loc}(\mathbf{R}^+)$. Now if $\Omega \subset \mathbf{R}^{\mathbf{n}}$ then

$$\begin{aligned} \int_{\widehat{\Omega}} d\mu(x,t) &= \int_\Omega \left(\int_{\{0<t<d(x,\ {}^c\Omega)\}} d\sigma(t) \right) W(x)\, dx \\ \leq &\|\sigma\| \int_\Omega W(x)\, dx \leq \|\sigma\| \int_0^{|\Omega|} w(t)\, dt \cdot \end{aligned}$$

REMARK 3.4.4:

(i) If $\alpha \geq 1$ then $\|\mu\|_{V^\alpha}$ is equivalent to taking the supremum in (20) over all cubes $Q \subset \mathbf{R}^{\mathbf{n}}$. In fact, given $\Omega \subset \mathbf{R}^{\mathbf{n}}$ we can find a Whitney decomposition $\{Q_k\}_k$ of Ω; that is, Q_k is a cube contained in Ω, $Q_k \cap Q_l = \emptyset$ if $k \neq l$, $\Omega = \cup_k Q_k$ and

$d(Q_k, {}^c\Omega) \le Cl_k$, where l_k is the length of Q_k. Set $\widetilde{Q}_k$ to be the cube centered at x_k (the center of Q_k) and whose side-length equals $(C+3)l_k$. Then

$$\widehat{\Omega} \subset \bigcup_k \widehat{\widetilde{Q}_k}.$$

In fact, let $(x,t) \in \widehat{\Omega}$. Then $B(x,t) \subset \Omega$. Let Q_k be such that $x \in Q_k$. If $z \in B(x,t)$, then

$$|z - x_k| \le |z - x| + |x - x_k| \le t + l_k.$$

Since $B(x,t) \subset \Omega$ we have $t \le 2l_k + Cl_k$. Therefore, $|z - x_k| \le (C+3)l_k$ and $B(x,t) \subset \widetilde{Q}_k$; that is, $(x,t) \in \widehat{\widetilde{Q}_k}$. We see that

$$|\mu|(\widehat{\Omega}) \le |\mu|\left(\bigcup_k \widehat{\widetilde{Q}_k}\right) \le \sum_k |\mu|(\widehat{\widetilde{Q}_k}) \le \sum_k C_\mu |\widetilde{Q}_k|^\alpha \le C_\mu \sum_k |Q_k|^\alpha \qquad (\alpha \ge 1)$$
$$\le C_\mu \left(\sum_k |Q_k|\right)^\alpha = C_\mu |\Omega|^\alpha.$$

(ii) Let $\varphi \in B_\Psi$ and $w(t) = \dfrac{1}{\varphi(t)}$. Then condition (20) is equivalent to

$$|\mu|(\widehat{\Omega}) \le C\|\mu\|_{V^w} \frac{|\Omega|}{\varphi(|\Omega|)},$$

since by Theorem 1.1.8-(13)

$$\int_0^{|\Omega|} w(t)\, dt = \int_0^{|\Omega|} \frac{t}{\varphi(t)} \frac{dt}{t} \le C \frac{|\Omega|}{\varphi(|\Omega|)}.$$

(iii) If we consider $\Lambda(w)$ as a rearrangement invariant space, then the quantity in the denominator in (20),

$$\int_0^s w(t)\, dt,$$

is the Fundamental function of $\Lambda(w)$; that is, the norm of the characteristic function of any set whose measure equals s (see [**KR-PE-SE**] for more details).

THEOREM 3.4.5. *Suppose w is a weight in $\mathbf{R}^+$ that is locally in $L(\mathbf{R}^+)$. Then*

$$(T_\infty(\Lambda(w)))^* = V^w \cdot$$

PROOF: Let $\mu \in V^w$ and $f \in T_\infty(\Lambda(w))$. We want to show that

$$\left| \int_{\mathbf{R}_+^{\mathbf{n}+1}} f(x,t)\, d\mu(x,t) \right| \leq C\|\mu\|_{V^w} \|f\|_{T_\infty(\Lambda(w))} \cdot$$

Let us assume that μ is a positive measure and $f \geq 0$. Then

$$\left| \int_{\mathbf{R}_+^{\mathbf{n}+1}} f(x,t)\, d\mu(x,t) \right| = \int_0^\infty f_\mu^*(s)\, ds,$$

where f_μ^* is the rearrangement of f with respect to μ; that is, if

$$\mu_f(s) = \mu(\{(x,t) \ : \ f(x,t) > s\})$$

then

$$f_\mu^* = \inf\{r \ : \ \mu_f(r) \leq s\} \cdot$$

But

$$\{(x,t) \ : \ f(x,t) > s\} \subset T(\{y \ : \ A_\infty(f)(y) > s\}) \cdot \tag{21}$$

In fact, if $f(x,t) > s$ and $z \in B(x,t)$ then

$$A_\infty(f)(z) = \sup_{|x'-z|<t'} f(x',t') \geq f(x,t) > s,$$

and so $z \in \{y \ : \ A_\infty(f)(y) > s\}$; thus, $B(x,t) \subset \{y \ : \ A_\infty(f)(y) > s\}$, which is equivalent to (21). Since f is a continuous function $\{y \ : \ A_\infty(f)(y) > s\}$ is an open set. Now

$$\begin{aligned} \mu_f(s) &\leq \mu(T(\{y \ : \ A_\infty(f)(y) > s\})) \qquad \text{(by (20))} \\ &\leq \|\mu\|_{V^w} \int_0^{|\{y \ : \ A_\infty(f)(y)>s\}|} w(r)\, dr = \|\mu\|_{V^w} \int_0^{m_{A_\infty(f)}(s)} w(r)\, dr \end{aligned}$$

where m is the Lebesgue measure in $\mathbf{R}^\mathbf{n}$. Since $A_\infty(f)$ is a continuous function then

$$\left(A_\infty(f)\right)^*(s) = \left(m_{A_\infty(f)}\right)^{-1}(s)$$

(if g is an invertible function, we will denote by g^{-1} the inverse function of g). Set

$$W(t) = \int_0^t w(r)\, dr$$

and

$$H(t) = W\left(m_{A_\infty(f)}(t)\right)\cdot$$

Then

$$H^{-1}(s) = (A_\infty(f))^*\left(W^{-1}(s)\right)\cdot$$

Now,

$$f_\mu^*(s) = \inf\{r \ : \ \mu_f(r) \le s\} \le \inf\{r \ : \ \|\mu\|_{V^w} H(r) \le s\} = H^{-1}\left(\frac{s}{\|\mu\|_{V^w}}\right)\cdot$$

Thus,

$$\int_0^\infty f_\mu^*(s)\, ds \le \int_0^\infty H^{-1}\left(\frac{s}{\|\mu\|_{V^w}}\right) ds$$

$$= \int_0^\infty (A_\infty(f))^*\left(W^{-1}\left(\frac{s}{\|\mu\|_{V^w}}\right)\right) ds \ \begin{cases} t = W^{-1}\left(\dfrac{s}{\|\mu\|_{V^w}}\right) \\ dt = \dfrac{1}{W'(t)}\dfrac{ds}{\|\mu\|_{V^w}} = \dfrac{1}{\|\mu\|_{V^w} w(t)} ds \end{cases}$$

$$= \int_0^\infty \|\mu\|_{V^w} (A_\infty(f))^*(t) w(t)\, dt = \|\mu\|_{V^w}\|f\|_{T_\infty(\Lambda(w))}.$$

Conversely, if $l \in (T_\infty(\Lambda(w)))^*$ by exhausting $\mathbf{R}_+^{\mathbf{n}+1}$ with an increasing sequence of compact sets K_n, we obtain a measure μ on $\mathbf{R}_+^{\mathbf{n}+1}$. If f is a continuous function with compact support, then

$$l(f) = \int_{\mathbf{R}_+^{\mathbf{n}+1}} f(x,t)\, d\mu(x,t)\cdot$$

Without loss of generality, we may assume that $\mu \ge 0$. Let Ω be any open and bounded set in $\mathbf{R}^{\mathbf{n}}$. If $\{f_m\}$ is a sequence of continuous functions with compact support such that $f_m \uparrow \chi_{\widehat{\Omega}}$. Then

$$\left|\int_{\mathbf{R}_+^{\mathbf{n}+1}} f_m(x,t)\, d\mu(x,t)\right| \le \|l\|\ \|f_m\|_{T_\infty(\Lambda(w))},$$

hence,

$$\left| \int_{\widehat{\Omega}} d\mu(x,t) \right| = \mu(\widehat{\Omega}) \leq \|l\| \, \|A_\infty(\chi_{\widehat{\Omega}})\|_{\Lambda(w)}\cdot$$

But

$$A_\infty(\chi_{\widehat{\Omega}})(x) = \chi_\Omega(x),$$

thus

$$\|A_\infty(\chi_{\widehat{\Omega}})\|_{\Lambda(w)} = \int_0^\infty (\chi_\Omega)^*(s) w(s)\, ds = \int_0^\infty \chi_{(0,|\Omega|)}(s) w(s)\, ds = \int_0^{|\Omega|} w(s)\, ds\cdot$$

That is,

$$\mu(\widehat{\Omega}) \leq \|l\| \int_0^{|\Omega|} w(s)\, ds \cdot \quad \blacksquare$$

COROLLARY 3.4.6. *If $0 < p < \infty$, then*

$$\left(T_\infty^{p,1}\right)^* = V^{1/p}.$$

PROOF: Set $w(t) = t^{1/p-1}$. Then $w \in L^1_{loc}(\mathbf{R}^+)$ and $T_\infty(\Lambda(w)) = T_\infty^{p,1}$, since $\Lambda(w) = L^{p,1}$. Thus, by the theorem and Example 3.4.3-(i)

$$\left(T_\infty^{p,1}\right)^* = (T_\infty(\Lambda(w)))^* = V^w = V^{1/p}\cdot \quad \blacksquare$$

REMARK 3.4.7: The previous corollary gives a generalization of Theorem (5.1) in [**AL - MI 1**], where $1 \leq p < \infty$. Our proof differs from theirs in the fact that we consider norms in terms of rearrangement of functions instead of the distribution function.

We want to also mention another duality result closely related to the above theorem, (see [**CO-ME-ST 2**] and [**AL-MI 1**]).

THEOREM 3.4.8. $(T_\infty^p)^* = V^{1/p}$, $0 < p \leq 1$.

(§5) Maximal functions over general domains.

Several authors have studied of the boundedness of maximal operators defined by means of general subsets. For example, in [**NA-ST**], a Hardy-Littlewood type operator is associated with a collection of subsets $\Omega(x) \subset \mathbf{R}_+^{\mathbf{n}+1}$, $x \in \mathbf{R}^{\mathbf{n}}$. The natural way to define the balls for these sets is to take the subset of $\Omega(x)$ at level t, that is the set of points $z \in \mathbf{R}^{\mathbf{n}}$ so that $(z,t) \in \Omega(x)$. Our idea is to also replace the sets $\Gamma(x)$ in the definition of the Tent spaces by a more general family of subsets of $\mathbf{R}_+^{\mathbf{n}+1}$. We will restrict ourselves to finding the dual space for a particular case and make some comments related to radial maximal functions and to the announcement of some negative results.

DEFINITION 3.5.1. *Let* $\Omega = \{\Omega(x)\}_{x \in \mathbf{R}^{\mathbf{n}}}$ *be a collection of measurable subsets,* $\Omega(x) \subset \mathbf{R}_+^{\mathbf{n}+1}$. *For a measurable function* f *in* $\mathbf{R}_+^{\mathbf{n}+1}$ *we define the maximal function of* f *with respect to* Ω *as*

$$A_\infty^\Omega(f)(x) = \sup_{(y,t) \in \Omega(x)} |f(y,t)|\cdot$$

We will always assume that Ω *is chosen so that* $A_\infty^\Omega(f)$ *is a measurable function. We also define*

$$T_\Omega = T_{\infty,\Omega}^1 = \left\{ f : \ A_\infty^\Omega(f) \in L^1(\mathbf{R}^{\mathbf{n}}) \right\},$$

where f *satisfies the conditions of Definition 3.2.3 for the case* $q = \infty$, *and*

$$\|f\|_{T_\Omega} = \|A_\infty^\Omega(f)\|_{L^1(\mathbf{R}^{\mathbf{n}})}\cdot$$

REMARK 3.5.2: It is clear that if $\Omega(x) = \Gamma(x)$ then T_Ω is precisely the Tent space T_∞^1. If $\Omega(x) = \{(x,t) : \ t > 0\}$ then $A_\infty^\Omega(f)$ is the radial maximal function of f.

DEFINITION 3.5.3. *Suppose* $\Omega = \{\Omega(x)\}_{x \in \mathbf{R}^{\mathbf{n}}}$ *is as above and* F *is any subset of* $\mathbf{R}^{\mathbf{n}}$. *We define the tent over* F, *with respect to* Ω, *as*

$$\widehat{F_\Omega} = \mathbf{R}_+^{\mathbf{n}+1} \setminus \bigcup_{x \notin F} \Omega(x)\cdot$$

For a measure μ in $\mathbf{R}_+^{\mathbf{n}+1}$ we say that μ is an Ω-Carleson measure and write $\mu \in V_\Omega$ if

$$\|\mu\|_{V_\Omega} = \sup_O \frac{|\mu|\left(\widehat{O_\Omega}\right)}{|O|} < \infty,$$

where the supremum is taken over all open and bounded $O \subset \mathbf{R}^\mathbf{n}$.

REMARK 3.5.4: If $\Omega(x) = \Gamma(x)$ then $\widehat{F_\Omega} = \widehat{F}$, the usual tent over F. If we choose $\Omega(x) = \{(x,t) : \ t > 0\}$ then $\widehat{F_\Omega} = F \times \mathbf{R}^+$ and it is denoted by $C(F)$.

LEMMA 3.5.5. *Suppose $F \subset \mathbf{R}^\mathbf{n}$ and $\Omega = \{\Omega(x)\}_{x \in \mathbf{R}^\mathbf{n}}$ are as above. Then*

(i) $A_\infty^\Omega\left(\chi_{\widehat{F_\Omega}}\right)(x) \leq \chi_F(x)$ *for all $x \in \mathbf{R}^\mathbf{n}$.*

(ii) $A_\infty^\Omega\left(\chi_{\widehat{F_\Omega}}\right)(x) = \chi_F(x)$ *if and only if $\Omega(x) \cap \widehat{F_\Omega} \neq \emptyset$ for all $x \in F$.*

PROOF:

(i) Observe that

$$\chi_{\widehat{F_\Omega}}(y,t) = \begin{cases} 1, & \text{if } (y,t) \notin \Omega(z), \text{ for all } z \notin F \\ 0, & \text{otherwise} \end{cases}. \tag{22}$$

Suppose $x \notin F$. Then if $(y,t) \in \Omega(x)$ we have that $\chi_{\widehat{F_\Omega}}(y,t) = 0$ (by (22)), and this shows (i).

