Discrete averaging for maps

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Motivational framework: beam dynamics in plasma physics

The analysis of accelerator-beam dynamics is an important topic in plasma physics. This is usually modelled by a (close-to-)symplectic return map.

In this setting, averaging for near-identity maps and the corresponding adiabatic invariants are used to control particle trapping into resonances.

Adiabatic trapping:

- Simplifies modeling: manageable equations to design of plasma devices.
- Explains confinement: when fields change slowly, adiabatic invariants let you predict how particles move and whether they stay trapped or escape.
- Evolution wrt parameters of adiabatic invariants is useful to split a particle beam into separate beams.

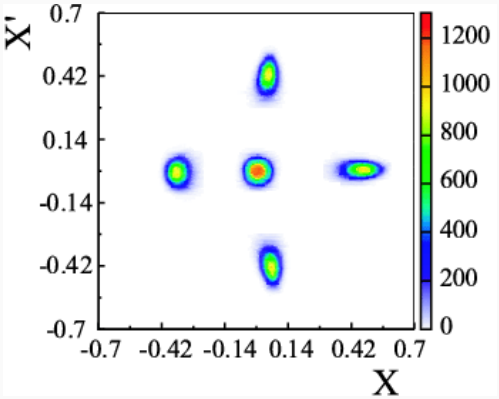
Drawbacks:

- Physics: Requires slow field variation; imperfect adiabaticity causes scattering or partial trapping; collisions and nonideal fields reduce effectiveness.
- Mathematics: Iterates can be experimentally recorded but the actual expression of the return map is approximated; the computation of the adiabatic invariant involves a transformation to a normal form.

M. Giovannozzi. Progresses in the studies of adiabatic splitting of charged particle beams by crossing nonlinear resonance. Physical Review Special Topics - Accelerators and Beams 12, 2009.

Beam dynamics, the Hénon map and the adiabatic invariant

The conservative Hénon map is the simplest standard model. Particular interest: the bifurcation near the 1:4 resonance (degenerate).



2D beam distribution of the split beam in the transfer line downstream of the Proton-Syncrhroton extraction point (Gilardoni et al. Phys.Rev.ST 9, 2006).

Previous works: Adiabatic invariant obtained via transformation to a higher-order normal form.

Alternative? We will show that discrete averaging allows one to determine the adiabatic invariant directly in the original coordinates, without recourse to normal form transformations, and to carry out a systematic analysis of the domain of validity of the adiabatic approximation.

Applications of discrete averaging go far beyond the context of this example.

A. Bazzani, C. Frye, M. Giovannozzi, and C. Hernalsteens. Analysis of adiabatic trapping for quasi-integrable area-preserving maps. Phys. Rev. E, 89:042915, 2014.

Classical averaging theory – overview

The averaging theory studies systems that can be written in the form

$$\dot{x} = a(x, \varphi), \qquad \varepsilon\dot{\varphi} = b(x, \varphi),$$

where a and b are periodic in φ with a period $T > 0$. If $|b| \geq c > 0$, the implicit function theorem allows to reduce the study to a non-autonomous equation of the form

$$x' = \varepsilon f(x, \tau),$$

with a non-autonomous time-periodic vector field f.

Averaged system: The solution of the (autonomous) averaged system

$$y' = \varepsilon \bar{f}(y), \qquad \text{where} \quad \bar{f}(y) = \frac{1}{T}\int_0^T f(y, \tau)d\tau,$$

gives the leading order approximation for trajectories starting from a point x_0 at $\tau = 0$: if $x(0, x_0) = y(0, x_0) = x_0$ then

$$x(\tau, x_0) = y(\tau, x_0) + O(\varepsilon) \qquad \text{for } |\tau| \leq C/\varepsilon, \quad C > 0.$$

J.A.Sanders, F.Verhulst and J.A.Murdock, Averaging methods in nonlinear dynamical systems, Applied Mathematical Sciences, 59, Springer, New York, (2007).

Classical averaging theory – adiabatic invariants

The integrals of motion of the averaged equation play the role of adiabatic invariants for the original system as they change slowly in time.

