Hamiltonian-Hopf bifurcation under a periodic forcing

Quasi-periodicity in splitting of separatrices

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Abstract

We consider the effect of a non-autonomous periodic perturbation on a 2-dof autonomous system obtained as a truncation of the Hamiltonian-Hopf normal form. We study the splitting of the invariant 2-dimensional stable/unstable manifolds of a fixed point. Due to the interaction of the intrinsic angle and the periodic perturbation the splitting behaves quasi-periodically on two angles. Different frequencies are considered: quadratic irrationals, frequencies having continuous fraction expansion with bounded and unbounded quotients, and "typical" frequencies in measure theoretical sense.

The model

We consider the $(2+\frac{1}{2})$-dof Hamiltonian system $H(\mathbf{x},\mathbf{y},t) = H_0(\mathbf{x},\mathbf{y}) + \epsilon H_1(\mathbf{x},\mathbf{y},t)$, being $(\mathbf{x},\mathbf{y}) = (x_1,x_2,y_1,y_2)$, where

$$H_0(\mathbf{x},\mathbf{y}) = \Gamma_1 + \nu(\Gamma_2 - \Gamma_3 + \Gamma_3^2), \qquad \Gamma_1 = x_1y_2 - x_2y_1, \quad 2\Gamma_2 = x_1^2 + x_2^2, \quad 2\Gamma_3 = y_1^2 + y_2^2,$$
$$H_1(\mathbf{x},\mathbf{y},t) = g(y_1)f(\theta), \qquad g(y_1) = y_1^5/(d-y_1), \quad f(\theta) = (c-\cos(\theta))^{-1}, \ \theta = \gamma t + \theta_0.$$

- We fix concrete values of c, d, γ and ϵ, and consider $\nu > 0$ as a perturbative parameter.
- The parameter $\theta_0 \in [0, 2\pi)$ is the initial time phase.
- Note that H_1 contains all powers y_1^k, $k \geq 5$, and all harmonics in θ.

The unperturbed system H_0. The functions $G_1 = \Gamma_1$ and $G_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2$ are **independent first integrals**. In polar coordinates $x_1 + i x_2 = R_1 e^{i\psi_1}$, $y_1 + i y_2 = R_2 e^{i\psi_2}$ the restriction to (R_1, R_2)-components is a Duffing Hamiltonian system (hence having figure-eight shape separatrices). On $W^{u/s}(\mathbf{0})$ one has $\psi_1 = \psi_2 \pm \pi$, $\psi_2 = t + \psi_0$. The **2-dimensional homoclinic surface** is foliated by homoclinic orbits $(x_1(t), x_2(t), y_1(t), y_2(t))$ given by

$$x_1(t) + i\,x_2(t) = -R_1(t)e^{i\,\psi(t)}, \qquad y_1(t) + i\,y_2(t) = R_2(t)e^{i\,\psi(t)},$$

being $\psi(t) = t + \psi_0$, $R_1(t) = \sqrt{2}\,\mathrm{sech}(\nu t)\tanh(\nu t)$, and $R_2(t) = \sqrt{2}\,\mathrm{sech}(\nu t)$.

Periodic forcing: ϵH_1. When restricted to the unperturbed $W^{u/s}(\mathbf{0})$, $g(y_1)$ has a factor 1-periodic in t while $f(\theta)$ is periodic in t with frequency γ. Hence, $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ leads to **quasi-periodic** phenomena.

The invariant manifolds $W^{u/s}(0)$ for different ν values

The angles (ψ_0, θ_0) are initial conditions on a fundamental domain (torus $\mathcal{T}$) of $W^{u/s}(\mathbf{0})$. Write $H_0 = G_1 + \nu G_2$, $G_1 = \Gamma_1$, $G_2 = \Gamma_2 - \Gamma_3 + \Gamma_3^2$, and consider the **Poincaré section** $\Sigma = \max(R_2)$.

