# *Resonant chaotic zones: dynamical consequences of the difference between the inner/outer splittings of separatrices*

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# Introduction: general framework

Let $F_\nu$ be a one-parameter family of APMs

$F_\nu(E_0) = E_0$ elliptic fixed point,

Spec$(DF_\nu)(E_0) = \{\lambda, \lambda^{-1}\}$, $\lambda = \exp(2\pi i \alpha)$.

$\rightarrow$ Assume that we are interested in the dynamics close to the $(q\!:\!m)$-resonance for $q, m \in \mathbb{N}$, with $1 \leq q < m$, gcd$(q, m) = 1$. Then, one can write $\alpha = q/m + \delta$ with $\delta \in \mathbb{R}$ (generically $\alpha'(\nu) \neq 0$) and we denote the family as $F_\delta$ (for arbitrary $q$ and $m$).

$\rightarrow$ $F_\delta : \mathcal{U} \rightarrow \mathbb{R}^2, \mathcal{U} \subset \mathbb{R}^2$ domain, is such that

1. $F_\delta$ real analytic in the $(x, y)$-coordinates of $\mathcal{U}$,
2. $\det DF_\delta(x, y) = 1$, for all $(x, y) \in \mathbb{R}^2$ and for all $\delta \in \mathbb{R}$, (APMs)
3. $F_\delta$ has a fixed point $E_0$ that will be assumed to be at the origin $\forall \delta \in \mathbb{R}$,
4. spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = \exp(2\pi i \alpha)$, $\alpha = q/m + \delta$, $q, m \in \mathbb{Z}$.

# Hénon map

As an example consider the Hénon map

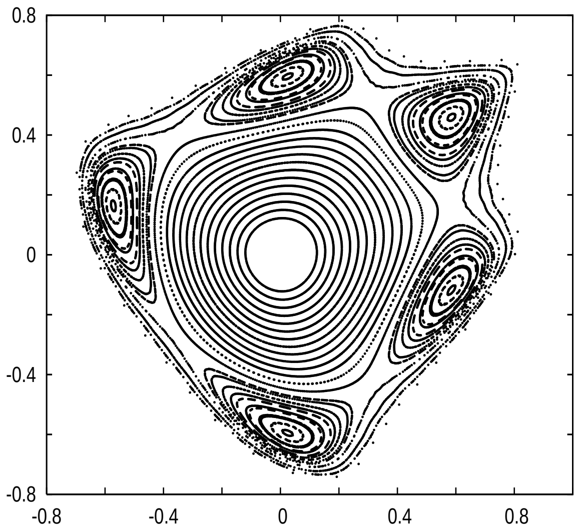
$$H_\alpha(x, y) = R_{2\pi\alpha}(x, y - x^2), \qquad \alpha \in (0, 1/2)$$

- It has two fixed points:

  the origin is an elliptic fixed point $E_0$,

  the point $P_h = (2\tan(\pi\alpha), 2\tan^2(\pi\alpha))$ is a hyperbolic fixed point.

- Reversible with respect to $y = x^2/2$ and $y = \tan(\pi\alpha)x$.



# I. The inner/outer splitting of separatrices for a resonant island

- We want to describe the dynamics in the **resonant chains** emanating from (but **relatively far** from) the elliptic fixed point $E_0$.
- Special interest in quantitative information concerning the splitting of separatrices and the chaotic zone.

Planning:

BNF $\longrightarrow$ Interp. Hamiltonian $\longrightarrow$ Simplified Model $\longrightarrow$ Splitting of separatrices

# BNF

$F_\delta$ one-parameter $\delta$-family of APMs with $F(E_0) = E_0$ elliptic fixed point.
Spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = e^{2\pi i\alpha}$, $\alpha = q/m + \delta$, $\delta$ small enough.
$(x, y)$-Cartesian coord., $(z, \bar{z})$-complex coord. ($z = x + iy$, $\bar{z} = x - iy$).
The Birkhoff NF to order $m$ around $E_0$ can be expressed as

$$\text{BNF}_m(F)(z) = R_{2\pi\frac{q}{m}}\left(\underbrace{e^{2\pi i\gamma(r)}z}_{\text{unavoidable res.}} + \underbrace{i\bar{z}^{m-1}}_{m\text{-order res.}}\right) + R_{m+1}(z, \bar{z}),$$

where

$$\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \ldots + b_s r^{2s}, \quad r = |z|,$$

being

$s = [(m-1)/2]$,
$b_i \in \mathbb{R}$ are the so-called Birkhoff coefficients,
$R_{m+1}(z, \bar{z})$ denotes the remainder which is of $\mathcal{O}(m+1)$.