(ii) $A_\infty^\Omega\left(\chi_{\widehat{F_\Omega}}\right)(x) = \chi_F(x)$ if and only if for all $x \in F$, $A_\infty^\Omega\left(\chi_{\widehat{F_\Omega}}\right)(x) = 1$ if and only if there exists $(y,t) \in \Omega(x)$ such that $(y,t) \in \widehat{F_\Omega}$ if and only if $\Omega(x) \cap \widehat{F_\Omega} \neq \emptyset\cdot$ ∎

THEOREM 3.5.6. *Let $\Omega = \{\Omega(x)\}_{x \in \mathbf{R}^\mathbf{n}}$ be as above. Then*

$$(T_\Omega)^* = V_\Omega\cdot$$

PROOF: As in the proof of Theorem 3.4.5 it is easy to see that if $l \in (T_\Omega)^*$ then there exists a measure μ on $\mathbf{R}_+^{\mathbf{n}+1}$ that represents l over functions in T_Ω with compact support. Thus, if $O \subset \mathbf{R}^\mathbf{n}$ is open and bounded (by the lemma)

$$|\mu|\left(\widehat{O_\Omega}\right) = \int_{\mathbf{R}_+^{\mathbf{n}+1}} \chi_{\widehat{O_\Omega}}(x,t)\, d|\mu|(x,t) \leq C_\mu \|\chi_{\widehat{O_\Omega}}\|_{T_\Omega}$$

$$\leq C_\mu \|\chi_O\|_{L^1} = C_\mu |O|\cdot$$

Conversely, if $f \in T_\Omega$ and if we set $F^\lambda = \{y \in \mathbf{R^n} :\ A_\infty^\Omega(f)(y) > \lambda\}$, then

$$\{(x,t) \in \mathbf{R}_+^{\mathbf{n}+1} :\ |f(x,t)| > \lambda\} \subset \widehat{F_\Omega^\lambda}. \tag{23}$$

In fact, if $|f(x,t)| > \lambda$, $A_\infty^\Omega(f)(z) \leq \lambda$, implies that $(x,t) \notin \Omega(z)$ and, hence,

$$(x,t) \in \mathbf{R}_+^{\mathbf{n}+1} \setminus \left(\bigcup_{z \notin F^\lambda} \Omega(z) \right) = \widehat{F_\Omega^\lambda}.$$

Therefore, for $\mu \in V_\Omega$, we have

$$\begin{aligned}
&\left| \int_{\mathbf{R}_+^{\mathbf{n}+1}} f(x,t)\, d\mu(x,t) \right| \\
&\leq \int_0^\infty |\mu|(\{(x,t) \in \mathbf{R}_+^{\mathbf{n}+1} :\ |f(x,t)| > \lambda\})\, d\lambda \text{ (by (23))} \\
&\leq \int_0^\infty |\mu|(\widehat{F_\Omega^\lambda})\, d\lambda \leq \|\mu\|_{V_\Omega} \int_0^\infty |F^\lambda|\, d\lambda = \|\mu\|_{V_\Omega} \|f\|_{T_\Omega} \cdot \ \blacksquare
\end{aligned}$$

As was proved in **[FE-ST]** the non-tangential maximal function and the radial maximal function of Poisson integrals of functions (distributions) in the Hardy space $H^p(\mathbf{R^n})$ have an equivalent L^p-"norm", $p > 0$. This leads us to consider how this result could be extended for all functions in the Tent spaces T_∞^p relative to both cones $\Gamma(x)$ and lines $\{(x,t) :\ t > 0\}$. From the point of view of the dual spaces we see that the latter is a much bigger space than the former. We give the details in what follows.

EXAMPLE 3.5.7: As in Example 3.4.3-(i), given a family of sets $\Omega = \{\Omega(x)\}_{x \in \mathbf{R^n}}$ and $\alpha > 0$, we can introduce the definition of (α, Ω)-Carleson measure; that is, a measure μ satisfying

$$|\mu| \left(\widehat{O_\Omega} \right) \leq C|O|^\alpha \qquad (\mu \in V_\Omega^\alpha) \tag{24}$$

for all open and bounded $O \subset \mathbf{R^n}$. In particular, if $\Omega(x) = \{(x,t) : t > O\}$ then $\widehat{O_\Omega} = C(O) = O \times \mathbf{R}^+$ and condition (24), for $\alpha \geq 1$, is equivalent to checking the inequality only for cubes $Q \subset \mathbf{R^n}$. Let us denote $V_{rad}^\alpha = V_\Omega^\alpha$, where $\Omega(x)$ is the vertical line above x.

First suppose that $0 < \alpha \le 1$, $f \in L^{1/(1-\alpha)}(\mathbf{R^n})$ and σ is a positive finite measure in $\mathbf{R}^+$. Then

$$d\mu(x,t) = f(x)\,dx\,d\sigma(t) \in V^\alpha_{rad}\cdot$$

In fact, if $O \subset \mathbf{R^n}$ then

$$\left| \int_{C(O)} d\mu(x,t) \right| \le \left(\int_O |f(x,t)|\,dx \right) \left(\int_0^\infty d\sigma(t) \right) \le \|\sigma\|\ \|f\|_{L^{1/(1-\alpha)}} |O|^\alpha\cdot$$

An example of a measure that is in V^α but not in V^α_{rad} is the Dirac mass at the point $(x_0, t_0) \in \mathbf{R}^{\mathbf{n}+1}_+$. This follows by considering a collection of cubes converging to x_0.

For the case $\alpha > 1$ we get the remarkable fact that

$$V^\alpha_{rad} = \{0\}\cdot$$

To show this fix a cube $Q \subset \mathbf{R^n}$ and $N \in \mathbf{Z}^+$. Decompose Q in 2^{nN} subcubes Q_i such that $\overset{\circ}{Q}_i \cap \overset{\circ}{Q}_j = \emptyset$, $i \ne j$, $Q = \cup_i Q_i$ and $|Q_i| = \dfrac{|Q|}{2^{nN}}$. Now, if $\mu \in V^\alpha_{rad}$ we have

$$\begin{aligned} |\mu|(C(Q)) &\le |\mu|\left(\cup_i C(Q_i)\right) \le \sum_i |\mu|(C(Q_i)) \le C_\mu \sum_i |Q_i|^\alpha \\ =&C_\mu \sum_{i=1}^{2^{nN}} \frac{|Q|^\alpha}{2^{\alpha n N}} = C_\mu |Q|^\alpha 2^{nN(1-\alpha)} \longrightarrow 0, \quad \text{as } N \longrightarrow \infty\cdot \end{aligned}$$

Hence $\mu \equiv 0$.

(§6) Interpolation of Tent spaces and Carleson measures.

As we did for the weighted Lorentz spaces, we will now study the interpolation results of Tent spaces for the real method with a function parameter and the complex method for families, and hence also for the method of Calderón. We first extend the results of **[CO-ME-ST 2]** to parameters in the class B_Ψ and then, by reiteration, we get the result for a general Tent space. Once the intermediate spaces for this method are known, we can apply our reiteration theorem, to families of Tent spaces parameterized by the unit circle (see Theorem 2.3.7), to find the interpolated spaces for the method in [**CCRSW 1**]. As is indicated in the previous chapter, one can easily get the interpolation results for the method of Calderón. Finally, by a duality argument, we can also interpolate the spaces of w-Carleson measures in some particular cases. We now give the proof of the main theorem.

THEOREM 3.6.1.
(i) If $1 < p_0 < p_1 < \infty$, $1 < q, r < \infty$, and $\varphi \in B_\Psi$, then

$$\left(T_q^{p_0}, T_q^{p_1}\right)_{\varphi,r} = T_q(\Lambda^r(w)),$$

where

$$w(t) = \frac{t^{1/p_0 - 1/r}}{\varphi(t^{1/p_0 - 1/p_1})}.$$

(ii) If $1 < p_0 < p_1 < \infty$, $1 \leq r < \infty$ and $\varphi \in B_\Psi$, then

$$(T_\infty^{p_0}, T_\infty^{p_1})_{\varphi,r} = T_\infty(\Lambda^r(w)),$$

with w as before.

PROOF:
(i) We first notice that, by definition, we have

$$A_q : T_q^{p_i} \longrightarrow L^{p_i} \qquad i = 0, 1$$

is a bounded, positive, sublinear operator. Since L^{p_i} is a lattice space, then A_q satisfies the interpolation property (see [**ME 2**]); that is,

$$A_q : \left(T_q^{p_0}, T_q^{p_1}\right)_{\varphi,r} \longrightarrow (L^{p_0}, L^{p_1})_{\varphi,r}$$

for all $\varphi \in B_\Psi$ and $0 < r \leq \infty$. Thus, by Theorem 2.3.13 and the observation made in the proof of Theorem 2.4.3, we have

$$A_q : \left(T_q^{p_0}, T_q^{p_1}\right)_{\varphi,r} \longrightarrow \Lambda^r(w),$$

boundedly, with

$$w(t) = \frac{t^{1/p_0 - 1/r}}{\varphi(t^{1/p_0 - 1/p_1})}\cdot$$

Thus,

$$(25) \qquad \left(T_q^{p_0}, T_q^{p_1}\right)_{\varphi, r} \hookrightarrow T_q(\Lambda^r(w))\cdot$$

For the converse, we make use of (25), for the conjugate indices; that is, if $\phi \in B_\Psi$ then

$$\left(T_{q'}^{p_0{}'}, T_{q'}^{p_1{}'}\right)_{\phi, r'} \hookrightarrow T_{q'}(\Lambda^{r'}(\tilde{w})),$$

where

$$\tilde{w}(t) = \frac{t^{1/p_0{}' - 1/r'}}{\phi(t^{1/p_0{}' - 1/p_1{}'})}\cdot$$

By duality, we have

$$(26) \qquad \left(\left(T_{q'}^{p_0{}'}, T_{q'}^{p_1{}'}\right)_{\phi, r'}\right)^* \hookleftarrow \left(T_{q'}(\Lambda^{r'}(\tilde{w}))\right)^*\cdot$$

Using Theorem 2.4 in [**PER**], we obtain

$$(27) \qquad \left(\left(T_{q'}^{p_0{}'}, T_{q'}^{p_1{}'}\right)_{\phi, r'}\right)^* = \left(T_q^{p_0}, T_q^{p_1}\right)_{\psi, r},$$

where $\psi(t) = 1/\phi(1/t)$ and, by Theorem 3.4.1, we also have that

$$(28) \qquad \left(T_{q'}(\Lambda^{r'}(\tilde{w}))\right)^* = T_q(\Lambda^r(1/\tilde{w}))\cdot$$

If we choose $\phi(t) = 1/\varphi(1/t)$, which is easily seen to be in B_Ψ, we obtain $\psi(t) = \varphi(t)$ and

$$\frac{1}{\tilde{w}(t)} = \frac{1}{\varphi(t^{-(1/p_0{}' - 1/p_1{}')})} / t^{1/p_0{}' - 1/r'} = \frac{t^{1/p_0 - 1/r}}{\varphi(t^{1/p_0 - 1/p_1})} = w(t)\cdot$$

Therefore, by (26), (27) and (28)

$$\left(T_q^{p_0}, T_q^{p_1}\right)_{\varphi, r} \hookleftarrow T_q(\Lambda^r(w))\cdot$$

(ii) Assume now that $q = \infty$, $1 \le r < \infty$. A first step is to interpolate $(T_\infty^p, L^\infty)_{\phi, s}$. To do this we use the following equivalence for the Peetre K- functional (see [**AL-MI 1**])

$$K\left(t, f; T_\infty^p, L^\infty\right) \approx K\left(t, A_\infty(f); L^p, L^\infty\right)\cdot$$

Hence,

$$\begin{aligned}\|f\|_{(T^p_\infty,L^\infty)_{\phi,s}} &= \left(\int_0^\infty \left(\frac{1}{\phi(t)}K\left(t,f;T^p_\infty,L^\infty\right)\right)^s \frac{dt}{t}\right)^{1/s}\\ &\approx \left(\int_0^\infty \left(\frac{1}{\phi(t)}K\left(t,A_\infty(f);L^p,L^\infty\right)\right)^s \frac{dt}{t}\right)^{1/s}\\ &= \|A_\infty(f)\|_{(L^p,L^\infty)_{\phi,s}} = \|f\|_{T_\infty(\Lambda^s(v))},\end{aligned}$$

where,

$$v(t) = \frac{t^{1/p-1/s}}{\phi(t^{1/p})} \tag{29}$$

since $(L^p, L^\infty)_{\phi,s} = \Lambda^s(v)$. In particular, if $\phi(t) = t^\theta$, $0<\theta<1$, we obtain

$$(T^p_\infty, L^\infty)_{\theta,p_\theta} = T^{p_\theta}_\infty,$$

with $1/p_\theta = (1-\theta)/p$. Thus,

$$T^{p_i}_\infty = \left(T^1_\infty, L^\infty\right)_{\frac{1}{p_i{}'},p_i}, \qquad i=0,1\cdot$$

By the reiteration theorem in **[PER]**

$$\begin{aligned}(T^{p_0}_\infty, T^{p_1}_\infty)_{\varphi,r} &= \left(\left(T^1_\infty, L^\infty\right)_{\frac{1}{p_0{}'},p_0}, \left(T^1_\infty, L^\infty\right)_{\frac{1}{p_1{}'},p_1}\right)_{\varphi,r}\\ &= \left(T^1_\infty, L^\infty\right)_{\phi,r} \text{ (where } \phi(t) = t^{1/p_0{}'}\varphi(t^{1/p_1{}'-1/p_0{}'}))\\ &= T_\infty(\Lambda^r(w)),\end{aligned}$$

where, by (29),

$$w(t) = \frac{t^{1-1/r}}{t^{1/p_0{}'}\varphi(t^{1/p_1{}'-1/p_0{}'})} = \frac{t^{1/p_0-1/r}}{\varphi(t^{1/p_0-1/p_1})}\cdot \quad \blacksquare$$

COROLLARY 3.6.2. *Suppose $\varphi_0, \varphi_1 \in B_\Psi$ and if we set $\tau(t) = \varphi_1(t)/\varphi_0(t)$, then*

$$\left|\frac{t\tau'(t)}{\tau(t)}\right| \geq \varepsilon > 0, \tag{30}$$

for all $t>0\cdot$ If $1<r_0, r_1, r, q<\infty$, $\varphi \in B_\Psi$ and $w_i(t) = t^{1-1/r_i}/\varphi_i(t)$, then

$$\left(T_q\left(\Lambda^{r_0}(w_0)\right), T_q\left(\Lambda^{r_1}(w_1)\right)\right)_{\varphi,r} = T_q\left(\Lambda^r(w)\right),$$

where
$$w(t)=\frac{t^{1-1/r}}{\varphi_o(t)\varphi(\tau(t))}\cdot$$
A similar result holds if $q=\infty$ *and* $1\leq r_i<\infty,\ i=0,1\cdot$

PROOF: This is a simple consequence of the theorem. We first find $1<p_0,p_1<\infty$ and $\phi_i\in B_\Psi$ such that

$$\left(T_q^{p_0},T_q^{p_1}\right)_{\phi_i,r_i}=T_q\left(\Lambda^{r_i}(w_i)\right)\cdot \tag{31}$$