Higher order adiabatic theory: a close to the identity time-periodic change of coordinates, $z = u_n(x, \tau)$, can be used to eliminate the time-dependence in the leading orders:

$$z' = \varepsilon f_n(z, \varepsilon) + \varepsilon^{n+1} g(z, \tau, \varepsilon).$$

Well-known property. An integral h_n of the truncated system $z' = \varepsilon f_n(z, \varepsilon)$ evaluated on a trajectory of the original system oscillates with an amplitude $O(\varepsilon)$ but stays near its initial value for substantially longer times $O(\varepsilon^{-n})$.

Relation with near-id maps: If $(x(t), \varphi(t))$ is a solution of $\dot{x} = a(x, \varphi), \varepsilon\dot{\varphi} = b(x, \varphi)$, there is an increasing sequence t_k that corresponds to consecutive intersections of the solution with the Poincaré section $\{\varphi = 0 \pmod{T}\}$. Then $x_k = x(t_k)$ is a trajectory of the first return map $F_\varepsilon : x_k \mapsto x_{k+1}$. Since the return time is small, the first return map is close to the identity, $F_\varepsilon = \mathrm{id} + O(\varepsilon)$.

Problem: Following this suspension+averaging approach, computing h_n is hard.

Discrete averaging – interpolating vector fields (IVFs)

Discrete averaging provides explicit expressions for autonomous IVFs $X_{n,\varepsilon}$ such that

$$F_\varepsilon = \Phi^1_{X_{n,\varepsilon}} + O(\varepsilon^{n+1}).$$

in terms of weighted averages over a trajectory segment. Then the adiabatic invariants can be obtained from the IVFs $X_{n,\varepsilon}$.

Interpolating vector fields. Let n_0 and n be two integers with $0 \leq n_0 \leq n$. The integer n defines the length of a trajectory segment and n_0 is used to select a starting point. There is a unique polynomial $P_n(t, x)$ of degree n in t which interpolates a segment of the orbit of a point x:

$$P_n(k, x) = F^k(x) \quad \text{for } k = -n_0, \ldots, n - n_0.$$

The IVF is defined as the derivative of $P_n(t, x)$ at $t = 0$:

$$X_n(x) = \frac{\partial P_n}{\partial t}(0, x).$$

IVFs are explicit and computationally inexpensive

$X_n(x)$ is a weighted sum of the iterates of $F(x)$: the Lagrange interpolation procedure gives

$$P_n(t,x) = \sum_{k=-n_0}^{n-n_0} b_k(t) F^k(x), \qquad \text{where} \qquad b_k(t) = \prod_{\substack{-n_0 \le j \le n-n_0 \\ j \neq k}} \frac{t-j}{k-j}.$$

Consequently,

$$X_n(x) = \sum_{k=-n_0}^{n-n_0} p_{nk} F^k(x),$$

where the coefficients $p_{nk} = b_k'(0)$ depend on n and n_0 but are independent of the map F. For example, for $(n_0, n) = (0,1)$ and $(1,2)$ we get respectively

$$X_1(x) = F(x) - x, \qquad \text{and} \qquad X_2(x) = \tfrac{1}{2}\big(F(x) - F^{-1}(x)\big).$$

Application: The Hénon map 1:4 resonance

The Hénon map $H_c : (x, y) \mapsto (c(1 - (x + 1)^2) + 2x + y, -x)$ has two fixed points $p_e = (0, 0)$ and $p_h = (-2, 2)$. When $c = 1$, p_e is at the 1:4 resonance. **Not generic:** for $c = 1 + \varepsilon$ with $\varepsilon > 0$ there is a 4-periodic hyperbolic orbit at a distance $\mathcal{O}(\sqrt{\varepsilon})$ from p_e whose invariant manifolds form a chain of four islands of stability containing a 4-periodic elliptic orbit at a distance $\mathcal{O}(\sqrt[4]{\varepsilon})$ from p_e. We use discrete averaging to study the bifurcation:

The central interpolation scheme applied to $F_\varepsilon = H_{1+\varepsilon}^4$ provides the IVF $X_2(x, y, \varepsilon) = \frac{1}{2}\left(H_{1+\varepsilon}^4(x, y) - H_{1+\varepsilon}^{-4}(x, y)\right)$, such that

$$H_{1+\varepsilon}^4 - \Phi_{X_2}^1 = O_9(x, y) + \varepsilon O_7(x, y) + O(\varepsilon^2).$$

We use integration to extract the Hamiltonian part of the vector field X_2. If

$$h_2(x, y, \varepsilon) = \int_0^1 X_2(tx, ty) \cdot (y, -x)\, dt,$$

then

$$X_2(x, y, \varepsilon) - \left(\frac{\partial h_2}{\partial y}, -\frac{\partial h_2}{\partial x}\right)(x, y, \varepsilon) = O_8(x, y) + \varepsilon O_5(x, y) + O(\varepsilon^2).$$

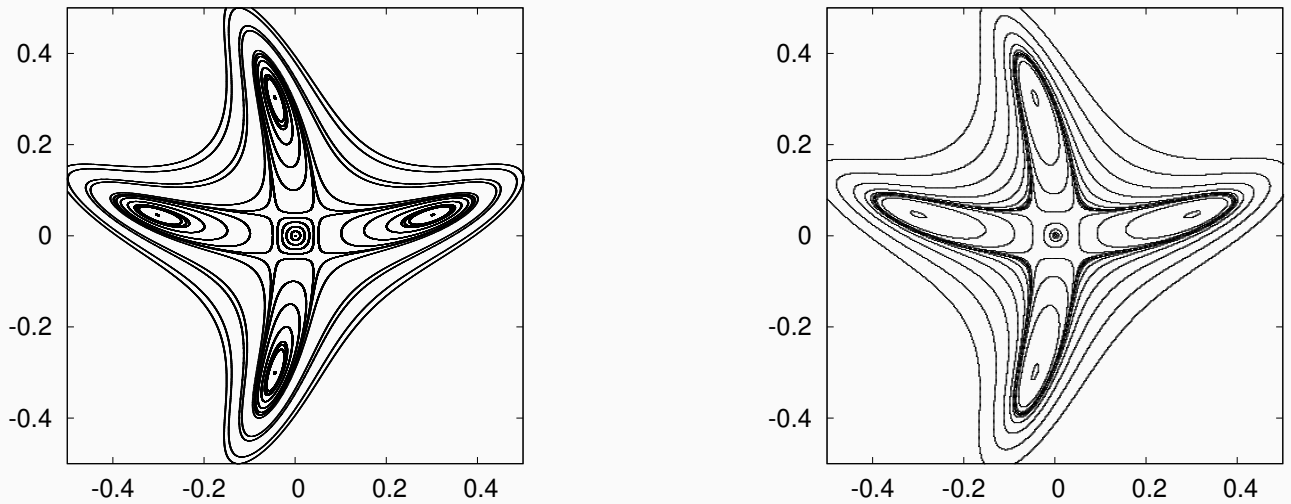
Moreover, one can check that h_2 is an adiabatic invariant **of** $H_{1+\varepsilon}$ such that

$$h_2 \circ H_{1+\varepsilon}(x, y, \varepsilon) - h_2(x, y, \varepsilon) = O_9(x, y) + \varepsilon O_7(x, y) + O(\varepsilon^2).$$

Application: The Hénon map 1:4 resonance – adiabatic invariant

One has $h_2 = \tilde{h}_2 + \mathcal{O}\left((x+y)^7, \varepsilon^2\right)$, with

$$\begin{aligned}\tilde{h}_2 = & -x^2y^2 + x^4y - xy^4 - \tfrac{x^6}{3} + 2x^3y^3 - \tfrac{y^6}{3} + x^2y^5 - x^5y^2 \\ & + \varepsilon\Big(2\left(x^2+y^2\right) + 2xy(y-x) + \left(x^2-y^2\right)^2 + 3x^4y - 2x^3y^2 + 2x^2y^3 - 3xy^4 \\ & + \tfrac{1}{3}\left(-4x^6 + 6x^5y - 17x^4y^2 + 24x^3y^3 - 17x^2y^4 + 6xy^5 - 4y^6\right)\Big)\end{aligned}$$