# Interpolating flow of the BNF

$(I, \varphi)$-Poincaré variables ($z = \sqrt{2I}\exp(i\varphi)$).

$$\mathcal{H}_{nr}(I) = \pi \sum_{n=0}^{s} \frac{b_n}{n+1}(2I)^{n+1} \quad \text{and} \quad \mathcal{H}_r(I,\varphi) = \frac{1}{m}(2I)^{\frac{m}{2}}\cos(m\varphi).$$

Let $r_*$ such that $\gamma(r_*) = 0$, that is $r_* \approx (-b_0/b_1)^{1/2}$, $b_0 = \delta$.

$\rightarrow$ The flow $\phi$ generated by the Hamiltonian

$$\mathcal{H}(I,\varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I,\varphi)$$

interpolates $K$ with an error of order $m+1$ with respect to the $(z, \bar{z})$-coordinates, that is,

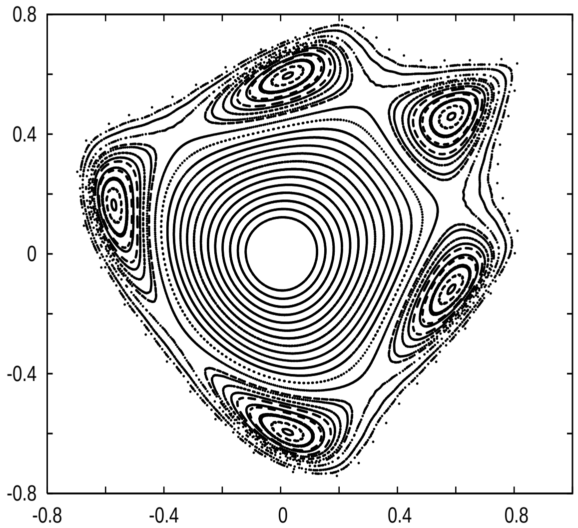
$$K(I,\varphi) = \phi_{t=1}(I,\varphi) + \mathcal{O}\left(I^{\frac{m+1}{2}}\right).$$

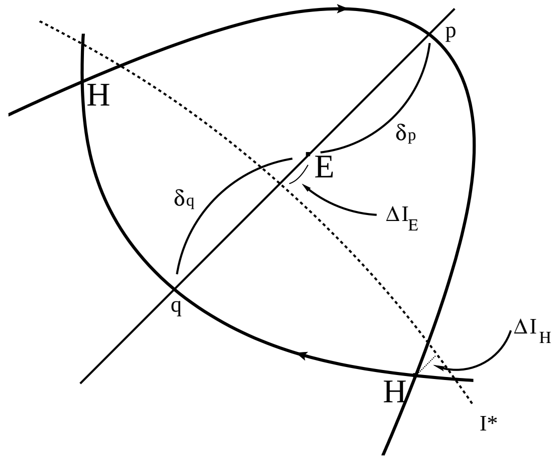
If we assume $b_1 \neq 0$ this approximation holds in an annulus centred in the resonance radius $r_*$ of width $r_*^{1+\nu}$, for $\nu > 0$.

# *Description of resonances*

Generic case: $\alpha = q/m + \delta$, $m > 5$, $\delta$ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then $F$ **has** a resonant island of order $m$.
- The resonant zone is determined by **two periodic orbits** of period $m$ located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.




"Outer splitting $\leftrightarrow$ p"

"Inner splitting $\leftrightarrow$ q"

# A model around a generic resonance

For a generic APM such that $\delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the $m$-resonance strip $(m \geq 5)$ can be modelled, after suitable scaling ($J \sim \delta^{-m/4}(I - I_*)$), by the time-$\log(\lambda)$ map of the flow generated by

$$\mathcal{H}(J, \psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos(\psi),$$

where $c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1})$. Bounding the errors, it is shown that it gives a “good” enough approximation of the dynamics in an annulus containing the $m$-islands.