Fix p_0 sufficiently close to 1 and p_1 close to ∞ and set $\phi_i(t)=t^{\frac{1}{\alpha}(\frac{1}{p_0}-1)}\varphi_i(t^{1/\alpha})$, where $\alpha=1/p_0-1/p_1\cdot$ Then (31) is given by the theorem. Therefore, by (30) and reiteration, and by Theorem 3.6.1,

$$\left(T_q\left(\Lambda^{r_0}(w_0)\right),T_q\left(\Lambda^{r_1}(w_1)\right)\right)_{\varphi,r}=\left(T_q^{p_0},T_q^{p_1}\right)_{\phi_0\varphi\left(\frac{\phi_1}{\phi_0}\right),r}=T_q(\Lambda^r(w)),$$

where,
$$w(t)=\frac{t^{1/p_0-1/r}}{\phi_0(t^\alpha)\varphi(\phi_1(t^\alpha)/\phi_0(t^\alpha))}=\frac{t^{1-1/r}}{\varphi_0(t)\varphi(\varphi_1(t)/\varphi_0(t))}\cdot \quad \blacksquare$$

COROLLARY 3.6.3. *Suppose* $\varphi_0,\varphi_1\in B_\Psi$ *satisfy condition* (30), $1<r_0,r_1,q<\infty$ *and* $0<\theta<1$. *Set* $1/r_\theta=(1-\theta)/r_0+\theta/r_1$ *and* $w_i=t^{1-1/r_i}/\varphi_i(t)\cdot$ *Then*

$$\left(T_q\left(\Lambda^{r_0}(w_0)\right),T_q\left(\Lambda^{r_1}(w_1)\right)\right)_{\theta,r_\theta}=T_q\left(\Lambda^{r_\theta}\left(w_0^{1-\theta}w_1^\theta\right)\right)\cdot$$

PROOF: Set $\varphi(t)=t^\theta$ and apply the previous corollary. Observe that
$$w(t)=\frac{t^{1-1/r_\theta}}{\varphi_0(t)\dfrac{\varphi_1^\theta(t)}{\varphi_0^\theta(t)}}=\frac{t^{(1-1/r_0)(1-\theta)}}{\varphi_0^{1-\theta}(t)}\frac{t^{(1-1/r_1)\theta}}{\varphi_1^\theta(t)}=w_0^{1-\theta}(t)w_1^\theta(t)\cdot \quad \blacksquare$$

COROLLARY 3.6.4. *Suppose* $\varphi_0,\varphi_1\in B_\Psi$ *satisfy condition* (30) *and* $w_i=1/\varphi_i(t)$, *for* $i=0,1\cdot$ *If* $\varphi\in B_\Psi$, *then*

$$\left(V^{w_0},V^{w_1}\right)_{\varphi,\infty}=V^w,$$

where $w(t)=w_0(t)\varphi(w_1(t)/w_0(t))\cdot$ *For the particular case,* $\varphi(t)=t^\theta$, *we obtain*

$$\left(V^{w_0},V^{w_1}\right)_{\theta,\infty}=V^{w_0^{1-\theta}w_1^\theta}.$$

PROOF: We know, by Theorem 3.4.5, that

$$V^{w_i} = \left(T_\infty\left(\Lambda(w_i)\right)\right)^*, \qquad i = 0, 1\cdot$$

By the duality theorem in **[PER]**,

$$\left(V^{w_0}, V^{w_1}\right)_{\varphi,\infty} = \Big(\left(T_\infty\left(\Lambda(w_0)\right), T_\infty\left(\Lambda(w_1)\right)\right)_{\phi,1}\Big)^*$$

where $\phi(t) = 1/\varphi(1/t)\cdot$ By Corollary 3.6.2,

$$\left(T_\infty\left(\Lambda(w_0)\right), T_\infty\left(\Lambda(w_1)\right)\right)_{\phi,1} = T_\infty\left(\Lambda(w)\right),$$

where,

$$w(t) = \frac{1}{\varphi_0(t)\phi(\varphi_1(t)/\varphi_0(t))} = \frac{\varphi(\varphi_0(t)/\varphi_1(t))}{\varphi_0(t)} = w_0(t)\varphi(w_1(t)/w_0(t))\cdot$$

Again, by Theorem 3.4.5,

$$\left(T_\infty\left(\Lambda(w)\right)\right)^* = V^w\cdot \quad \blacksquare$$

We move now to complex interpolation. The main tool we use is Theorem 2.3.7. First, we parameterize a family of Tent spaces, according to the index set $\mathbf{T}$. As in Theorem 2.3.14, we fix two measurable functions $r : \mathbf{T} \longrightarrow (p_0, p_1)$, where $1 < p_0 < p_1 < \infty$, and $w : \mathbf{T} \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ such that the function

$$\varphi_\theta(t) = \frac{t^{1-1/r(\theta)}}{w(\theta, t)}$$

belongs to B_Ψ, for every $\theta \in \mathbf{T}$. We will also need to assume that the auxiliary function $\phi_\theta(t) = t^{(1/p_0-1)/\alpha}\varphi_\theta(t^{1/\alpha})$, $\alpha = 1/p_0 - 1/p_1$ satisfies condition (A) of Definition 2.3.6. Fix $1 < q < \infty$. Our family of spaces is then defined by

$$\mathcal{T} = \left\{T_q\left(\Lambda^{r(\theta)}(w(\theta,\cdot))\right)\right\}_{\theta\in\mathbf{T}}\cdot \tag{32}$$

Recall that

$$\left[T_q\left(\Lambda^{r(\theta)}(w(\theta,\cdot))\right)\right]_z, \qquad |z| < 1$$

denotes the intermediate space given by the complex interpolation method for families, as in **[CCRSW 1]**.

THEOREM 3.6.5. *Let $\mathcal{T}$ be defined as in (32). Then $\mathcal{T}$ is an interpolation family of Banach spaces and*

$$\left[T_q\left(\Lambda^{r(\theta)}(w(\theta,\cdot))\right)\right]_z = T_q\left(\Lambda^{r(z)}(w(z,\cdot))\right)$$

with equivalent norms, where

$$\frac{1}{r(z)} = \int_{\mathbf{T}} \frac{1}{r(\theta)} P_z(\theta)\, d\theta$$

and

$$w(z,t) = \exp\left(\int_{\mathbf{T}} (\log w(\theta,t)) P_z(\theta)\, d\theta\right) \cdot$$

PROOF: By Theorem 3.6.1 we have

$$T_q\left(\Lambda^{r(\theta)}(w(\theta,\cdot))\right) = \left(T_q^{p_0}, T_q^{p_1}\right)_{\phi_\theta, r(\theta)}$$

and, hence, Theorem 2.3.7 shows that $\mathcal{T}$ is an interpolation family. Again, using this theorem we get (by Theorem 3.6.1)

$$\begin{aligned}\left[T_q\left(\Lambda^{r(\theta)}(w(\theta,\cdot))\right)\right]_z &= \left(T_q^{p_0}, T_q^{p_1}\right)_{\phi_z, r(z)} \\ &= T_q\left(\Lambda^{r(z)}(w(z,\cdot))\right),\end{aligned}$$

where

$$w(z,t) = \frac{t^{1/p_0 - 1/r(z)}}{\phi_z(t^{1/p_0 - 1/p_1})}, \qquad \frac{1}{r(z)} = \int_{\mathbf{T}} \frac{1}{r(\theta)} P_z(\theta)\, d\theta$$

and

$$\phi_z(t) = \exp\left(\int_{\mathbf{T}} (\log \phi(\theta,t)) P_z(\theta)\, d\theta\right) \cdot$$

Finally,

$$\begin{aligned} w(z,t) = \frac{t^{1/p_0 - 1/r(z)}}{\phi_z(t^{1/p_0 - 1/p_1})} &= t^{1/p_0} \frac{\exp\left(\int_{\mathbf{T}} \frac{-1}{r(\theta)} (\log t) P_z(\theta)\, d\theta\right)}{\exp\left(\int_{\mathbf{T}} \log(t^{1/p_0 - 1} \varphi_\theta(t)) P_z(\theta)\, d\theta\right)} \\ &= \exp\left(\int_{\mathbf{T}} \left(\left(1 - \frac{1}{r(\theta)}\right) \log t - \log \varphi_\theta(t)\right) P_z(\theta)\, d\theta\right) \\ &= \exp\left(\int_{\mathbf{T}} \log\left(\frac{t^{1-1/r(\theta)}}{\varphi_\theta(t)}\right) P_z(\theta)\, d\theta\right) = \exp\left(\int_{\mathbf{T}} \log w(\theta,t) P_z(\theta)\, d\theta\right) \cdot \ \blacksquare \end{aligned}$$

COROLLARY 3.6.6. *Suppose $\varphi_0, \varphi_1 \in B_\Psi$, $1 < r_0 < r_1 < \infty$ and $1 < q < \infty$. Fix $0 < \theta < 1$ and set $w_i(t) = t^{1-1/r_i}/\varphi_i(t)$, $i = 0, 1$. Then,*

$$[T_q(\Lambda^{r_0}(w_0)), T_q(\Lambda^{r_1}(w_1))]_\theta = T_q\left(\Lambda^{r_\theta}(w_0^{1-\theta} w_1^\theta)\right),$$

where $1/r_\theta = (1-\theta)/r_0 + \theta/r_1 \cdot$

PROOF: We use the same argument given in the proof of Corollary 2.3.11. Fix two numbers $1 < p_0 < r_0 < r_1 < p_1 < \infty$ and set $\phi_i(t) = t^{(1/p_0-1)/\alpha}\varphi_i(t^{1/\alpha})$, $1/\alpha = 1/p_0 - 1/p_1 \cdot$ Then,

$$T_q(\Lambda^{r_i}(w_i)) = \left(T_q^{p_0}, T_q^{p_1}\right)_{\phi_i, r_i} \cdot$$

Hence,

$$\begin{aligned}[T_q(\Lambda^{r_0}(w_0)), T_q(\Lambda^{r_1}(w_1))]_\theta &= \left[\left(T_q^{p_0}, T_q^{p_1}\right)_{\phi_0, r_0}, \left(T_q^{p_0}, T_q^{p_1}\right)_{\phi_1, r_1}\right]_\theta \\ &= \left(T_q^{p_0}, T_q^{p_1}\right)_{\phi_\theta, r_\theta} = T_q(\Lambda^{r_\theta}(w)),\end{aligned}$$

where,

$$\phi_\theta(t) = (\phi_0(t))^{1-\theta}(\phi_1(t))^\theta, \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$$

and $w(t) = t^{1/p_0 - 1/r_\theta}/\phi_\theta(t^\alpha)$. It only remains to show that $w \equiv w_0^{1-\theta} w_1^\theta$. But

$$\phi_\theta(t^\alpha) = t^{(1/p_0-1)(1-\theta)}\varphi_0^{1-\theta}(t)t^{\theta(1/p_0-1)}\varphi_1^\theta(t),$$

and so,

$$\begin{aligned}w(t) &= \frac{t^{1/p_0-(1-\theta)/r_0-\theta/r_1}}{t^{(1/p_0-1)(1-\theta)}\varphi_0^{1-\theta}(t)t^{\theta(1/p_0-1)}\varphi_1^\theta(t)} \\ &= \frac{t^{(1-1/r_0)(1-\theta)}}{\varphi_0^{1-\theta}(t)} \frac{t^{(1-1/r_1)\theta}}{\varphi_1^\theta(t)} = w_0^{1-\theta}(t) w_1^\theta(t) \cdot \quad \blacksquare\end{aligned}$$

Chapter IV

Applications to the theory of Hardy spaces and weighted inequalities

In this final chapter, we will present a variety of results, dealing with the Hardy spaces, that are closely related to the theory developed in the previous chapters. These spaces have been studied intensively from different points of view, (see, for example, [**FE-ST**]). We will be concerned with their definition in terms of maximal functions, specially those given by a not necessarily smooth kernel. The main tools are the properties that the Tent spaces $T_\infty(\Lambda(w))$ satisfy.

(§1) The maximal Hardy spaces.

In [**WE**], G. Weiss proposed the consideration of the following spaces:

For a function $\varphi \in L^1 \cap L^\infty$ with $\int_{\mathbf{R^n}} \varphi(x)\, dx \neq 0$ we can define the non-tangential maximal function, with respect to φ, of a function f as

$$m_\varphi(f)(x) = \sup_{|x-y|<t} |(f * \varphi_t)(y)| \tag{1}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$ is the usual dilation of φ. Observe that in our previous notation

$$m_\varphi(f) \equiv A_\infty(f * \varphi_t)\cdot \tag{2}$$

The Hardy space is defined as

$$H^1_\varphi = \left\{ f \in L^1(\mathbf{R^n}) :\ m_\varphi(f) \in L^1(\mathbf{R^n}) \right\}$$

with

$$\|f\|_{H^1_\varphi} = \|m_\varphi(f)\|_{L^1(\mathbf{R^n})} \left(= \|f * \varphi_t(\cdot)\|_{T^1_\infty} \right) \cdot$$

In **[FE-ST]** the authors showed that if φ satisfies certain smoothness conditions; e.g., φ has compact support and satisfies the Dini condition, then H^1_φ coincides with the space real-H^1. That not every φ satisfies this equality is easily proved by choosing φ to be the characteristic function of the unit ball in $\mathbf{R^n}$. It turns out that the space H^1_φ consists only of the zero function.

The question raised in **[WE]** was whether there exists a function φ for which H^1_φ is neither the trivial space nor H^1. A complete answer, for $n = 1$, was given in **[UC-WI]** where the authors showed that it is possible to find such a function. Moreover they proved that if we know that $H^1_\varphi \neq \{0\}$, then H^1_φ contains the so called special atom function

$$a(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ -1, & \text{if } -1 < x < 0 \\ 0, & \text{otherwise} \end{cases}$$

which is equivalent to saying that H^1_φ contains the space of special atoms of O'Neil and de Souza, which coincides with the Besov space $\dot{B}^{0,1}_1$, (see **[SO]**). Another treatment of this fact can be also found in **[HA]**.

We generalize this result to higher dimensions, for which we use the discrete characterization of the Besov spaces given in **[FR-JA 1]**. We also show how the minimality property of the space $\dot{B}^{0,1}_1$ can be related to some other embeddings between the Besov spaces and the Lorentz spaces $L^{p,q}$. We now define the maximal Hardy spaces with respect to $\Lambda(w)$.