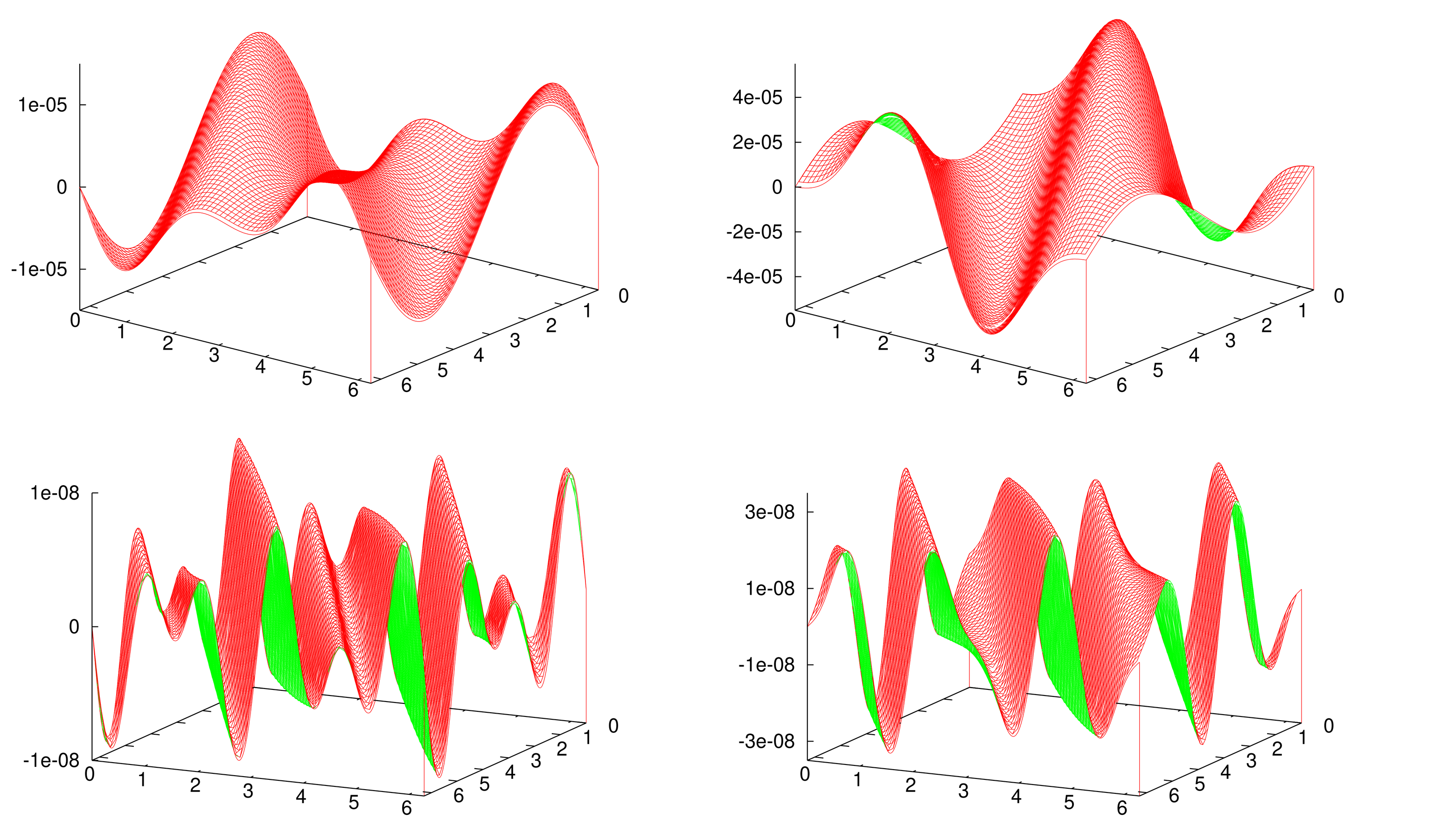

Figure 1: Splitting of the invariant manifolds: ΔG_1 (left) and ΔG_2 (right) for $\nu = 2^{-4}$ (top) and $\nu = 2^{-6}$ (bottom). We have considered $c = 5$, $d = 7$, $\epsilon = 10^{-3}$, and $\gamma = \gamma_0 = (\sqrt{5}-1)/2$.