$\longrightarrow$ Then, we have the following... [a]

[a] The details of the proof (singularities, suitable Hamiltonian,...) can be found in: *Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps.* Nonlinearity 22, 5:1191–1245, 2009.

# Main result: the hypothesis

- **A1.** $b_1(\delta)$ is non-zero for $\delta = 0$.
- **A2.** $F$ maybe meromorphic but the possible singularities remain at a finite distance as $|\delta| \searrow 0$.
- **A3.** $P_h^r$ – hyperbolic $m$-periodic point on a resonant zone close to $E_0$,
  $\gamma(t)$ – separatrix of the interp. Hamiltonian flow $\varphi_t$,
  Assume that the closest singularities of $\gamma(t)$ to $\mathbb{R}$ have $|\text{Im}(t)| = \tau$.
  Represent $W^u_{P_h^r}$ and $W^s_{P_h^r}$ as functions of $t$, close to $\gamma(t)$.
  $\mathcal{E}(t)$ – distance $W^u_{P_h^r}(t) - W^s_{P_h^r}(t)$ (periodic in $t$).
  $G(t)$ – restriction of $\mathcal{E}(t)$ to $t + i(\tau - \mathcal{O}(\delta^q)),\ t \in \mathbb{R}$, $q > 0$.
  We require that there exist constants $k_1, k_2 > 0$ and $j_2 \leq j_1$ such that for all $\delta$, $0 < \delta < \delta_0$, one has $k_1\delta^{j_1} < |G| < k_2\delta^{j_2}$ and that the first harmonic $c_1$ of the Fourier expansion of $G(t)$ verifies $|c_1| > \alpha|G|$, with $\alpha > 0$ a constant independent of $\delta$.

# *Main result*

**Theorem.** Let $F$ be an APM. Assume that it has an $m$-order resonance strip, $m > 4$, located at an average distance $I = I_* = \mathcal{O}(\delta)$ from the elliptic fixed point and $\delta$ is sufficiently small. Under the assumptions **A1**, **A2** and **A3**,

a) The outer splitting $\sigma_+$ is larger than the inner one $\sigma_-$. The difference between the position of the corresp. nearest singularities is $\mathcal{O}(\delta^{m/4-1})$.

b) Neither the inner nor the outer splittings oscillate.

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- It should be adapted to strong resonances (e.g. 1:4 res. of Hénon map).
- It does not apply if too far from the origin (e.g. the 2:11 res. of Hénon map).
- $\mathcal{H}(J, \psi)$ plays the role of a "limit" Hamiltonian in Fontich-Simó thm. on exp. small upper bounds of the splitting $\rightarrow$ singularities $\tau_\pm = \frac{\pi}{2} \pm d + \ldots$.
- $\sigma_\pm = \exp\left(-\frac{2\pi \,\text{Im}\, \tau_\pm - \eta_\pm}{\log(\lambda(\epsilon))}\right)\left(\cos\left(\frac{2\pi \,\text{Re}\, \tau_\pm}{\log(\lambda(\epsilon))} - \phi_\pm\right) + o(1)\right)$.

# II. A heuristic justification of why upper bounds are expected to be generic

The theorem states that, under assumptions **A1**, **A2** and **A3**, $\sigma_+ > \sigma_-$.

Note that:

- **A1** is a generic assumption.
- Concerning **A2**, a suitable scaling to study the resonance zone moves the possible singularities of $F$ to a distance $\mathcal{O}(\delta^{-m/4})$.
- **A3** guarantees that the splitting of separatrices behaves exponentially small w.r.t. $\delta$ (as Fontich-Simó upper bound).

$\longrightarrow$ Question: How to proceed to check assumption **A3**?