DEFINITION 4.1.1. *Suppose $\varphi \in L^1 \cap L^\infty$ satisfies $\displaystyle\int_{\mathbf{R^n}} \varphi(x)\,dx \neq 0$, and w is a non-negative weight in $L^1_{loc}(\mathbf{R}^+)$. We define*

$$H_{\varphi,w} = \left\{ f \in \Lambda(w) :\ m_\varphi(f) \in \Lambda(w) \right\}$$

and

$$\|f\|_{H_{\varphi,w}} = \|m_\varphi(f)\|_{\Lambda(w)}\cdot$$

REMARK 4.1.2:

(i) It is clear that if $w \equiv 1$ then $H_{\varphi,w} = H^1_\varphi$. Also, if φ is the characteristic function of the unit ball then

$$m_\varphi(f)(x) \le Mf(x)\cdot \tag{3}$$

Thus, if w is a weight for which the Hardy-Littlewood maximal operator is bounded on $\Lambda(w)$, then

$$\|m_\varphi(f)\|_{\Lambda(w)} \le \|M(f)\|_{\Lambda(w)} \le C\|f\|_{\Lambda(w)},$$

which shows that $H_{\varphi,w} = \Lambda(w)$. An example of such a weight is given by

$$w(t) = \frac{1}{\phi(t)}, \qquad \phi \in B_\Psi, \tag{4}$$

(see Theorem 1.2.7). In particular (4) holds for $w(t) = t^{-\theta},\ 0 < \theta < 1$. Another way to show this equivalence is given by the fact that

$$\|M(f)\|_{\Lambda^q(w)} \approx \|f\|_{\Lambda^q(w)}, \qquad w(t) = \frac{t^{1-1/q}}{\phi(t)}\cdot \tag{5}$$

To see this, recall that

$$(M(f))^*(t) \approx t\,S(f^*)(t) = \int_0^t f^*(s)\,ds,$$

and hence,

$$\begin{aligned}\|f\|^q_{\Lambda^q(w)} &= \int_0^\infty (f^*(t)w(t))^q\,dt \le \int_0^\infty (tS(f^*)(t)w(t))^q\,dt\\ &\le C\int_0^\infty \left((M(f))^*(t)w(t)\right)^q dt = C\|M(f)\|^q_{\Lambda^q(w)} \le C\|f\|^q_{\Lambda^q(w)}\cdot\end{aligned}$$

For the case $w(t) = t^{1/p-1/q}$, we have $\|M(f)\|_{L^{p,q}} \approx \|f\|_{L^{p,q}}$, if $1 < p < \infty$ and $1 \le q \le \infty\cdot$ As a side remark, we notice that (5) has an L^1-version (see **[SJ]**), namely

$$\|f\|_{L^1} \approx \|M(f)\|_{L^{1,\infty}}\cdot$$

In fact, fix a cube $Q \subset \mathbf{R^n}$ and set $i_Q = \inf_{x \in Q} Mf(x)$. Then $i_Q > 0$, if $f \not\equiv 0$, and

$$\int_Q |f(x)|\,dx \le |Q| i_Q \le |\{x \in \mathbf{R^n} : \ Mf(x) \ge i_Q\}| i_Q \le \|Mf\|_{L^{1,\infty}}.$$

Thus, if we let Q increase to $\mathbf{R^n}$,

$$\|f\|_{L^1} \le \|Mf\|_{L^{1,\infty}} \le C\|f\|_{L^1}.$$

(ii) If $\Lambda(w)$ is a Banach space (e.g, if $1/w(t) \in B_\Psi$), it is easy to show that $H_{\varphi,w}$ is also a Banach space (see **[HA]**). Also, as in the case of H^1_φ, one can show the following property:

PROPOSITION 4.1.3. *Suppose $\phi \in B_\Psi$ and $w(t) = 1/\phi(t)$. Then the convolution with an L^1 function is a bounded operator in $H_{\varphi,w}$.*

PROOF: Let $g \in L^1(\mathbf{R^n})$ and set $C_g(f) \equiv f * g$. Then we trivially have that $C_g : L^1 \longrightarrow L^1$ and $C_g : L^\infty \longrightarrow L^\infty$, boundedly. Hence, by real interpolation $((L^1, L^\infty)_{\phi,1} = \Lambda(w))$ we get that $C_g : \Lambda(w) \longrightarrow \Lambda(w)$. Now, if $f \in H_{\varphi,w}$, it is easy to show that

$$m_\varphi(f * g)(x) \le (|g| * m_\varphi(f))(x),$$

and hence,

$$\begin{aligned}\|f * g\|_{H_{\varphi,w}} = \|m_\varphi(f * g)\|_{\Lambda(w)} &\le \| \, |g| * m_\varphi(f)\|_{\Lambda(w)} \\ &\le C\|g\|_1 \|m_\varphi(f)\|_{\Lambda(w)} = C\|g\|_1 \|f\|_{H_{\varphi,w}}. \ \blacksquare\end{aligned}$$

We now introduce briefly the (homogeneous) Besov spaces $\dot{B}^{\alpha,q}_p$. Instead of given their rather complicated definition, we will directly assume as our starting point the discrete characterization provided by the work of M. Fraizer and B. Jawerth (see **[FR-JA 1]**). We will restrict ourselves to those cases that will be needed in the proofs of the results. For a good reference about the Besov spaces see **[PE 2]**.

Fix $\alpha \ge 0$, $1 \le p, q \le \infty$. Let $\psi \in \mathcal{S}$ satisfy

$$(6) \qquad \begin{cases} \operatorname{supp} \hat{\psi}(\xi) \subset \{\xi \in \mathbf{R^n} : \ \frac{1}{\pi} \le |\xi| \le \pi\} \\ \hat{\psi}(\xi) \ge c > 0, \qquad \text{if } \frac{1}{2} \le |\xi| \le 2 \end{cases}.$$

We parameterize the family of all dyadic cubes $Q \subset \mathbf{R^n}$ as follows

$$Q_{\nu,k} = \{x \in \mathbf{R^n} :\ k_i 2^{-\nu} \le x_i < (k_i+1)2^{-\nu},\ i = 1, \cdots, n\}$$

for each $\nu \in \mathbf{Z}$ and $k \in \mathbf{Z^n}$ and we set

$$\psi_Q(x) = |Q|^{-1/2}\psi(2^\nu x - k), \qquad Q = Q_{\nu,k}.$$

Then, a function f belongs to the (homogeneous) Besov space $\dot{B}_p^{\alpha,q}$ if it admits a representation of the form

$$f \equiv \sum_Q s_Q \psi_Q, \tag{7}$$

and

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \inf\left\{\left(\sum_{\nu\in\mathbf{Z}}\left(\sum_{l(Q)=2^{-\nu}}\left(|Q|^{-\alpha/n+1/p-1/2}|s_Q|\right)^p\right)^{q/p}\right)^{1/q}\right\} < \infty,$$

where the infimum is taken over all possible sequences $\{s_Q\}_Q$ satisfying (7). Observe that if $p = q$ then

$$\|f\|_{\dot{B}_p^{\alpha,p}} = \inf\left\{\left(\sum_Q\left(|Q|^{-\alpha/n+1/p-1/2}|s_Q|\right)^p\right)^{1/p}\right\}.$$

The following lemmas are the key arguments to understand the proof of the minimality theorem.

LEMMA 4.1.4. (Wiener, see Theorem 7.2.4 in [**RU**])

Suppose $f, g \in L^1(\mathbf{R^n})$ satisfy that supp $\hat{f}$ *is compact and $\hat{g}(\xi) \neq 0$ for all $\xi \in$* supp $\hat{f}$*. Then there exists a function $h \in L^1(\mathbf{R^n})$ such that $g * h \equiv f$.*

LEMMA 4.1.5. *Suppose $\varphi \in L^1 \cap L^\infty$ is a radial and real-valued function satisfying $\int_{\mathbf{R^n}} \varphi(x)\, dx \neq 0$. Then the space H^1_φ is invariant under orthogonal transformations; that is, if $f \in H^1_\varphi$ and $\sigma \in O(n)$ and if we set $f_\sigma(x) = f(\sigma(x))$ then,*

$$f_\sigma \in H^1_\varphi \qquad \text{and} \qquad \|f_\sigma\|_{H^1_\varphi} = \|f\|_{H^1_\varphi}.$$

PROOF: Suppose $f \in H^1_\varphi$ and $\sigma \in O(n)$. Then, since φ is radial

$$\varphi_t * f_\sigma = (\varphi_t * f)_\sigma \cdot$$

Hence,

$$\begin{aligned} m_\varphi(f_\sigma)(x) &= \sup_{|x-y|<t} |(\varphi_t * f_\sigma)(y)| = \sup_{|x-y|<t} |(\varphi_t * f)(\sigma(y))| \\ &= \sup_{|\sigma(x)-y|<t} |(\varphi_t * f)(y)| = m_\varphi(f)(\sigma(x)), \end{aligned}$$

and so,

$$\|f_\sigma\|_{H^1_\varphi} = \|m_\varphi(f_\sigma)\|_{L^1} = \|m_\varphi(f)(\sigma(\cdot))\|_{L^1} = \|f\|_{H^1_\varphi} \cdot \quad \blacksquare$$

THEOREM 4.1.6. *Suppose $\varphi \in L^1 \cap L^\infty$ is a radial and real-valued function and $\int_{\mathbf{R^n}} \varphi(x)\,dx \neq 0$. If H^1_φ contains a non-zero function, then*

$$\dot{B}_1^{0,1} \hookrightarrow H^1_\varphi \cdot$$

PROOF: We first observe that since $\dot{B}_1^{0,1} \subset H^1 \subset L^1(\mathbf{R^n})$, we only need to show the norm estimate. Let $f \in H^1_\varphi$ such that $f \not\equiv 0$. Our main goal is to construct another function $g \in H^1_\varphi$ satisfying the condition $\hat{g}(\xi) \neq 0$ if $1/\pi \leq |\xi| \leq \pi$. We will achieve this by rotating and dilating the function f and then pasting together all the pieces we get. Recall that for $r > 0$ we denote $f_r(x) = r^{-n} f(x/r)$ and hence, $\widehat{f_r}(\xi) = \hat{f}(r\xi) \cdot$ Also recall that if $\sigma \in O(n)$ then $\widehat{f_\sigma}(\xi) = \hat{f}(\sigma(\xi)) \cdot$ Since $f \not\equiv 0$ then there are $\xi_0 \in \mathbf{R^n}$ and $r_0 > 0$ such that for some $C_0 > 0$,

$$|\hat{f}(\xi)| \geq C_0 > 0 \qquad \text{if } \ |\xi - \xi_0| \leq r_0 \tag{8}$$

(we may assume that $0 \notin B(\xi_0, r_0)$). We can choose r_0 small enough so that for all $N \in \mathbf{Z}^+$ and all $\xi_1, \cdots, \xi_N \in B(\xi_0, r_0)$ we have

$$|\hat{f}(\xi_1) + \cdots + \hat{f}(\xi_N)| \geq C_0 > 0 \tag{9}$$

with the same constant C_0 as in (8). Choose $\varepsilon > 0$ sufficiently small and a bump function $\eta \in \mathcal{S}$ such that

$$0 \leq \hat{\eta} \leq 1,\ \hat{\eta}(\xi) = 1 \text{ if } |\xi - \xi_0| \leq r_0 - \varepsilon \text{ and } \ \hat{\eta}(\xi) = 0 \text{ if and only if } |\xi - \xi_0| \geq r_0 \cdot$$

Since the space H^1_φ is invariant under convolution with L^1 functions we have that $h \equiv f * \eta \in H^1_\varphi$,

$$\hat{h}(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } |\xi - \xi_0| \leq r_0 - \varepsilon \\ 0, & \text{if } |\xi - \xi_0| \geq r_0 \end{cases},$$

and

$$(10) \qquad |\hat{h}(\xi_1) + \cdots + \hat{h}(\xi_N)| > 0 \qquad \text{for all } \xi_1, \cdots, \xi_N \in B(\xi_0, r_0)\cdot$$

This follows immediately from (9). Set $s_0 = r_0 - \varepsilon$. Suppose first that we have the following condition:

$$A = \left\{\xi : \frac{1}{\pi} \leq |\xi| \leq \pi\right\} \subset \bigcup_{\sigma \in O(n)} B(\sigma(\xi_0), s_0)\cdot$$

Since A is compact we could find a finite number of those $\sigma \in O(n)$, say $\sigma_1, \cdots, \sigma_M$ such that

$$(11) \qquad \left\{\xi : \frac{1}{\pi} \leq |\xi| \leq \pi\right\} \subset \bigcup_{i=1}^{M} B(\sigma_i(\xi_0), s_0)\cdot$$

Consider now the function

$$g \equiv h_{\sigma_1^{-1}} + \cdots + h_{\sigma_M^{-1}}\cdot$$

We know, by the last lemma, that $g \in H^1_\varphi$ and if $1/\pi \leq |\xi| \leq \pi$ then

$$\hat{g}(\xi) = \hat{h}(\sigma_1^{-1}(\xi)) + \cdots + \hat{h}(\sigma_M^{-1}(\xi))\cdot$$

Clearly $\hat{h}(\sigma_i^{-1}(\xi)) \neq 0$ if and only if $\sigma_i^{-1}(\xi) \in B(\xi_0, r_0)\cdot$ Since, by (11), there exists at least one $i_0 \in \{1, \cdots, M\}$ for which this happens then, by (10) we have that $\hat{g}(\xi) \neq 0$.

Suppose now that (11) does not hold. Then we have to dilate h on both directions; (i.e., for $r > 1$ and also $r < 1$) until we find a set that will satisfy (11), in place of $B(\xi_0, r_0)$. Here are the details.

First observe that if $r > 0$ then

$$\text{supp } \widehat{h_r} = \overline{B}\left(\frac{\xi_0}{r}, \frac{r_0}{r}\right)\cdot$$

In fact, since $\widehat{h_r}(\xi) = \hat{h}(r\xi)$ then

$$\xi \in \operatorname{supp} \widehat{h_r} \Longleftrightarrow |\xi_0 - r\xi| \leq r_0 \Longleftrightarrow \xi \in \overline{B}\left(\frac{\xi_0}{r}, \frac{r_0}{r}\right)\cdot$$

Secondly, we choose a finite sequence of numbers

$$0 < a_M < a_{M-1} < \cdots < a_1 < a_0 = 1 = b_0 < b_1 < b_2 < \cdots < b_J < \infty,$$

such that the following properties are satisfied:

$$\text{(12)} \qquad \begin{aligned} &B\left(\frac{\xi_0}{a_j}, \frac{r_0}{a_j}\right) \bigcap B\left(\frac{\xi_0}{a_{j-1}}, \frac{r_0}{a_{j-1}}\right) \neq \emptyset \qquad j = 1, \cdots, M \\ &B\left(\frac{\xi_0}{b_j}, \frac{r_0}{b_j}\right) \bigcap B\left(\frac{\xi_0}{b_{j-1}}, \frac{r_0}{b_{j-1}}\right) \neq \emptyset \qquad j = 1, \cdots, J \end{aligned}$$

$$\text{(13)} \qquad \left|\frac{\xi_0}{b_J}\right| < \frac{1}{\pi} \qquad \text{and} \qquad \left|\frac{\xi_0}{a_M}\right| > \frac{1}{\pi}\cdot$$

To show that one can find this family of balls, we observe that if $1 > a_{j-1} > a_j$, then

$$\begin{aligned} B\left(\frac{\xi_0}{a_j}, \frac{r_0}{a_j}\right) \bigcap B\left(\frac{\xi_0}{a_{j-1}}, \frac{r_0}{a_{j-1}}\right) \neq \emptyset &\Longleftrightarrow \frac{|\xi_0| + r_0}{a_{j-1}} > \frac{|\xi_0| - r_0}{a_j} \\ &\Longleftrightarrow a_j > a_{j-1}\frac{|\xi_0| - r_0}{|\xi_0| + r_0}\cdot \end{aligned}$$

Thus if we take

$$\begin{aligned} a_1 &= \frac{|\xi_0| - r_0}{|\xi_0| + r_0} + \delta < 1 \\ a_2 &= a_1\frac{|\xi_0| - r_0}{|\xi_0| + r_0} + \delta = \left(\frac{|\xi_0| - r_0}{|\xi_0| + r_0}\right)^2 + P_2(\delta) \\ &\vdots \\ a_j &= a_{j-1}\frac{|\xi_0| - r_0}{|\xi_0| + r_0} + \delta = \left(\frac{|\xi_0| - r_0}{|\xi_0| + r_0}\right)^j + P_j(\delta) \end{aligned}$$

(where $P_j(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0^+$), we see that for j big enough and δ small, these numbers satisfy conditions (12) and (13). Similarly one can find the sequence of b_j's.