The Poincaré-Melnikov function

For simplicity, we discuss on the G_1-splitting (similar for the G_2-splitting). Recall that $H_1 = g(y_1)f(\theta)$, where $g(y_1) = y_1^5(d-y_1)^{-1}$ and $f(\theta) = (c - \cos(\theta))^{-1}$. Let c_j (resp. d_k) be the coefficients of the Fourier (resp. Taylor) expansion of f (resp. g'), that is,

$$f(\theta) = \sum_{j\geq 0} c_j \cos(j\theta), \qquad g'(y_1) = \sum_{k\geq 0} d_k y_1^{5+k}.$$

If $\zeta^0(s)$ is a solution of the system when $\epsilon = 0$, then one has $\psi = t + \psi_0$, $\theta = \gamma t + \theta_0$, $(\psi_0, \theta_0) \in \mathcal{T}$, and the distance

$$G_1^u(\psi_0,\theta_0) - G_1^s(\psi_0,\theta_0) = \Delta G_1 + \mathcal{O}(\epsilon^2),$$

is given by

$$\Delta G_1 = \epsilon \int_{-\infty}^{\infty} \{G_1, H_1\} \circ \zeta^0(s)\, ds = 4\epsilon \int_{-\infty}^{\infty} \sin(t+\psi_0)\, f(\gamma t + \theta_0) \sum_{k\geq 0} \frac{\sqrt{2^{k+1}}\, d_k\, (\cos(t+\psi_0))^{4+k}}{(\cosh(\nu t))^{5+k}}\, dt.$$

After some algebra one obtains

$$\Delta G_1 = \epsilon \sum_{j\geq 0} c_j \sum_{k\geq 0} 2^{\frac{3+k}{2}} d_k \sum_{0\leq 2i\leq 4+k} b_{4+k,i} \sum_{l=\pm 1} I_1 \sin((k+5-2i)\psi_0 + lj\theta_0) = \epsilon \sum_{m_1\geq 0} \sum_{m_2\in\mathbb{Z}} C^{(1)}_{m_1,m_2} \sin(m_1\psi_0 - m_2\theta_0), \qquad \text{where}$$

$$I_1 = I_1(k+5-2i+lj\gamma, \nu, k+5), \quad I_1(s,\nu,n) = \int_{\mathbb{R}} \frac{\cos(st)}{(\cosh(\nu t))^n}\, dt, \quad b_{m,i} = \frac{m+1-2i}{2^m(m+1)} \binom{m+1}{i}.$$

Main result

Assume that $\epsilon > 0$, $c > 1$, $d > \sqrt{2}$, $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ and $\nu < \nu_M \ll 1$. Let m_1/m_2 be an approximant of γ, and let $c_s \in \mathbb{R}$ be the constant such that $c_s m_1 |m_1 - \gamma m_2| = 1$.

Theorem. There exists a "universal" (almost independent of γ) function $\psi_1(L)$ s.t. **the contribution of the harmonic** associated to m_1/m_2 to the splitting satisfies

$$\psi_i(L)|_{L=c_s\nu m_1^2} \approx \sqrt{c_s\nu}\log|C^{(i)}_{m_1,m_2}|, \quad \text{when } \nu \to 0,$$

where $\Psi_2(L) = \Psi_1(L) - \sqrt{L}\log L/m_1$, $\Psi_i(L) \leq \Psi_M \approx -4.860298$.

In particular, if m_1/m_2 corresponds to a **dominant best approximant harmonic (BA)** of ΔG_1 (resp. ΔG_2) for $\nu \in (\nu_0, \nu_1)$, $\nu_0, \nu_1 \ll 1$, then $\Delta G_i \approx \exp\left(\psi_i(L)|_{L=\nu m_1^2 c_s}/\sqrt{c_s\nu}\right)$, $i = 1, 2$.