# *Numerical check of* **A3**

For a fixed $\delta$:

1. Compute the parametrisation $g_u$ of $W^u(P_h)$ (resp. $g_s$ for $W^s(P_h)$):
   $F(x(s), y(s)) = (x(\lambda s), y(\lambda s))$, $s \in \mathbb{C}$.
2. Introduce $t = \log(s)$, then $g_u(t+h) = g_u(t)$, $h = \log(\lambda)$.
3. Using BNF around $P_h$ define an energy $E(x, y)$ and transport it along the manifolds.
4. Measure the difference between $W^u(P_h)$ and $W^s(P_h)$ in a fundamental domain. This gives a periodic function $\mathcal{E}(t)$.
5. Restrict $\mathcal{E}(t)$ to a suitable line $t_r + i\sigma$, with a suitable $\sigma < \tau$. This gives a periodic function $G(t)$.
6. Carry out the Fourier analysis to check **A3**.

Repeat the process for different $\delta$ values ($0 < \delta < \delta_0$). **Ok but Expensive!!**.

# Comments

To check **A3** directly is difficult (for a general map).

However, if $F$ is given by a closed-form expression, we can check directly the exponentially small behaviour w.r.t. $\delta$ in a simple way.

**Remark.** Any finite order jet of $F$ is useful to analyse beyond-all-orders phenomena: the ignored terms become relevant close to a singularity.

$\longrightarrow$ We show how to proceed in a concrete example which also "justifies" why we expect that the behaviour of the inner/outer splittings of a resonant island is (generically!) given by the exponentially small upper bound. [a]

[a] Some of the details can be found in the appendix of *Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.* Physica D, 240(8), 2011.

# An improved model around a resonant island

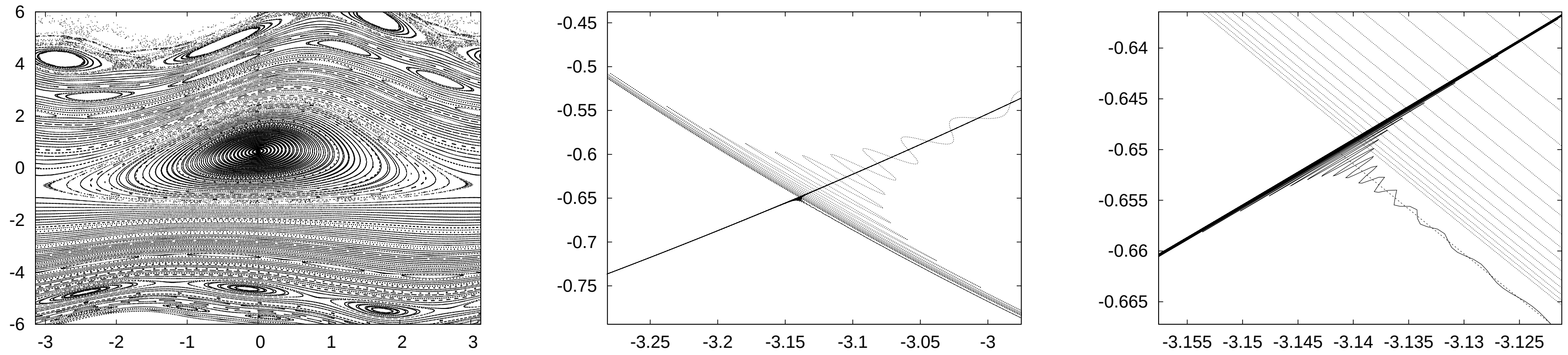
$m$-resonance

$\mathcal{H}(q,p) = p^2/2 - (1+dp)\cos(q)$, $d = \mathcal{O}(\delta^{m/4-1})$.

$\varphi^{\mathcal{H}}_{t=\gamma}$, $\gamma \approx \log(\lambda) = \mathcal{O}(\delta^{m/4})$, approx. the dynamics around the $m$-res.

An approximation of $\varphi^{\mathcal{H}}_{t=\gamma}$ is given by

$$\text{MSTM: } \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} q + \gamma(\bar{p} - d\cos(q)) \\ (p - \gamma\sin(q))/(1+\gamma d \sin(q)) \end{pmatrix}$$




$\delta = 0.65$, $m = 8$, $d = \delta^{m/4-1}$, $\gamma = \delta^{m/4}$.