Now define the function

$$l(x) = \sum_{j=0}^{M} h_{a_j}(x) + \sum_{j=1}^{J} h_{b_j}(x)$$

and set

$$U = \bigcup_{j=0}^{M} B\left(\frac{\xi_0}{a_j}, \frac{r_0}{a_j}\right) \cup \bigcup_{j=1}^{J} B\left(\frac{\xi_0}{b_j}, \frac{r_0}{b_j}\right)\cdot$$

Then $l \in H^1_\varphi$ and $\hat{l}(\xi) \neq 0$ for all $\xi \in U$:

That $l \in H^1_\varphi$ is clear, since it is the sum of dilations of functions in H^1_φ. Now if $\xi \in U$ then

$$\widehat{h_{a_j}}(\xi) \neq 0, \ \left(\widehat{h_{b_j}}(\xi) \neq 0\right) \qquad \text{if and only if} \qquad a_j\xi \in B(\xi_0, r_0), \ (b_j\xi \in B(\xi_0, r_0))\cdot$$

Thus, by (10)

$$\hat{l}(\xi) = \sum_{j=0}^{M} \hat{h}(a_j\xi) + \sum_{j=1}^{J} \hat{h}(b_j\xi) \neq 0\cdot$$

Notice that l also satisfies property (10), namely

$$\hat{l}(\xi_1) + \cdots + \hat{l}(\xi_N) \neq 0 \qquad \text{for all} \ \ \xi_1, \cdots, \xi_N \in U\cdot \tag{14}$$

By (12) and (13) and by a similar argument as in the previous case, we can find a finite sequence $\sigma_1, \cdots, \sigma_N \in O(n)$ such that

$$\left\{\xi : \ \frac{1}{\pi} \leq |\xi| \leq \pi\right\} \subset \bigcup_{j=1}^{N} \sigma_j(U)\cdot$$

Hence, if we set

$$g \equiv l_{\sigma_1^{-1}} + \cdots + l_{\sigma_N^{-1}},$$

we have that $g \in H^1_\varphi$ and so if $1/\pi \leq |\xi| \leq \pi$,

$$\hat{g}(\xi) = \hat{l}(\sigma_1^{-1}(\xi)) + \cdots + l(\sigma_N^{-1}(\xi)) \neq 0,$$

by (14). If we choose the functions ψ as in (6) and g as above then we are under the hypotheses of our Lemma 4.1.4 and we can find a function $K \in L^1(\mathbf{R^n})$ so that $g * K \equiv \psi$, and hence, applying Lemma 4.1.5 we have that $\psi \in H^1_\varphi$.

We want now to estimate the maximal function, with respect to φ, of a function in $\dot{B}_1^{0,1}$, (recall that if $f \equiv \sum_Q s_Q \psi_Q$ then $\|f\|_{\dot{B}_1^{0,1}} = \inf\{\sum_Q |s_Q|\, |Q|^{-1/2}\}$). We start with the function ψ_Q, $Q = Q_{\nu,k}$:

$$(\varphi_t * \psi_Q)(y) = \int_{\mathbf{R^n}} \frac{1}{t^n} 2^{\nu n/2} \varphi(u/t)\psi(2^\nu(y-u)-k)\, du \qquad \begin{cases} v &= 2^\nu u \\ dv &= 2^{n\nu} du \end{cases}$$

$$= \int_{\mathbf{R^n}} \frac{2^{-n\nu/2}}{t^n} \varphi(v/(2^\nu t))\psi(2^\nu y - k - v)\, dv = 2^{n\nu/2}(\varphi_{2^\nu t} * \psi)(2^\nu y - k)\cdot$$

Thus,

$$\begin{aligned} m_\varphi(\psi_Q)(x) &= \sup_{|x-y|<t} |(\varphi_t * \psi_Q)(y)| \\ &= \sup_{|x-y|<t} 2^{n\nu/2} |(\varphi_{2^\nu t} * \psi)(2^\nu y - k)| = \sup_{|x - \frac{u+k}{2^\nu}| < 2^{-\nu} s} 2^{n\nu/2} |(\varphi_s * \psi)(u)| \\ &= \sup_{|(2^\nu x - k) - u| < s} 2^{n\nu/2} |(\varphi_s * \psi)(u)| = 2^{n\nu/2} m_\varphi(\psi)(2^\nu x - k)\cdot \end{aligned}$$

Therefore,

$$\begin{aligned} \|\psi_Q\|_{H^1_\varphi} = \|m_\varphi(\psi_Q)\|_{L^1} &= 2^{n\nu/2} \int_{\mathbf{R^n}} m_\varphi(\psi)(2^\nu x - k)\, dx \\ &= 2^{-n\nu/2} \int_{\mathbf{R^n}} m_\varphi(\psi)(x)\, dx = 2^{-n\nu/2} \|\psi\|_{H^1_\varphi}\cdot \end{aligned}$$

Hence, if $f \in \dot{B}_1^{0,1}$, $f \equiv \sum_Q s_Q \psi_Q$ we have

$$\|f\|_{H^1_\varphi} \le \sum_Q |s_Q|\ \|\psi_Q\|_{H^1_\varphi} = \|\psi\|_{H^1_\varphi} \sum_Q |s_Q|\, |Q|^{-1/2}.$$

Thus, taking the infimum over all possible decompositions we get

$$\|f\|_{H^1_\varphi} \le \|\psi\|_{H^1_\varphi} \|f\|_{\dot{B}_1^{0,1}}\cdot \quad \blacksquare$$

REMARK 4.1.7: As we mentioned in the introduction we can relate this result to some embeddings between the Besov spaces and the Lorentz spaces $L^{p,q}$. With more generality we can consider the Besov spaces with a function parameter $\phi \in B_\Psi$. See **[CO-DF]** for a detailed study of these spaces. Set

$$w(t) = \frac{1}{\phi(t)} \qquad \text{and} \qquad \overline{w}(t) = \sup_{s>0} \frac{w(st)}{w(s)} = \overline{\phi}(\frac{1}{t})\cdot$$

We say that a function $f \in \dot{B}_w$ if $f \equiv \sum_Q s_Q \psi_Q$ and

$$\|f\|_{\dot{B}_w} = \inf \left\{ \sum_Q 2^{-n\nu/2} \overline{w}(2^{-n\nu}) |s_Q| \right\} < \infty$$

where ψ is chosen as in (6). Observe that if $\phi(t) = t^\theta,\ 0 < \theta < 1$, then $\overline{w}(t) = t^{-\theta}$ and hence,

$$\begin{aligned} \|f\|_{\dot{B}_w} &= \inf \left\{ \sum_Q 2^{-n\nu/2} 2^{\theta n\nu} |s_Q| \right\} \\ &= \inf \left\{ \sum_Q |Q|^{1/2-\theta} |s_Q| \right\} = \|f\|_{\dot{B}_1^{\theta n,1}} \cdot \end{aligned}$$

By Remark 4.1.2 we know that if φ is the characteristic function of the unit ball then $H_{\varphi,w} = \Lambda(w)$ and, in particular, we obtain directly that $\psi \in H_{\varphi,w}$, since

$$\psi \in L^1 \cap L^\infty \subset (L^1, L^\infty)_{\phi,1} = \Lambda(w) \cdot$$

Using the last estimate of the proof of the theorem, but with $\Lambda(w)$ in place of L^1, we get that

$$\|f\|_{H_{\varphi,w}} \leq \|\psi\|_{H_{\varphi,w}} \|f\|_{\dot{B}_w};$$

i.e., $\dot{B}_w \hookrightarrow \Lambda(w) \cdot$ In particular, for $w(t) = t^{-\theta},\ 0 < \theta < 1$ this gives

$$\dot{B}_1^{\theta n,1} \hookrightarrow L^{\frac{1}{1-\theta},1}. \tag{15}$$

Much more can be said about this inclusion. We will use the techniques already described of **[FR-JA 1]** plus the following interpolation result to get (15) for a bigger class of spaces, (see **[JO]** for another proof).

THEOREM 4.1.8. (See **[PE 2]**)

If $0 \leq \alpha_0 < \alpha_1 < \infty$, $0 < \theta < 1$, $1 \leq p < \infty$ *and* $1 \leq q_0, q_1, q \leq \infty$, *then*

$$(\dot{B}_p^{\alpha_0,q_0}, \dot{B}_p^{\alpha_1,q_1})_{\theta,q} = \dot{B}_p^{\alpha,q},$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.

THEOREM 4.1.9. *If* $0 < \theta < 1$ *and* $1 \le q \le \infty$, *then*

$$\dot{B}_1^{\theta n, q} \hookrightarrow L^{\frac{1}{1-\theta}, q}.$$

PROOF: We first prove that $\dot{B}_1^{0,1} \hookrightarrow L^1$ and $\dot{B}_1^{n,1} \hookrightarrow L^\infty$. That $\dot{B}_1^{0,1} \hookrightarrow L^1$ was already mentioned in the proof of Theorem 4.1.6. If $f \in \dot{B}_1^{n,1}$, then as in (7), $f \equiv \sum_Q s_Q \psi_Q$ and so

$$|f(x)| \le \sum_Q |s_Q||\psi_Q(x)| \le \sum_Q |s_Q|\ |Q|^{-1/2}.$$

Taking the infimum over all decompositions and then the supremum with respect to x we obtain

$$\|f\|_\infty \le \|f\|_{\dot{B}_1^{n,1}}.$$

By the previous result with $\alpha_0 = 0, \alpha_1 = n$ and $p = q_0 = q_1 = 1$ we get

$$\dot{B}_1^{\theta n, q} = (\dot{B}_1^{0,1}, \dot{B}_1^{n,1})_{\theta, q} \hookrightarrow (L^1, L^\infty)_{\theta,q} = L^{\frac{1}{1-\theta}, q}. \quad \blacksquare$$

We consider now some properties of the dual space of H^1_φ, using the duality results for the Tent spaces T^1_∞. The idea is not new and goes back to the work in [**HA**]. There, it is proved that a new representation of functions in BMO, is given by the action of a Carleson measure on a modified version of the function $\varphi_t(\cdot - y)$, for $(y,t) \in \mathbf{R}^{\mathbf{n}+1}_+$, (see Theorem (2.27) in [**HA**] for details). The starting point is the isometric identification of H^1_φ with a closed subspace of T^1_∞. In fact, if $\varphi \in L^1 \cap L^\infty$, $\int_{\mathbf{R^n}} \varphi(x)\, dx \neq 0$, $\varphi \ge 0$ and continuous then the mapping

$$H^1_\varphi \xrightarrow{\Phi} T^1_\infty$$
$$f \longrightarrow (f * \varphi_t)(x)$$

is clearly an isometry onto its image. Notice that since $\{\varphi_t\}_{t>0}$ is an approximation of the identity, we have that

$$\Phi^{-1}(g)(x) = \lim_{t \to 0} g(x,t),$$

for all $g \in \Phi(H^1_\varphi)\cdot$ Thus, the dual of the space H^1_φ, can be identified as a quotient space of $V^1 = (T^1_\infty)^*$. In more detail, if we denote by

$$N_\varphi = \left\{ \mu \in V^1 : \int_{\mathbf{R}^{\mathbf{n}+1}_+} (f * \varphi_t)(x)\, d\mu(x,t) = 0, \quad \text{for all } f \in H^1_\varphi \right\}$$

then we have that

$$(H^1_\varphi)^* \approx \left(\Phi(H^1_\varphi)\right)^* = V^1/N_\varphi\cdot$$

For the particular case where $n = 1$ and φ is smooth enough so that $H^1_\varphi = H^1$, it was proved in **[HA]** that if one considers the function

$$\tilde{\varphi}(x,t) = \begin{cases} \varphi_t(x) & \text{if } |x| > 1 \text{ or } t > 1 \\ 0 & \text{otherwise} \end{cases}$$

then we have that the function

$$\varphi_t(x-y) - \tilde{\varphi}(x,t)$$

is μ-integrable ($\mu \in V^1$), in the variables (x,t), for almost every $y \in \mathbf{R}^{\mathbf{n}}$. So, by Fubini's theorem and the fact that $\int_{\mathbf{R}} f(x)\, dx = 0$ for all $f \in H^1$, one gets, for $\mu \in N_\varphi$:

$$\begin{aligned} \int_{\mathbf{R}^2_+} (f * \varphi_t)(x)\ d\mu(x,t) &= \int_{\mathbf{R}^2_+} \left(\int_{\mathbf{R}} f(y)(\varphi_t(x-y) - \tilde{\varphi}(x,t))\ dy \right)\ d\mu(x,t) \\ &= \int_{\mathbf{R}} f(y) \left(\int_{\mathbf{R}^2_+} (\varphi_t(x-y) - \tilde{\varphi}(x,t))\ d\mu(x,t) \right)\ dy = 0\cdot \end{aligned}$$

Thus, we conclude that

$$\int_{\mathbf{R}^2_+} (\varphi_t(x-y) - \tilde{\varphi}(x,t))\, d\mu(x,t)$$

must be constant, for a.e. $y \in \mathbf{R}$.

We will find a class of Carleson measures N that is contained in all of the above N_φ. Thus, if we now take the quotient V^1/N, we find that it satisfies a maximality condition; namely, the dual of H^1_φ can be identified as a closed subspace of V^1/N. Notice that a similar result holds if we recall that if φ is radial and H^1_φ is not trivial then $\dot{B}^{0,1}_1 \hookrightarrow H^1_\varphi$, and therefore, since the dual of $\dot{B}^{0,1}_1$ is the Bloch space $\dot{B}^{0,\infty}_\infty$ (see **[PE 2]**), we also have that $(H^1_\varphi)^* \hookrightarrow \dot{B}^{0,\infty}_\infty$. We give now the definition of a class of measures that contains the class N.

DEFINITION 4.1.10. *Suppose μ is a Borel measure in $\mathbf{R}^{\mathbf{n}+1}_+$. We say that μ is invariant under horizontal translations if for all measurable sets $E \subset \mathbf{R}^{\mathbf{n}+1}_+$ and all $x \in \mathbf{R}^{\mathbf{n}}$, if we denote by*

$$x + E = \{(x+y,t) : \ (y,t) \in E\},$$

we then have that $\mu(x+E) = \mu(E)$. For $\alpha \geq 1$ we define

$$N^\alpha = \big\{\mu \in V^\alpha : \ \mu \ \text{ is invariant under horizontal translations}\big\}\cdot$$

Our goal is to completely characterize the class N^α. For this, we need the following definition.