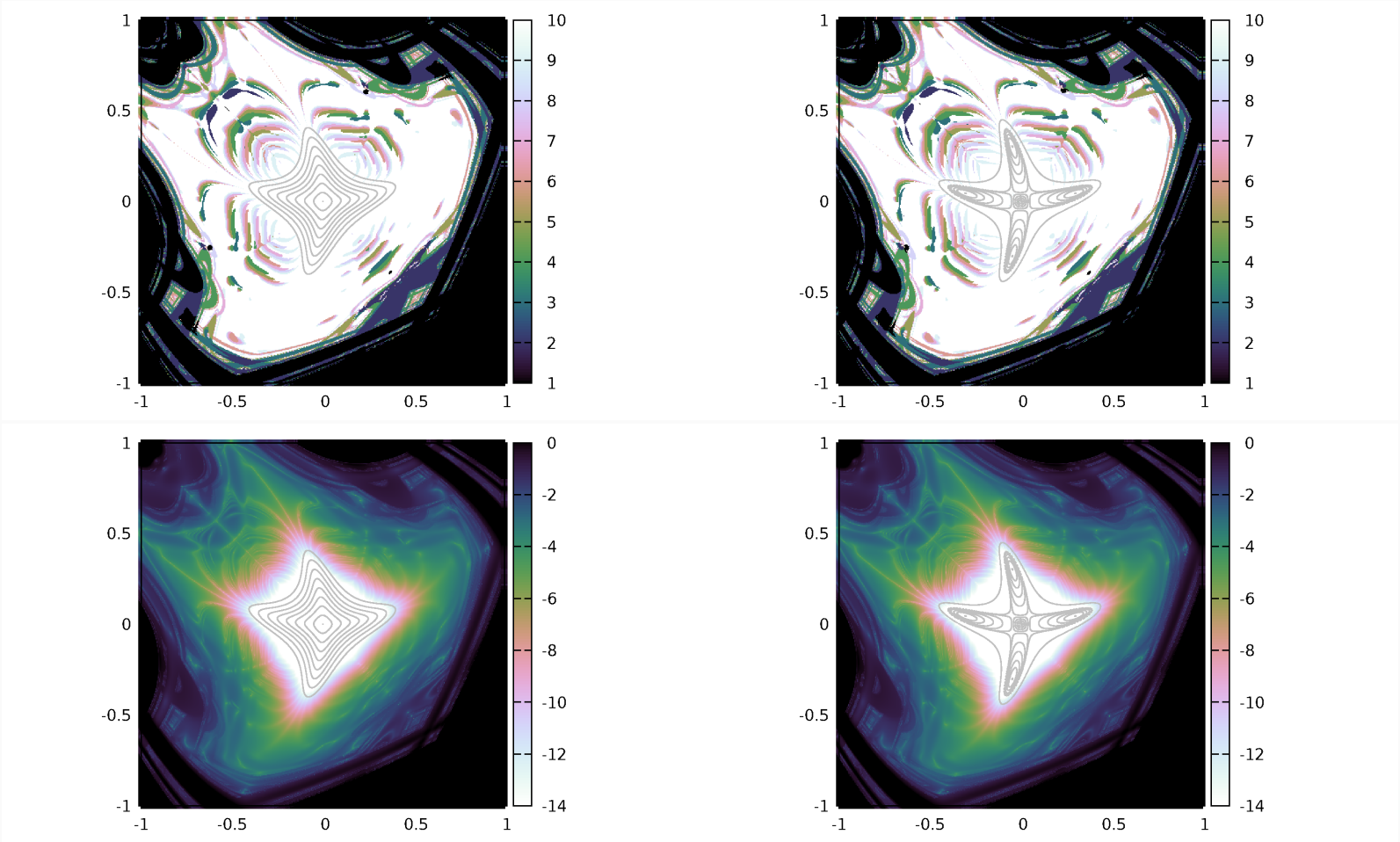
Iterates of the Henon map (left) and level lines of h_2 (right) for $c = 1 + \varepsilon$, $\varepsilon = 10^{-3}$.