# The limit (inner or semi) map

Singularities: $i\pi/2 \pm d + \mathcal{O}(d^2)$. Introducing $q = i\,(A+u)$, $p = B\,v$, with $A = \log(-2i/\gamma d)$ and $B = i/\gamma$ we get the limit map (for $\gamma \to 0$):

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} u + \bar{v} + e^u \\ v/(1+e^u) \end{pmatrix} \quad \text{indep. of parameters}$$

Fixed points: $v = 0$, Re$u = -\infty$ (Im$u$ arbitrary). Introduce $w = e^u$, then

$$\begin{pmatrix} w \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \bar{w} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} w\exp(w + \bar{v}) \\ v/(1+w) \end{pmatrix}$$

Fixed points: $w = 0$. The f.p. $(0,0)$ is parabolic with inv. manifolds $v = g(w)$ with slopes $0$, $-2$ (in $\mathbb{C}^2$).

$\longrightarrow$ It is enough to show that the inv. manifolds of the limit map do not coincide.

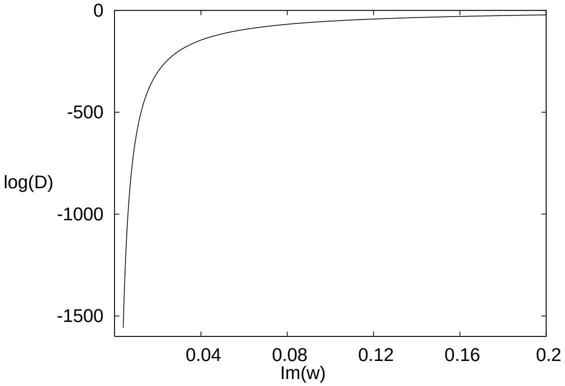
# The inv. manifolds distance

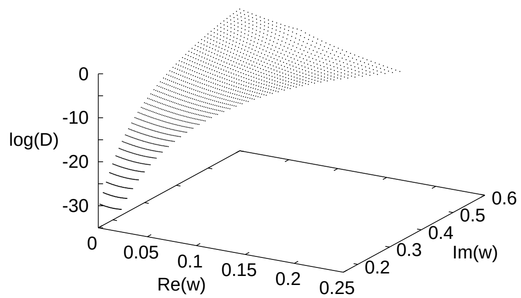
We look for the distance between the inv. manifolds in the complex domain $\mathbb{C}^2$.

$\longrightarrow$ The inv. manifold with slope $0$ corresponds to $v = 0$: On it $w \mapsto we^w$ and (locally!) $w = 0$ is foliated by homoclinic invariant curves.

$\longrightarrow$ For the inv. manifold $v = -2w + \ldots$ the $W^u/W^s$ branches *do not coincide*.

We use a graph repr. $v = g(w)$ of $W^u/W^s$ around $w = 0$ (locally) and we compute the distance between $W^u$ and $W^s$ on Re$(w) = 0$.




Right: Considering a fundamental domain we observe that there are not homoclinic points (for $0 \leq$ Im$(w) \leq 0.16$).

# Some remarks, work in progress...

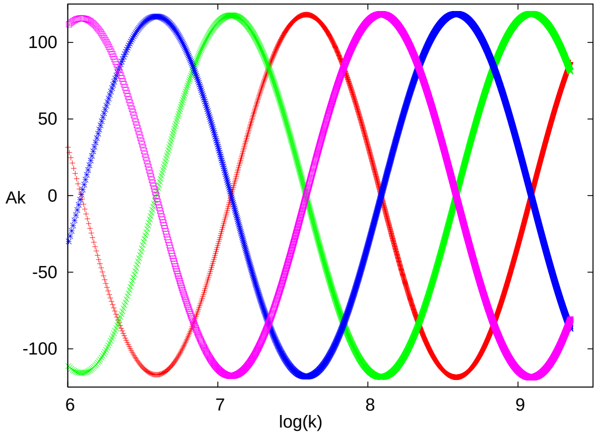
- By continuity the MSTM map has an exponentially small splitting.
- There is a strong numerical evidence supporting the following facts:

1. The inv. manifold $v = g(w) = \sum_k a_k w^k$ has a Gevrey-1 character.
2. The radius of convergence of the (scaled) Borel transform

$$A(\xi) := \sum_{k\geq 1} A_k \xi^k, \quad A_k = \frac{a_k}{k!}(2\pi)^k,$$

   is $\pm i$. It has an essential singularity: $A(i-\xi) \sim \xi^{\pi i}/\xi$ for $|\xi| << 1$.