DEFINITION 4.1.11. *Suppose σ is a Borel measure in $\mathbf{R}^+$. We say that σ is a measure of order β, with $\beta \geq 0$, if there exists a constant $C > 0$ such that*

$$\int_0^t d|\sigma| \leq Ct^\beta, \qquad \text{for all } \ t > 0\cdot \tag{16}$$

In this case, we write $\sigma \in M^\beta$ and also

$$\|\sigma\|_{M^\beta} = \inf\{C : \ C \text{ satisfies } (16)\}\cdot$$

Note that if $\beta = 0$ then M^0 is the space of finite measures in $\mathbf{R}^+$. Also if $\beta > 0, \ 1 \leq p \leq \infty, \ \gamma = \beta - 1/p'$ and $f \in L^p(\mathbf{R}^+, dt)$ then, by Hölder's inequality we have that $d\sigma(t) = t^\gamma f(t)\,dt \in M^\beta$ and $\|\sigma\|_{M^\beta} \leq C_{\beta,p}\|f\|_p$. We now give the characterization of N^α.

THEOREM 4.1.12. *Suppose $\alpha \geq 1$ and μ is a Borel measure in $\mathbf{R}^{\mathbf{n}+1}_+$. Then $\mu \in N^\alpha$ if and only if μ is a product measure of the form $dxd\sigma(t)$, for some $\sigma \in M^{n(\alpha-1)}$. In this case we have that $\|\mu\|_{V^\alpha} \approx \|\sigma\|_{M^{n(\alpha-1)}}\cdot$*

PROOF: Suppose first that $d\mu(x,t) = dxd\sigma(t)$ for some $\sigma \in M^{n(\alpha-1)}$. Then, it is clear that μ is invariant under horizontal translations. By Remark 3.4.4-(i), since $\alpha \geq 1$ we need only consider cubes $Q \subset \mathbf{R}^{\mathbf{n}}$. Now, $\widehat{Q} \subset Q \times (0, l(Q))$ and hence,

$$\begin{aligned} |\mu|(\widehat{Q}) &\leq \left(\int_Q dx\right)\left(\int_0^{l(Q)} d|\sigma|\right) \\ &\leq |Q| \ \|\sigma\|_{M^{n(\alpha-1)}}(l(Q))^{n(\alpha-1)} \leq C_n\|\sigma\|_{M^{n(\alpha-1)}}|Q|^\alpha; \end{aligned}$$

that is, $\|\mu\|_{V^\alpha} \leq C_n \|\sigma\|_{M^{n(\alpha-1)}}$.

Conversely, if $\mu \in N^\alpha$ we want to show that for all measurable sets $A \subset \mathbf{R^n}$ and $B \subset \mathbf{R}^+$ we have that

$$\mu(A \times B) = |A|\sigma(B),$$

where $\sigma \in M^{n(\alpha-1)}$. Without loss of generality we may assume that $\mu \geq 0$. Fix $B \subset \mathbf{R}^+$ measurable, and define

$$\nu_B(A) = \mu(A \times B), \qquad A \subset \mathbf{R^n}\cdot$$

Then, it is obvious that ν_B is a positive Borel measure in $\mathbf{R^n}$ and ν_B is invariant under translations, since

$$\nu_B(y + A) = \mu((y + A) \times B) = \mu(y + (A \times B)) = \mu(A \times B) = \nu_B(A)\cdot$$

Hence, by the uniqueness of the Haar measure in $\mathbf{R^n}$ (see [**RU**]), there exists a constant $C_B \geq 0$ so that

$$\nu_B(A) = C_B|A|\cdot \tag{17}$$

Thus, if we fix $A \subset \mathbf{R^n}$, with $0 < |A| < \infty$, we find that

$$C_B = \frac{\mu(A \times B)}{|A|} \qquad \text{(independently of } A)\cdot \tag{18}$$

Define $\sigma(B) = C_B$, so that $\mu(A \times B) = |A|\sigma(B)$. Then by (18), it is also obvious that σ is a positive Borel measure in $\mathbf{R}^+$. Thus, it only remains to show that $\sigma \in M^{n(\alpha-1)}$. Fix $t > 0$ and choose a cube $Q \subset \mathbf{R^n}$ such that $l(Q) = t$. Then, by (18) with $A = Q$ and the fact that $\mu \in N^\alpha$ we have that

$$\begin{aligned}\int_0^t d|\sigma| = |\sigma|(0,t) = \frac{|\mu|(Q \times (0,t))}{|Q|} &\leq \frac{|\mu|(\widehat{Q^*})}{|Q|} \\ &\leq c_n \|\mu\|_{V^\alpha} |Q|^{\alpha-1} = C_n \|\mu\|_{V^\alpha} t^{n(\alpha-1)};\end{aligned}$$

that is, $\|\sigma\|_{M^{n(\alpha-1)}} \leq C_n \|\mu\|_{V^\alpha}$. ∎

Finally, if we set $N = N^1$ we find the following minimality property.

COROLLARY 4.1.13. *Suppose* $\varphi \in L^1 \cap L^\infty$, $\int_{\mathbf{R^n}} \varphi(x)\,dx \neq 0$, $\varphi \geq 0$ *and* φ *has compact support. Then* $N \subset N_\varphi$.

PROOF: Let $\mu \in N$, that is, $d\mu(x,t) = dx d\sigma(t)$ with σ a finite measure in $\mathbf{R}^+$ (by the above theorem). Let $f \in H^1_\varphi$. Then by Fubini's theorem,

$$\int_{\mathbf{R^{n+1}_+}} (f * \varphi_t)(x)\,dx\,d\sigma(t) = \int_{\mathbf{R^{n+1}_+}} \left(\int_{\mathbf{R^n}} f(y)\varphi_t(x-y)\,dy \right) dx\,d\sigma(t)$$

$$= \int_{\mathbf{R^{n+1}_+}} \left(\int_{\mathbf{R^n}} \varphi_t(x-y)\,dx \right) f(y)\,dy\,d\sigma(t) = \|\varphi\|_1 \int_{\mathbf{R^+}} \left(\int_{\mathbf{R^n}} f(y)\,dy \right) d\sigma(t) = 0$$

since, as we pointed out earlier if φ has compact support then $H^1_\varphi \subset H^1$ and hence $\int_{\mathbf{R^n}} f(y)\,dy = 0$. ∎

We now continue the study of embedding properties for the spaces H^1_φ. We will first give a brief description of the (homogeneous) Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_p$ along the same lines as our comments about the Besov spaces. As in that case, we will take as the definition the characterization obtained now in [**FR-JA 2**]. Fix $\psi \in \mathcal{S}$ satisfying (6) and the additional condition

$$\sum_{\nu \in \mathbf{Z}} |\widehat{\psi}(2^\nu \xi)|^2 = 1 \qquad \text{if } \ \xi \neq 0\cdot \tag{19}$$

For $\alpha \geq 0$, $1 \leq p, q \leq \infty$ we say that a sequence $s = \{s_Q\}_Q$, (where Q runs over all dyadic cubes $Q = Q_{\nu,k} \subset \mathbf{R^n}$), belongs to the space $\dot{f}^{\alpha,q}_p$ if

$$\|s\|_{\dot{f}^{\alpha,q}_p} = \left\| \left(\sum_Q \left(|Q|^{-\alpha/n} |s_Q| \widetilde{\chi}_Q \right)^q \right)^{1/q} \right\|_{L^p} < \infty, \qquad (p \neq \infty)$$

where $\widetilde{\chi}_Q \equiv |Q|^{-1/2}\chi_Q$, and for $p = \infty$,

$$\|s\|_{\dot{f}^{\alpha,q}_\infty} = \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \sum_{Q \subset P} \left(|Q|^{-\alpha/n} |s_Q| \widetilde{\chi}_Q(x) \right)^q \,dx \right)^{1/q} < \infty\cdot$$

Notice, that if $1 \leq q < \infty$ then $\|s\|^q_{\dot{f}^{\alpha,q}_\infty}$ is equivalent to the V^1 norm of the measure

$$\sum_Q \left(|Q|^{-\alpha/n - 1/2} |s_Q| \right)^q |Q| \ \delta_{(2^{-\nu}k, 2^{-\nu})}, \tag{20}$$

where δ is the Dirac mass. We say that a function f belongs to the (homogeneous) Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ if f admits a decomposition of the form

$$(21)\qquad f \equiv \sum_Q s_Q \psi_Q, \qquad \|f\|_{\dot{F}_p^{\alpha,q}} = \inf\left\{ \|\{s_Q\}\|_{\dot{f}_p^{\alpha,q}} \right\},$$

where the infimum is taken over all sequences satisfying (21). It is easy to show, just by looking at the definitions, that $\dot{B}_1^{0,1} = \dot{F}_1^{0,1}$. Also in **[FR-JA 2]** it is proved that $\dot{F}_1^{0,2} = H^1$. On the one hand we know that under some conditions on φ, H^1_φ is non-trivial if and only if it contains $\dot{B}_1^{0,1} = \dot{F}_1^{0,1}$. On the other hand, if φ has compact support, we always have that $H^1_\varphi \subset H^1 = \dot{F}_1^{0,2}$. Thus, we see that the spaces $\dot{F}_1^{0,1}$ and $\dot{F}_1^{0,2}$ play an important end-point condition for the Hardy spaces H^1_φ. We address now the following question: under which conditions on the kernel φ can one get that $\dot{F}_1^{0,q} \subset H^1_\varphi$ where $1 \leq q \leq 2$. This is the right scale of spaces to consider, since

$$\dot{F}_1^{0,1} \subset \dot{F}_1^{0,q} \subset \dot{F}_1^{0,2} \qquad 1 \leq q \leq 2\cdot$$

We will need to use the following duality result (see **[FR-JA 2]**).

LEMMA 4.1.14. *If $1 \leq p, q < \infty$, then*

$$\left(\dot{f}_p^{0,q}\right)^* = \dot{f}_{p'}^{0,q'}\cdot$$

For a kernel φ and ψ as above, and for each dyadic cube $Q \subset \mathbf{R^n}$, we define the linear functional, T^Q_φ, acting on the space of Carleson measures:

$$(22)\qquad T^Q_\varphi(\mu) = \int_{\mathbf{R}^{\mathbf{n}+1}_+} (\varphi_t * \psi_Q)(x)\, d\mu(x,t)\cdot$$

We notice that T^Q_φ is well defined if $H^1_\varphi \neq \{0\}$ and φ is radial, since then $\psi_Q \in H^1_\varphi$ and hence $|T^Q_\varphi(\mu)| \leq \|\psi_Q\|_{H^1_\varphi} \|\mu\|_{V^1}\cdot$

We now give the characterization:

THEOREM 4.1.15. *Suppose $\varphi \in L^1 \cap L^\infty$, $\int_{\mathbf{R^n}} \varphi(x)\, dx \neq 0$,* supp φ *compact and T^Q_φ is as in (22). Fix $1 \leq p \leq 2$. Then the following are equivalent:*

(i) $\dot{F}_1^{0,p} \hookrightarrow H^1_\varphi$.

(ii) $\{T^Q_\varphi(\mu)\}_Q$ *is uniformly bounded in* $\dot{f}^{0,p'}_\infty$*; i.e, there exists a* $C > O$ *such that*

$$\|\{T^Q_\varphi(\mu)\}_Q\|_{\dot{f}^{0,p'}_\infty} \leq C\|\mu\|_{V^1}\cdot$$

(iii) *(If* $1 < p \leq 2$*). The mapping*

$$\mu \xrightarrow{T_\varphi} \sum_Q \left(2^{n\nu/2} T^Q_\varphi(\mu)\right)^{p'} 2^{-n\nu}\ \delta_{(2^{-\nu}k,2^{-\nu})}$$

is "bounded" on V^1*; i.e, there exists a constant* $C > 0$ *such that*

$$\|T_\varphi(\mu)\|_{V^1} \leq C\|\mu\|^{p'}_{V^1}\cdot$$

PROOF: That (ii) and (iii) are equivalent, for $1 < p \leq 2$ is a consequence of (20), since $|Q| = 2^{-n\nu}$, $\alpha = 0$ and $p' < \infty$ we have

$$\|T_\varphi(\mu)\|_{V^1} = \left\|\sum_Q \left(2^{n\nu/2} T^Q_\varphi(\mu)\right)^{p'} 2^{-n\nu}\ \delta_{(2^{-\nu}k,2^{-\nu})}\right\|_{V^1}$$

$$= \left\|\sum_Q \left(|Q|^{-1/2}|T^Q_\varphi(\mu)|\right)^{p'} |Q|\ \delta_{(2^{-\nu}k,2^{-\nu})}\right\|_{V^1} \leq C\|\{T^Q_\varphi(\mu)\}_Q\|^{p'}_{\dot{f}^{0,p'}_\infty} \leq C\|\mu\|^{p'}_{V^1}\cdot$$

To prove that (i) is equivalent to (ii) we choose $f \in \dot{F}^{0,p}_1$, with $f \equiv \sum_Q s_Q\psi_Q$ and $s = \{s_Q\}_Q \in \dot{f}^{0,p}_1$. Then

$$\|f\|_{H^1_\varphi} = \|f * \varphi_t\|_{T^1_\infty} = \sup_{\|\mu\|_{V^1}\leq 1} \left|\int_{\mathbf{R}^{\mathbf{n}+1}_+} (\varphi_t * f)(x)\, d\mu(x,t)\right|\cdot$$

But

$$\int_{\mathbf{R}^{\mathbf{n}+1}_+} (\varphi_t * f)(x)\, d\mu(x,t) = \sum_Q s_Q T^Q_\varphi(\mu)\cdot \tag{23}$$

Hence, if (i) holds we have that

$$\sup_{\|\mu\|_{V^1}\leq 1} \Big|\sum_Q s_Q T^Q_\varphi(\mu)\Big| \leq C\|f\|_{\dot{F}^{0,p}_1} \leq C\|s\|_{\dot{f}^{0,p}_1}$$

and, so $\{T_\varphi^Q(\mu)\}_Q \in \left(\dot{f}_1^{0,p}\right)^* = \dot{f}_\infty^{0,p'}$, with $\|\{T_\varphi^Q(\mu)\}_Q\|_{\dot{f}_\infty^{0,p'}} \leq C$, if $\|\mu\|_{V^1} \leq 1$, from which (ii) holds.

Conversely, if (ii) is satisfied, then by (23) we have that

$$\begin{aligned}\|f\|_{H^1_\varphi} &\leq \sup_{\|\mu\|_{V^1}\leq 1} \Big|\sum_Q s_Q T_\varphi^Q(\mu)\Big| \\ &\leq \sup_{\|\mu\|_{V^1}\leq 1} \|s\|_{\dot{f}_1^{0,p}} \|\{T_\varphi^Q(\mu)\}_Q\|_{\dot{f}_\infty^{0,p'}} \leq C\|s\|_{\dot{f}_1^{0,p}}.\end{aligned}$$

Taking the infimum over all such $s = \{s_Q\}_Q$, we get (i). ∎

Notice that for $p = 2$, the above theorem characterizes those functions φ, with compact support, for which $H^1_\varphi = H^1$.

(§2) Carleson measures and pointwise estimates.

We are going to consider now the study of pointwise estimates for a class of operators, using the properties we have shown for certain types of Carleson measures. In particular, as a consequence of this argument we can get the boundedness for a family of bilinear operators defined on the product of L^q and some space of measures, into a Lipschitz space, we give yet another proof of the pointwise boundedness for the Fourier transform of distributions in H^p and we improve and generalize the Féjer-Riesz inequality for harmonic extensions of H^p functions.

We begin with a simple but very useful observation.