- Computations where done directly in original coordinates.
- A higher accuracy approximation can be achieved using X_n, $n > 2$.
- No need of algebraic manipulators to numerically evaluate X_n or h_n.

Application: The Hénon map 1:4 resonance – bifurcation

Q: Where can the discrete averaging be used to analyze the dynamics?

For a mesh of points $(x, y) \in [-1, 1]^2$, we compute the integer n minimizing $G_{(x,y)}(n) = |X_{2n}(x, y) - X_{2n+2}(x, y)|_2$. Plot: We display n and $\log_{10} G_{(x,y)}(n)$.



IVF theory: $G_{(x,y)}(n) = \mathcal{O}(\epsilon^{2n+1})$ where $\epsilon =$ distance of the map to the identity.

IVF theory - approximation by autonomous flow

Let $D \subset \mathbb{R}^d$ be an open domain and consider a smooth family of maps $F_\varepsilon : D \to \mathbb{R}^d$ that is *tangent to identity*, i.e.,

$$F_\varepsilon(x) = x + \varepsilon f(x,\varepsilon), \quad f(x,\varepsilon) = \sum_{k=0}^{n-1} \varepsilon^k f_k(x) + \varepsilon^n r_n(x,\varepsilon), \quad r_n \text{ bounded.}$$

Theorem 1. If a tangent to identity family $F_\varepsilon \in \mathcal{C}^{n+1}\big(D \times [-\varepsilon_0, \varepsilon_0]\big)$ for some $\varepsilon_0 > 0$, then there is a unique polynomial in ε vector field $g(x,\varepsilon) = \sum_{k=0}^{n-1} \varepsilon^k g_k(x)$ such that

$$F_\varepsilon = \Phi^1_{\varepsilon g_\varepsilon} + O(\varepsilon^{n+1})$$

uniformly on every compact subset of D. Moreover, for $1 \le k \le m \le n$

$$\frac{1}{k!} \left.\frac{\partial^k X_m}{\partial \varepsilon^k}\right|_{\varepsilon=0} = g_{k-1},$$

where X_m is an interpolating vector field of order m and consequently

$$F_\varepsilon = \Phi^1_{X_m} + O(\varepsilon^{m+1}).$$

Comment: The error decays faster for larger m but there is no information on the constant in the O-term. Therefore it is not possible to use this theorem to decide what order of interpolation leads to the most accurate approximation for a fixed value of ε.

IVF theory - uniform bounds

Let $\epsilon, \delta > 0$. For a set $D_0 \subset \mathbb{C}^d$, let D_δ be a δ-neighbourhood of D_0. We denote by $\mathcal{C}^\omega(D_\delta, \mathbb{C}^d)$ the space of all analytic maps $F : D_\delta \to \mathbb{C}^d$ equipped with the supremum norm $\|\cdot\|_{D_\delta}$. Let $B_\epsilon(D_\delta) \subset \mathcal{C}^\omega(D_\delta, \mathbb{C}^d)$ be the closed ball of radius ϵ centered at the identity map: an analytic map $F \in B_\epsilon(D_\delta)$ iff

$$\|F - \mathrm{id}\,\|_{D_\delta} \leq \epsilon.$$

Let $M_\alpha = \lfloor c_0/\alpha \rfloor + 1$ where $c_0 = 1/(6\mathrm{e}) = 0.061\ldots$

Theorem 2. Let $\epsilon, \delta > 0$ be such that $\epsilon/\delta \leq c_0$ and $D_0 \subset \mathbb{C}^d$. Then for any map $F \in B_\epsilon(D_\delta)$ the IVF of order $m \in [2, M_{\epsilon/\delta}]$ satisfies $\|X_m\|_{D_{\delta/3}} \leq 2\epsilon$ and

$$\left\|\Phi^1_{X_m} - F\right\|_{D_0} \leq 3\epsilon \left(\frac{6(m-1)\epsilon}{\delta}\right)^m.$$

In particular, for $m = M_{\epsilon/\delta}$, one has $\left\|\Phi^1_{X_m} - F\right\|_{D_0} \leq 3\,\epsilon \exp\left(-c_0\,\delta/\epsilon\right).$

Comments.