3. The coefficients $A_k$ behave as




red, green, blue, magenta

$$\updownarrow$$

$$k \equiv 0, 1, 2, 3 (\text{mod } 4)$$

## III. Dynamical consequences of the difference between the inner/outer splittings

- The splitting of separatrices creates a chaotic zone (CZ).
- In a resonant island both inner/outer splittings play a role.

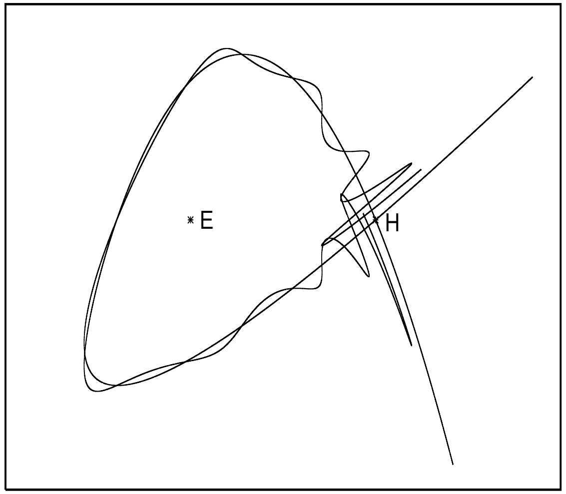
$\longrightarrow$ Question: Can we estimate the size of the CZ?

Planning:

1. Size of CZ if only one splitting plays a role (*open* case)?
2. How to take into account the effect of both splittings (*figure eight* case)?

Main tool: return maps (SM + aprox. by STM)

# *Open case*




$$SM : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b \log |y'| \\ y + \sin(2\pi x) \end{pmatrix}$$

where $b = 1/\log(\lambda)$, $\lambda$ the dominant eigenvalue of $DF(h)$ and $a$ is a “shift”.

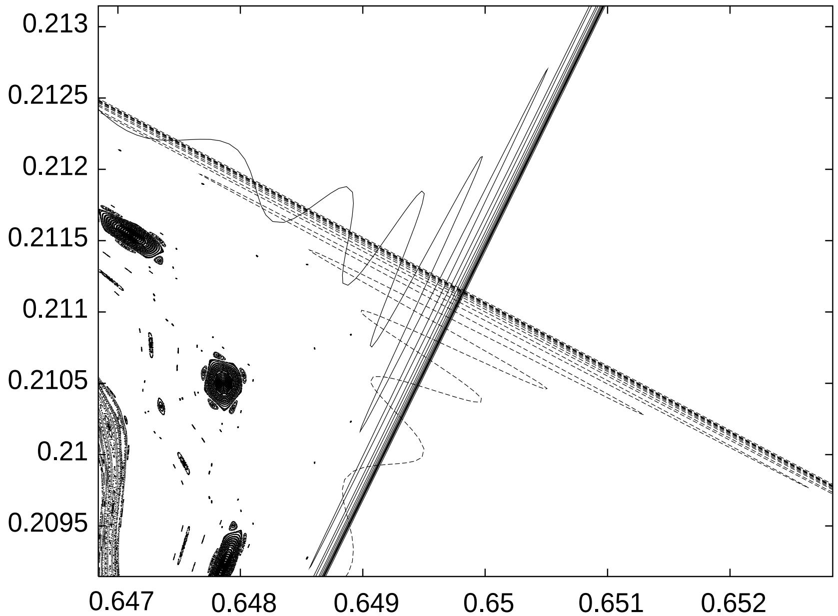
The $y$-vble. is scaled by the amplitude of the splitting.

We deal with an **a priori stable** case: $\log(\lambda) = \mathcal{O}(\epsilon)$ and $a = \mathcal{O}(1/\epsilon) \Rightarrow$ $A = \mathcal{O}(\exp(-ctant/\epsilon^r)$. Here $\epsilon$ is a “distance-to-integrable” parameter.