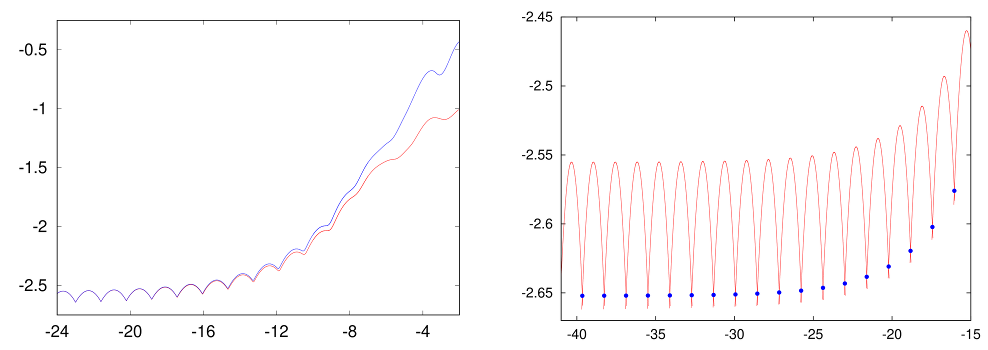

Figure 2: For $\gamma = (\sqrt{5}-1)/2$, $\epsilon = 10^{-4}$ we represent $\sqrt{\nu}\log|C^{(1)}_{m_1,m_2}/\epsilon|$ as a function of $\log_2(\nu)$. In the right plot, the points correspond to the values ν_j where the dominant harmonic changes. As expected, dominant harmonics are associated to best approximants: from $m_1 = F_j \to F_{j+1}$, where $\{F_j\}_j$ denotes the Fibonacci sequence. The rightmost change corresponds to $m_1 = 55 \to m_1 = 89$, while the leftmost to $m_1 = 196418 \to m_1 = 317811$.

Other frequencies: hidden/not hidden best approximants

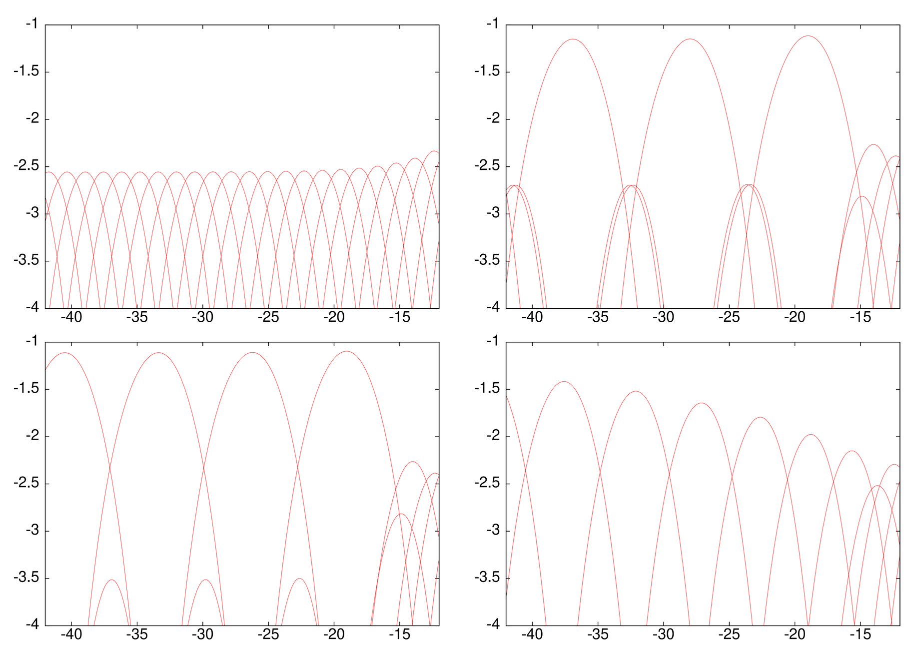

Figure 3: We display $\sqrt{\nu}(\log(\Delta G_1)/\epsilon)$ as a function of $\log_2(\nu)$.
Top left : $\gamma_0 = (\sqrt{5}-1)/2 = [0; 1, 1, 1, 1, 1, \ldots] \approx 0.618033988749894$.
Top right : $\gamma_1 = [0; 10 \times 1, 1, 10, 1, 1, 10, 1, 1, 10, 1, \ldots] \approx 0.618051226819253$.
Bottom left : $\gamma_2 = [0; 10 \times 1, 1, 10, 1, 10, 1, 10, 1, 10 \ldots] \approx 0.618051374461158$.
Bottom right: $\gamma_3 = [0; 10 \times 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots] \approx 0.618020663293438$.