LEMMA 4.2.1. *Let $f \in L^q(\mathbf{R^n})$, $1 \leq q \leq \infty$, $\sigma \in M^\alpha$, with $\beta = \dfrac{1}{q'} + \dfrac{\alpha}{n} \geq 1$ and set $d\mu(x,t) = f(x)dxd\sigma(t)$. Then*

$$\mu \in V^\beta \qquad \text{and} \qquad \|\mu\|_{V^\beta} \leq \|\sigma\|_{M^\alpha}\|f\|_{L^q}.$$

PROOF: Since $\beta \geq 1$, we only need to consider cubes $Q \subset \mathbf{R^n}$ to show that $\mu \in V^\beta$. Now,

$$\begin{aligned}|\mu|(\widehat{Q}) &\leq \left(\int_Q |f(x)|\,dx\right)\left(\int_0^{|Q|^{1/n}} d|\sigma|(t)\right) \\ &\leq \|f\|_{L^q}|Q|^{1/q'}\|\sigma\|_{M^\alpha}|Q|^{\alpha/n} = \|f\|_{L^q}\|\sigma\|_{M^\alpha}|Q|^\beta\end{aligned}$$

and so, $\|\mu\|_{V^\beta} \leq \|f\|_{L^q}\|\sigma\|_{M^\alpha}\cdot$ ∎

For our next result, we need to introduce a densely defined bilinear functional. We will restrict the action of this operator, when considering distributions in $H^p(\mathbf{R^n})$, to the dense subspace $\mathcal{S}_0$ of those functions in $\mathcal{S}$ with mean zero.

DEFINITION 4.2.2. *Fix $1 \leq q \leq \infty$. Suppose $F : \mathbf{R^n} \times \mathbf{R^n} \longrightarrow \mathbf{C}$ is a measurable function such that if we set $F_z(x) = F(z,x)$, $z, x \in \mathbf{R^n}$ then $F_z \in L^q(\mathbf{R^n})$. Let $\alpha \geq 0$. For $g \in \mathcal{S}_0$, set*

$$(R_F(g))(x,z) = \int_{\mathbf{R^n}} g(y)F(z,y+x)\,dy\cdot$$

We define, for $\sigma \in M^\alpha$,

$$T_F(g,\sigma)(z) = \int_0^\infty \left((R_F(g))(\cdot,z) * P_t\right)(0)\,d\sigma(t)$$

where P is the Poisson kernel in $\mathbf{R^n}$.

EXAMPLE 4.2.3: Suppose $q = \infty$ and $F(z,x) = e^{-ixz}$. Then $\|F\|_\infty = 1$ and if $g \in \mathcal{S}_0$ we have that

$$\left(R_F(g)\right)(x,z) = \int_{\mathbf{R^n}} g(y)e^{-i(x+y)z}\,dy = e^{-ixz}\hat{g}(z)\cdot$$

Hence,

$$\left(R_F(g)(\cdot,z) * P_t\right)(0) = \int_{\mathbf{R^n}} e^{-ixz}\hat{g}(z)P_t(x)\,dx = \hat{g}(z)\widehat{P}_t(z)\cdot$$

If $0 < p < 1$ and we consider the measure $d\sigma(t) = t^{n(\frac{1}{p}-1)-1}\,dt$, then $\sigma \in M^{n(\frac{1}{p}-1)}$, since

$$\int_0^t d|\sigma|(t) = \frac{1}{n(\frac{1}{p}-1)}t^{n(\frac{1}{p}-1)},$$

and so,

$$\|\sigma\|_{M^{n(\frac{1}{p}-1)}} = \frac{1}{n(\frac{1}{p}-1)}\cdot$$

Therefore,

$$T_F(g,\sigma)(z) = \int_0^\infty \hat{g}(z)\widehat{P}_t(z)t^{n(\frac{1}{p}-1)-1}\,dt = c_n\hat{g}(z)\int_0^\infty e^{-2\pi t|z|}t^{n(\frac{1}{p}-1)-1}\,dt$$

and the integral is finite since $n(\frac{1}{p}-1) > 0$.

THEOREM 4.2.4. *Suppose* $1 \leq q \leq \infty$, $\alpha \geq n/q$ *and* $1/p = \alpha/n + 1/q'$, *so that* $0 < p \leq 1$. *Then*

$$|T_F(g,\sigma)(z)| \leq c_n \|\sigma\|_{M^\alpha} \|F_z\|_{L^q(\mathbf{R^n})} \|g\|_{H^p(\mathbf{R^n})},$$

for all $\sigma \in M^\alpha$ *and* $g \in \mathcal{S}_0\cdot$

PROOF: The proof is a simple consequence of the previous lemma and the fact that $\|g\|_{H^p(\mathbf{R^n})} \approx \|PI(g)\|_{T^p_\infty}$, where $PI(g)(x,t) = (P_t * g)(x)\cdot$ To estimate this quantity we use Theorem 3.4.8: $(T^p_\infty)^* = V^{1/p},\ 0 < p \leq 1\cdot$ First notice that, since P is an even function,

$$\begin{aligned} T_F(g,\sigma)(z) &= \int_0^\infty \left(\int_{\mathbf{R^n}} P_t(u)(R_F(g))(u,z)\, du \right) d\sigma(t) \\ &= \int_{\mathbf{R}^{\mathbf{n}+1}_+} g(y) \left(\int_{\mathbf{R^n}} P_t(u) F(z, y+u)\, du \right) dx\, d\sigma(t) \\ &= \int_{\mathbf{R}^{\mathbf{n}+1}_+} g(y) \left(\int_{\mathbf{R^n}} P_t(v-y) F(z,v)\, dv \right) dx\, d\sigma(t) \\ &= \int_{\mathbf{R}^{\mathbf{n}+1}_+} PI(g)(v,t) F(z,v)\, dv\, d\sigma(t)\cdot \end{aligned}$$

For a fixed z, consider the measure

$$d\mu(v,t) = F_z(v)\ dv\, d\sigma(t)\cdot$$

Then, by the lemma, we have that $\mu \in V^{1/p}$ and $\|\mu\|_{V^{1/p}} \leq \|\sigma\|_{M^\alpha} \|F_z\|_{L^q}\cdot$ Thus

$$\begin{aligned} |T_F(g,\sigma)(z)| &\leq \int_{\mathbf{R}^{\mathbf{n}+1}_+} |PI(g)(v,t)|\, d|\mu|(v,t) \\ &\leq \|PI(g)\|_{T^p_\infty} \|\mu\|_{V^{1/p}} \leq c_n \|\sigma\|_{M^\alpha} \|F_z\|_{L^q} \|g\|_{H^p}\cdot \quad \blacksquare \end{aligned}$$

COROLLARY 4.2.5. *If* $0 < p \leq 1$ *and* $g \in \mathcal{S}_0(\mathbf{R^n})$, *then*

$$|\hat{g}(z)| \leq C_{n,p} |z|^{n(\frac{1}{p}-1)} \|g\|_{H^p},$$

for all $z \in \mathbf{R^n}\cdot$

PROOF: It suffices to consider the case $0 < p < 1$ and $z \neq 0$. We recall that by Example 4.2.3 we have

$$T_F(g,\sigma)(z) = c_n \hat{g}(z) \int_0^\infty e^{-2\pi t|z|} t^{n(\frac{1}{p}-1)-1}\, dt\cdot$$

But,

$$\int_0^\infty e^{-2\pi t|z|}t^{n(\frac{1}{p}-1)-1}\,dt = C|z|^{-n(\frac{1}{p}-1)}\int_0^\infty e^{-2\pi u}u^{n(\frac{1}{p}-1)-1}\,du = C_{n,p}|z|^{-n(\frac{1}{p}-1)}.$$

Hence, by the theorem,

$$|T_F(g,\sigma)(z)| \leq c_n\|\sigma\|_{M^\alpha}\|F_z\|_\infty\|g\|_{H^p(\mathbf{R^n})};$$

that is,

$$C_{n,p}|\hat{g}(z)|\ |z|^{-n(\frac{1}{p}-1)} \leq \frac{c_n}{n(\frac{1}{p}-1)}\|g\|_{H^p},$$

which gives the result. ∎

REMARK 4.2.6: Corollary 4.2.5 was first proved in **[FE-ST]**, using a different approach. Later in **[TA-WE]**, it was also proved using the atomic characterization of H^p. We want to yet give another simple proof using now the duality of the H^p spaces. In **[DU-RO-SH]** it is shown that $(H^p(\mathbf{R^n}))^* = \dot{B}_\infty^{n(\frac{1}{p}-1),\infty}$, $0 < p < 1$, where the norm on this Besov space coincides with the Lipschitz norm of order $n(1/p-1)$, (see **[ST]**); namely,

$$\|f\|_{\dot{B}_\infty^{n(\frac{1}{p}-1),\infty}} = \sup_{\substack{x\in\mathbf{R^n}\\ h\in\mathbf{R^n}\setminus\{0\}}}\frac{|(\Delta_h^k f)(x)|}{|h|^{n(\frac{1}{p}-1)}},$$

where, $k\in\mathbf{N},\ k>\alpha$ and

$$(\Delta_h^k f)(x) = \sum_{r=0}^k\binom{k}{r}(-1)^r f(x+rh)\cdot$$

Now, we have the following

LEMMA 4.2.7. *Fix $y\in\mathbf{R^n}$, $\alpha>0$. Then*

$$\|e^{-iy\cdot}\|_{\dot{B}_\infty^{\alpha,\infty}} \approx |y|^\alpha\cdot$$

PROOF: Let $k\in\mathbf{N},\ k>\alpha$ and suppose $y\in\mathbf{R^n}\setminus\{0\}$. Then, for $h\in\mathbf{R^n}$

$$\begin{aligned}(\Delta_h^k e^{-iy\cdot})(x) &= \sum_{r=0}^k\binom{k}{r}(-1)^r e^{-iy(x+rh)}\\ &= e^{-iyx}\sum_{r=0}^k\binom{k}{r}(-1)^r e^{-iryh} = e^{-iyx}(1-e^{-iyh})^k.\end{aligned}$$

Hence,

$$|(\Delta_h^k e^{-iy\cdot})(x)|^2 = (2 - 2\cos(yh))^k\cdot$$

Thus,

$$\sup_{\substack{x\in\mathbf{R^n}\\ h\in\mathbf{R^n}\setminus\{0\}}} \frac{|(\Delta_h^k e^{-iy\cdot})(x)|}{|h|^\alpha} = \sup_{h\in\mathbf{R^n}\setminus\{0\}} 2^{k/2}\frac{(1-\cos(yh))^{k/2}}{|h|^\alpha}$$

$$\le C_k|y|^\alpha \sup_{u\in\mathbf{R}^+} \frac{(1-\cos u)^{k/2}}{u^\alpha} \le C_k \sup_{u\in\mathbf{R}^+} \frac{(1-\cos u)^{\alpha/2}}{u^\alpha}(1-\cos u)^{\frac{k-\alpha}{2}}|y|^\alpha$$

$$\le C_{k,\alpha}|y|^\alpha,$$

since $k > \alpha$. Conversely, we want to show that for any $y \in \mathbf{R^n}\setminus\{0\}$, there exists an $h \in \mathbf{R^n}\setminus\{0\}$ such that $|y| = |h|^{-1}$ and $1-\cos(yh) = 1-\cos(1) > 0\cdot$ In fact, if $h = y/|y|^2$ then trivially $|y| = |h|^{-1}$ and $y\cdot h = 1$. Hence

$$\|e^{-iy\cdot}\|_{\dot B_\infty^{\alpha,\infty}} \ge 2^{k/2}(1-\cos 1)^{k/2}|y|^\alpha\cdot \quad \blacksquare$$

Thus, by the duality between H^p and $\dot B_\infty^{n(\frac{1}{p}-1),\infty}$, $0 < p < 1$, and using this lemma, we find that if $g \in \mathcal{S}_0$

$$|\hat g(y)| = \left|\int_{\mathbf{R^n}} g(x)e^{-iyx}\,dx\right| \le \|g\|_{H^p}\|e^{-iy\cdot}\|_{\dot B_\infty^{n(\frac{1}{p}-1),\infty}} \le C_{n,p}|y|^{n(\frac{1}{p}-1)}\|g\|_{H^p}\cdot$$

As a curiosity, and from the proof of Corollary 4.2.5, we see that

$$\|e^{-iy\cdot}\|_{\dot B_\infty^{\alpha,\infty}} \approx \left(\int_0^\infty \widehat{P}_t(y)t^{\alpha-1}\,dt\right)^{-1}, \qquad \alpha > 0\cdot$$

One can also get very easily that, for $s > 0$, $1 < q \le \infty$ we have

$$\|e^{-iy\cdot}\|_{\dot B_\infty^{s,q}} \approx |y|^s\cdot$$

Hence (see **[TR]**), since

$$(\dot B_p^{s,q})^* = \dot B_\infty^{-s+n(\frac{1}{p}-1),q'} \qquad 0 < p \le 1,\ 0 < q < \infty,\ 0 < s < n(\frac{1}{p}-1),$$

and

$$(\dot{F}_p^{s,q})^* = \dot{B}_\infty^{-s+n(\frac{1}{p}-1),\infty} \qquad 0 < p < 1,\ 0 < q < \infty,\ 0 < s < n(\frac{1}{p} - 1),$$

where $q' = \infty$ if $0 < q \leq 1$, then, by a similar argument as above, we obtain

$$|\hat{f}(y)| \leq C|y|^{-s+n(\frac{1}{p}-1)}\|f\|_{\dot{B}_p^{s,q}}, \qquad 0 < p \leq 1,\ 0 < q < \infty,\ 0 < s < n(\frac{1}{p} - 1),$$

and

$$|\hat{f}(y)| \leq C|y|^{-s+n(\frac{1}{p}-1)}\|f\|_{\dot{F}_p^{s,q}}, \qquad 0 < p < 1,\ 0 < q < \infty,\ 0 < s < n(\frac{1}{p} - 1)\cdot$$

COROLLARY 4.2.8. *Suppose* $1 \leq q \leq \infty$, $\alpha \geq n/q$ *and* $1/p = \alpha/n + 1/q'$. *For* $f \in L^q(\mathbf{R^n})$ *and* $\sigma \in M^\alpha$ *define*

$$K(f,\sigma)(y) = \int_0^\infty (P_t * f)(y)\, d\sigma(t)\cdot$$

(i) *If* $0 < p < 1$ *then*

$$K : L^q(\mathbf{R^n}) \times M^\alpha \longrightarrow \dot{B}_\infty^{n(\frac{1}{p}-1),\infty},$$

and

$$\|K(f,\sigma)\|_{\dot{B}_\infty^{n(\frac{1}{p}-1),\infty}} \leq C_n\|\sigma\|_{M^\alpha}\|f\|_{L^q(\mathbf{R^n})}\cdot$$

(ii) *If* $p = 1$ *then*

$$K : L^q(\mathbf{R^n}) \times M^\alpha \longrightarrow BMO,$$

and

$$\|K(f,\sigma)\|_{BMO} \leq C_n\|\sigma\|_{M^\alpha}\|f\|_{L^q(\mathbf{R^n})}\cdot$$

PROOF: We will only show (i), because the proof of (ii) follows similarly. Since $(H^p(\mathbf{R^n}))^* = \dot{B}_\infty^{n(\frac{1}{p}-1),\infty}$, then to show that $K(f,\sigma) \in \dot{B}_\infty^{n(\frac{1}{p}-1),\infty}$ we only need to see that

$$\left|\int_{\mathbf{R^n}} g(y)K(f,\sigma)(y)\, dy\right| \leq C_n\|\sigma\|_{M^\alpha}\|f\|_{L^q}\|g\|_{H^p},$$

for all $g \in \mathcal{S}_0$. Set $F(z,x) = f(x)$, for all $z \in \mathbf{R^n}$. Then,

$$\int_{\mathbf{R^n}} g(y)K(f,\sigma)(y)\,dy = \int_{\mathbf{R^n}} g(y) \int_0^\infty (P_t * F_z)(y)\,d\sigma(t)dy = T_F(g,\sigma)(z),$$

for all $z \in \mathbf{R^n}$. Hence, by the last theorem,

$$\left| \int_{\mathbf{R^n}} g(y)K(f,\sigma)(y)\,dy \right| \leq C_n \|\sigma\|_{M^\alpha} \|f\|_{L^q} \|g\|_{H^p} \cdot \quad \blacksquare$$

We give now another application of our duality techniques to estimate harmonic extensions to $\mathbf{R}_+^{\mathbf{n}+1}$ of functions in H^p. The next theorem gives, as a particular case a generalization to higher dimensions of the Féjer-Riesz inequality (see **[GA-RU]** Theorems I-4.5 and III-7.57, for the case $p = 1$), and shows that it can be also proved in all cases $0 < p \leq 1$. Moreover, in the previous theorems, the authors work with the atomic characterization of H^1 and some extra conditions on the kernel are required, that will not be needed in our proof. This inequality gives the behavior in the vertical t-direction for the extension $\varphi_t * f(x)$, relative to a kernel φ, with $f \in \mathcal{S}_0$, instead of the well known growth on the x-direction for the harmonic extension $u \equiv PI(f)$; namely,

$$\sup_{t>0} \int_{\mathbf{R^n}} |u(x,t)|^p\,dx \leq C\|f\|_{H^p}^p.$$

The proof is based in finding the right pairing for an appropriate Carleson measure.