# Open case: results

- Distance to invariant curves from the separatrix: $d_c \sim |b|/k^*$ (SM is approximated by STM, $k^* \approx 0.97/(2\pi)$ Greene value).
  - When coming back to the original variables: $D_c \sim \sigma\ell/(2\pi k^* \log(\lambda))$,
  - If measured from the hyperbolic point, assuming the map close to the time-$\epsilon$ flow of $H(x,y) = y^2/2 - \alpha x^3 - \beta x^2$, one has: $D_c^h \approx (3LD_c/2)^{1/2}$, where $L$ is the distance between the hyperbolic and the elliptic point inside the “fish”. This result can be improved using higher order interpolating Hamiltonians.
- Distance to islands from the separatrix: $d_i \sim |b|/\tilde{k}$, $\tilde{k} = 2/\pi$.
- Expected number of “central” islands before the r.i.c. $\#\{islands\} \approx 1.415 \times b$.

# Hénon map $H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2)$



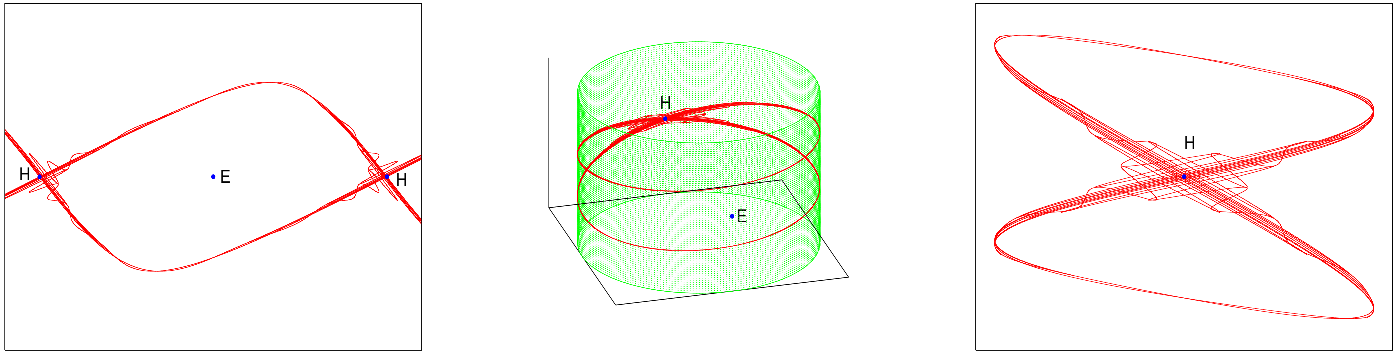

$\alpha = 0.1$

Experimental values: $(D_c^H)_e \approx 2.94 \times 10^{-3}$, $(D_i^H)_e \approx 2.08 \times 10^{-3}$

“Fish” interpolating Hamiltonian: $D_c^H \approx 2.47 \times 10^{-3}$, $D_i^H \approx 1.85 \times 10^{-3}$

5-order interp. Hamiltonian: $D_c^H \approx 2.731 \times 10^{-3}$, $D_i^H \approx 2.050 \times 10^{-3}$

# Figure eight case




$$DSM: \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{\bar{s}} + b \log|\bar{y}| \pmod 1 \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix},$$

where $\nu$ is such that $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, being $A_1$ and $A_{-1}$ the amplitudes of the outer/inner splittings resp. of the resonant island.

- It is defined on a domain $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ (upper/lower domains around the outer/inner separatrices of the resonance).
- $y > 0$ means we are outside the stable manifold (either in $\mathcal{U}$ or $\mathcal{D}$).

# *Main result on CZ*

**Theorem.** [a] Consider a generic resonance ($m \geq 5$) rel. close to the origin ($\delta$ rel. small). Assume $b_1\delta < 0$ and that the hypothesis **A1**, **A2** and **A3** of the theorem concerning the difference of the inner/outer splittings hold. Then,

- The width of the outer chaotic zone is larger than the width of the inner chaotic one if, and only if, sign $b_1 \cdot$ sign $b_2 < 0$.
- Both amplitudes of the stochastic layer are of the order of magnitude of the outer splitting (the largest one).