Hidden BA (HBA) and "typical" measure-theoretical properties

Assume that (our system satisfies these assumptions):

- The perturbation is the product of two functions $f(x_1, x_2, y_1, y_2)$ and $g(\theta)$. Denote by $\mathcal{P}_1(t,\psi)$ and $\mathcal{P}_2(\theta)$ their contribution to the Poincaré-Melnikov integral.
- The homoclinic connections tend to zero when $t \to \pm\infty$ as $\mathrm{sech}(\nu t)$.
- $\mathcal{P}_1(t,\psi)$ is of the form $\sum A_j(t)\sin(j\psi)$, $\psi = t + \psi_0$, where A_j depend on powers of $\mathrm{sech}(t)$ and $\|A_j\| \sim \exp(-j\rho_1)$, $\rho_1 > 0$,
- $\mathcal{P}_2(\theta)$ is of the form $B\sum_{j\geq 1}\exp(-j\rho_2)\cos(j\theta)$, $\theta = \gamma t + \theta_0$, $\rho_2 > 0$.

Then **minus the logarithm of the contribution** of the harmonic related to the BA N_k/D_k to the Poincaré-Melnikov function is

$$T(\nu, D_k) \approx D_k + s_k/\nu,$$

where $s_k = |N_k - \gamma D_k|$ and where we have approximated $N_k = \gamma D_k + \mathcal{O}(D_k^{-2})$. The role of CFE appears as $s_k^{-1} = D_k\left(c_k^+ + 1/c_k^-\right)$, $c_k^+ = [q_{k+1}; q_{k+2}, \ldots]$, $c_k^- = [q_k; q_{k-1}, \ldots, q_1]$. We are interested in **minimizing** $T(\nu, D_k)$ for a given ν.

Theorem. (1) **Two consecutive** harmonics associated to BA **cannot** be hidden.
(2) If the $k+1$-th harmonic associated to BA is hidden then $q_{k+2} = 1$.

The following properties related to the CFE of γ hold for numbers in a **set of full measure**:

- The geometric mean of CFE quotients tends to the Khinchin constant KC ≈ 2.685452.
- If D_n are the BA denominators, then $\lim_{n\to\infty}\log(D_n)/n \to \mathrm{LC} = \pi^2/(12\log(2))$ (Levy constant).
- The Gauss map $x \to 1/x - [1/x]$ is ergodic and the probability of having k as a quotient is given by the Gauss-Kuzmin law: $P(k) = \log_2(1 + 1/(k^2+2k))$. For a "typical" number, its CFE is a sequence of realizations of **not independent** identically distributed random variables.

Conjecture: Under the stated assumptions on the homoclinic and the perturbation, for a set of ratios of two frequencies $(1,\gamma)$ of full measure, **the distribution of HBA follows a normal law.**

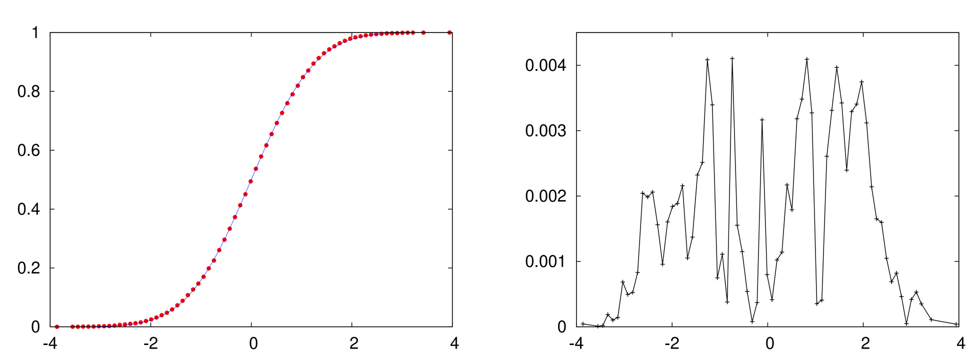

Figure 4: We display the results for $\gamma = \pi - 3$. Counting the HBA in blocks of 1000 consecutive BA, we obtain that the CDF is $N(\mu,\sigma)$ with $\mu \approx 279.118$ and $\sigma \approx 9.604$ (more than 1/4th of the BA are HBA). One has 2785810 HBA from the first 10^7 quotients. Similar results were obtained for the "typical" frequencies $e^{\gamma_0} - 1$, $e^{\sqrt{2}} - 4$, $e^{\sqrt{3}} - 5$, $e^{\sqrt{5}} - 9$, and $e^{\sqrt{7}} - 14$.

References

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