THEOREM 4.2.9. *If* $0 < p \leq 1$, $F \in T_\infty^p$ *and* $\sigma \in M^{n/p}$, *then*

$$\sup_{x\in\mathbf{R^n}} \int_0^\infty |F(x,t)|\,d|\sigma|(t) \leq \|\sigma\|_{M^{n/p}} \|F\|_{T_\infty^p}.$$

PROOF: Fix $x \in \mathbf{R^n}$ and set $d\mu(y,t) = \delta_x(y)d\sigma(t)$, where δ_x is the Dirac mass in $\mathbf{R^n}$ at the point x. Then $\mu \in V^{1/p}$ and $\|\mu\|_{V^{1/p}} \leq \|\sigma\|_{M^{n/p}}$. In fact, since $p \leq 1$, then if Q is a cube in $\mathbf{R^n}$ we have that

$$|\mu|(\widehat{Q}) \leq \left(\int_Q \delta_x(y)\right)\left(\int_0^{|Q|^{1/n}} d|\sigma|(t)\right) \leq |Q|^{1/p}\|\sigma\|_{M^{n/p}}.$$

Therefore, since $(T^p_\infty)^* = V^{1/p}$, we get that

$$\int_0^\infty |F(x,t)|\, d|\sigma|(t) \le \int_{\mathbf{R}^{n+1}_+} |F(y,t)|\, d|\mu|(y,t)$$

$$\le \|F\|_{T^p_\infty} \|\mu\|_{V^{1/p}} \le \|\sigma\|_{M^{n/p}} \|F\|_{T^p_\infty} \cdot \blacksquare$$

For the next result we introduce the following notation: if $f \in \mathcal{S}_0$, $0 < p \le 1$ and we choose $\varphi \in L^1 \cap L^\infty$, $\int_{\mathbf{R^n}} \varphi(x)\, dx \ne 0$ then we say that $f \in H^p_\varphi$ if $\|f\|_{H^p_\varphi} = \|\varphi_t * f\|_{T^p_\infty} < \infty \cdot$

COROLLARY 4.2.10. *Let φ be as above, $0 < p \le 1$.*

(i) (Féjer-Riesz inequality, if φ is the Poisson kernel).

If $f \in H^p_\varphi$, then

$$\sup_{x \in \mathbf{R^n}} \int_0^\infty |(\varphi_t * f)(x)| t^{\frac{n}{p}-1}\, dt \le C_{n,p} \|f\|_{H^p_\varphi} \cdot$$

(ii) *With more generality, if $p \le q \le 1$, then for $f \in H^p_\varphi$ we have*

$$\sup_{x \in \mathbf{R^n}} \int_0^\infty |(\varphi_t * f)(x)|^q\ t^{\frac{qn}{p}-1}\, dt \le C_{n,p} \|f\|^q_{H^p_\varphi} \cdot$$

PROOF:

(i) Consider the function $F(x,t) = (\varphi_t * f)(x)$ and the measure $d\sigma(t) = t^{\frac{n}{p}-1}\, dt$. Then $F \in T^p_\infty$ and $\sigma \in M^{n/p}$. Hence, by the theorem,

$$\sup_{x \in \mathbf{R^n}} \int_0^\infty |(\varphi_t * f)(x)| t^{\frac{n}{p}-1}\, dt = \sup_{x \in \mathbf{R^n}} \int_0^\infty |F(x,t)|\, d|\sigma|(t) \le C_{n,p} \|f\|_{H^p_\varphi} \cdot$$

(ii) Let $p \le q \le 1$ and consider now the function $F(x,t) = |(\varphi_t * f)(x)|^q$. Then $F \in T^{p/q}_\infty$ with $\|F\|_{T^{p/q}_\infty} = \|f\|^q_{H^p_\varphi}$. Also, if we set $d\sigma(t) = t^{\frac{qn}{p}-1}\ dt$ then $\sigma \in M^{qn/p}$ and hence, since $p/q \le 1$,

$$\sup_{x \in \mathbf{R^n}} \int_0^\infty |(\varphi_t * f)(x)|^q\ t^{\frac{qn}{p}-1}\, dt \le C_{n,p} \|F\|_{T^{p/q}_\infty} = C_{n,p} \|f\|^q_{H^p_\varphi} \cdot \quad \blacksquare$$

(§3) Weighted inequalities for maximal functions.

We will present an application of the atomic decomposition of the Tent spaces to get some weighted estimates for a maximal operator that generalizes the fractional maximal function.

DEFINITION 4.3.1. *Suppose $\varphi \in B_\Psi$. We define the fractional maximal function with respect to φ as*

$$M_\varphi f(x) = \sup_{x \in Q} \frac{1}{\varphi(|Q|)} \int_Q |f(y)|\, dy \cdot$$

For our main result we need to introduce the following notation. If $\varphi \in B_\Psi$ then, by Proposition 1.1.12-(20) we have that if $\rho_\varphi(t) = t/\varphi(t)$, then, $\rho_\varphi \in B_\Psi$. Hence, the function $\sigma_\varphi \equiv (\rho_\varphi)^{-1}$ is well defined and it is an increasing convex function.

THEOREM 4.3.2. *Suppose φ, φ_0, $\varphi_1 \in B_\Psi$ and let $1 \leq p < \infty$ be chosen such that $\varphi_0^p/t^{p-1} \in B_\Psi$. Let $w_i(t) = t^{1-1/p}/\varphi_i(t)$, $i = 0, 1$. Suppose μ is a measure in $\mathbf{R^n}$ and let $\Lambda^p(w_1, \mu)$ be the weighted Lorentz space, with respect to the measure μ. Suppose finally that μ satisfies the following property*

$$\mu(\Omega) \leq C\sigma_1 \left(\frac{\varphi(|\Omega|)}{\varphi_0(|\Omega|)} \right) \tag{24}$$

for all open and bounded $\Omega \subset \mathbf{R^n}$, where $\sigma_1(t) = \sigma_{\varphi_1}(t)$. Then

$$M_\varphi : \Lambda^p(w_0) \longrightarrow \Lambda^p(w_1, \mu) \tag{25}$$

as a bounded sublinear operator.

PROOF: Suppose $f \in \Lambda^p(w_0)$, $f \geq 0$ and set

$$f(x,t) = \frac{1}{t^n} \int_{|x-y|<t} f(y)\, dy, \qquad F(x,t) = f^p(x,t) \cdot$$

Then $A_\infty F(x) \leq C\, (Mf(x))^p$ and also

$$(M_\varphi f(x))^p \leq C \sup_{t>0} \left(\left(\frac{t^n}{\varphi(t^n)} \right)^p F(x,t) \right) \cdot$$

By Theorem 1.2.7 we have that M is bounded on $\Lambda^p(w_0)$ and so,

$$\|F\|_{T_\infty(\Lambda^1(w_0^p))} \leq C\|(Mf)^p\|_{\Lambda^1(w_0^p)} = C\|Mf\|^p_{\Lambda^p(w_0)} \leq C\|f\|^p_{\Lambda^p(w_0)} < \infty\cdot$$

Therefore $F \in T_\infty(\Lambda^1(w_0^p))$ and by hypothesis and Theorem 3.3.13 we can find a decomposition $F \equiv \sum_j \lambda_j b_j$ where,

$$\text{supp } b_j \subset T(\Omega_j), \quad \|b_j\|_\infty \leq C\frac{1}{|\Omega_j| w_0^p(|\Omega_j|)}, \quad \text{and} \quad \sum_j |\lambda_j| \leq C\|F\|_{T_\infty(\Lambda^1(w_0^p))}\cdot$$

Therefore,

$$|M_\varphi f(x)|^p \leq C\sum_j |\lambda_j| \sup_{t>0}\left\{\left(\frac{t^n}{\varphi(t^n)}\right)^p |b_j(x,t)|\right\}\cdot$$

Set

$$A_j(x) = \sup_{t>0}\left\{\left(\frac{t^n}{\varphi(t^n)}\right)^p |b_j(x,t)|\right\}\cdot$$

Then, supp $A_j \subset \Omega_j$ and

$$\|A_j\|_\infty \leq C\left(\frac{|\Omega_j|}{\varphi(|\Omega_j|)}\right)^p \left(\frac{\varphi_0(|\Omega_j|)}{|\Omega_j|}\right)^p = C\left(\frac{\varphi_0(|\Omega_j|)}{\varphi(|\Omega_j|)}\right)^p\cdot \tag{26}$$

Using (24) we also get

$$\int_0^{\mu(\Omega_j)} w_1^p(t)\,dt \approx \frac{\mu^p(\Omega_j)}{\varphi_1^p(\mu(\Omega_j))} \leq C\left(\frac{\varphi(|\Omega_j|)}{\varphi_0(|\Omega_j|)}\right)^p\cdot \tag{27}$$

These estimates allow us to show the following uniform bound for the functions A_j; namely,

$$\|A_j\|_{\Lambda^1(w_1^p,\mu)} \leq \|A_j\|_\infty \int_0^{\mu(\Omega_j)} w_1^p(t)\,dt \leq C\cdot$$

Thus

$$\begin{aligned}\|M_\varphi f\|_{\Lambda^p(w_1,\mu)} &= \|(M_\varphi f)^p\|^{1/p}_{\Lambda^1(w_1^p,\mu)} \\ &\leq C\Big(\sum_j |\lambda_j|\Big)^{1/p} \leq C\|F\|^{1/p}_{T_\infty(\Lambda^1(w_0^p))} \leq C\|f\|_{\Lambda^p(w_0)}\cdot \quad \blacksquare\end{aligned}$$

REMARKS 4.3.3:

(i) If we choose $\varphi(t) = t^{1-\alpha/n}$ with $0 < \alpha < n$, then $M_\varphi f = M_\alpha f$, is the fractional maximal operator of order α. If we set $\varphi_0(t) = t^{1-1/r}$, $\varphi_1(t) = t^{1-1/q}$ and $p = 1$ in the conditions of the theorem, we get, that if

$$\mu(\Omega) \leq C|\Omega|^{q(1/r-\alpha/n)},$$

then M_α is a bounded operator from $L^{r,1}$ into $L^{q,1}$. This result is a weaker version of Proposition 3.10 in [**BO-JO**].

(ii) Another interesting example is given by the following choice of the function parameters: Suppose $0 < \theta < 1$ and $0 \leq \gamma < \min(2\theta, 2(1-\theta))$ and set

$$f^{(\theta,\gamma)}(t) = t^\theta(1 + (\log t)^2)^{-\gamma/2}\cdot$$

Then $f^{(\theta,\gamma)} \in B_\Psi$ and in fact

$$0 < \alpha_{f^{(\theta,\gamma)}} = \theta - \frac{\gamma}{2} \leq \theta + \frac{\gamma}{2} = \beta_{f^{(\theta,\gamma)}} < 1\cdot$$

Since the function $f^{(\theta,\gamma)}$ is pointwise equivalent to $t^\theta(1+|\log t|)^{-\gamma}$ and by the example given after Definition 2.1.3 we have that if $1 \leq q \leq \infty$ and $w(t) = t^{1-1/q}/f^{(\theta,\gamma)}(t)$ then $\Lambda^q(w) = L^{p,q}(\log L)^\gamma$, where $p = 1/(1-\theta)\cdot$ We choose numbers $0 < \theta, \theta_0, \theta_1 < 1$ and

$$0 \leq \gamma < \min(2\theta, 2(1-\theta))$$
$$0 \leq \gamma_0 < \min(2\theta_0, 2(1-\theta_0))$$
$$0 \leq \gamma_1 < \min(2\theta_1, 2(1-\theta_1)),$$

and we define the functions:

$$\varphi = f^{(\theta,\gamma)},\ \varphi_0 = f^{(\theta_0,\gamma_0)},\ \varphi_1 = f^{(\theta_1,\gamma_1)}.$$

As in the theorem, we define the maximal function $M_{\theta,\gamma}f \equiv M_\varphi f$ and if $1 \leq p < \infty$ we also define the weights $w_i(t) = t^{1-1/p}/\varphi_i(t)$, $i = 0, 1$. Since we need to assume the condition $\varphi_0^p(t)/t^{p-1} \in B_\Psi$ we easily find that this is equivalent to also assuming that $p < 2/(2 - 2\theta_0 + \gamma_0)\cdot$ Now, condition (24) is equivalent to

$$\mu(\Omega)^{1-\theta_1}(1 + (\log \mu(\Omega))^2)^{\gamma_1/2} \leq C|\Omega|^{\theta-\theta_0}(1 + (\log|\Omega|)^2)^{(\gamma_0-\gamma)/2}. \tag{28}$$

Thus, under the above conditions, if μ satisfies (28), we have that

$$M_{\theta,\gamma} : L^{p_0,p}(\log L)^{\gamma_0} \longrightarrow L^{p_1,p}_{\mu}(\log L)^{\gamma_1}.$$

It is clear that if

$$\mu(\Omega) \leq C|\Omega|^{(\theta-\theta_0)/(1-\theta_1)},$$

with $\theta_0 \leq \theta$ and $\gamma_1 \leq \gamma_0 - \gamma$, then μ satisfies (28).

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