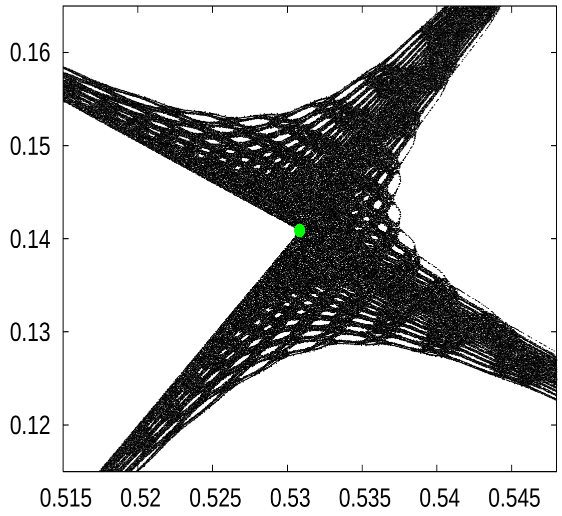
[a] Details and also examples of non-generic situations (strong resonances), can be found in: *Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.* Physica D, 240(8), 2011.

# Pendulum-like islands: comments

The idea is to construct an interpolating Hamiltonian of the map (in a domain containing the resonance) and to use preservation of energy to see how the distance to the rotational invariant curves changes when measuring from the upper $\mathcal{U}$ and the lower $\mathcal{D}$ domains. This can be done computing the ratio

$$f = \nabla\mathcal{H}(J_M)/\nabla\mathcal{H}(J_m)$$

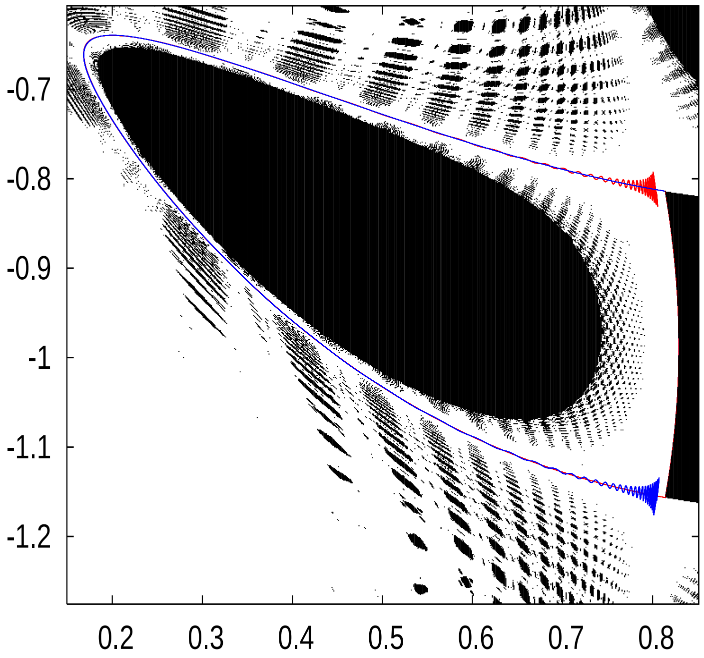
where $J_M$ and $J_m$ are the maximum (minimum) of the outer (inner) separatrix of the Hamiltonian. For close to the origin resonances $f = 1 + \mathcal{O}(\delta^{m/4})$.




$$\alpha = 0.21,$$
$$\sigma_- = \mathcal{O}(10^{-12}),$$
$$\sigma_+ = \mathcal{O}(10^{-3}).$$

# The 1:4 resonance of the Hénon map

- The same idea applies to resonances far from the origin as well as for strong resonances.
- An adapted interpolating Hamiltonian must be considered in each case.
- The “inner/outer amplitudes” of CZ can be of different order of magnitude.




$c = 1.015$,

$\sigma_+ = \mathcal{O}(10^{-54})$, $\sigma_- = \mathcal{O}(10^{-1})$.

Experimentally, $f \approx -5$. Using interp. Ham. up to order $\delta \approx c - 1$ we obtain $f \approx -5.64$.

But $\delta = 0.015$ is too large. For $\delta$ small we obtain better results (even we can predict # tiny islands).

**Thanks for your attention!!**