- Useful for individual maps (not necessarily embedded in a near-identity family).
- The error is controlled by the ratio ϵ/δ where δ characterizes the size of the complex neighbourhood where the map is ϵ-close to the identity.

Sketch of the proof of Theorem 2.

We consider

$$f_\mu = (1-\mu)\,\mathrm{id} + \mu F, \quad \mu \in \mathbb{C}.$$

One has

$$X_m(x) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \Delta_k(x),$$

$$\Delta_0(x) = x,\ \Delta_k(x) = \Delta_{k-1}(F(x)) - \Delta_{k-1}(x),\ k \geq 1.$$

1. **Limit flow.** For $|\mu| \leq \mu_1 = \delta/\epsilon$,

$$\|X_{1,\mu}\|_{D_\delta} = \|f_\mu - \mathrm{id}\,\| = \|\mu(F-\mathrm{id})\|_{D_\delta} \leq |\mu|\epsilon \leq \delta$$

$$\Longrightarrow \left\|\Phi^1_{X_{1,\mu}} - f_\mu\right\|_{D_0} \leq \left\|\Phi^1_{X_{1,\mu}} - \mathrm{id}\right\|_{D_0} + \left\|\mathrm{id} - f_\mu\right\|_{D_0} \leq 2|\mu|\epsilon.$$

   Then, the MMP on the disk $|\mu| \leq \mu_1$ implies that $\left\|\Phi^1_{X_1} - F\right\|_{D_0} \leq 2\epsilon^2/\delta$.

2. **Bounds for the finite differences in $D_{\delta/3}$.** Let $m \geq 2$ and $\mu_m = \frac{2\delta}{3\epsilon(m-1)}$.

   $\Delta_m(x_0) = \sum_{k=0}^{m-1} (-1)^{m-k-1} \binom{m-1}{k} \Delta_1(x_k) \Longrightarrow \|\Delta_m\|_{D_{\delta/3}} \leq 2^{m-1}|\mu|\epsilon\,.$

   Then, the MMP on $\mu \leq \mu_m$ implies $\|\Delta_m\|_{D_{\delta/3}} \leq 2^{m-1}\mu_m\epsilon\,(|\mu|/\mu_m)^m\,.$

Maximum modulus principle (MMP): If g analytic in the open disk $|\mu| < r$ and $g^{(k)}(0) = 0$, $0 \leq k < m$, then $|g(\mu)| \leq (|\mu|/r)^m \sup_{|\mu|<r} |g(\mu)|$.

Sketch of the proof of Theorem 2.

3. Bounds for $\left\|X_{m,\mu}\right\|_{D_{\delta/3}}$ with $2 \le m \le M_{\epsilon/\delta}$. For $|\mu| \le \mu_m/4$,

$$\left\|X_{m,\mu}\right\|_{D_{\delta/3}} \le \sum_{k=1}^{m} \frac{\left\|\Delta_k\right\|_{D_{\delta/3}}}{k} \le \epsilon \sum_{k=1}^{m} \frac{2^{k-1}|\mu|^k}{k\mu_k^{k-1}} \le \epsilon \sum_{k=1}^{m} \frac{2^{k-1}|\mu|^k}{\mu_m^{k-1}} \le 2\epsilon|\mu|.$$

Since $\mu_m > 4$ for $2 \le m \le M_{\epsilon/\delta}$ we can substitute $\mu = 1$ and get

$$\|X_m\|_{D_{\delta/3}} \le 2\epsilon.$$

4. Approximation of F by the time-one map of X_m in D_0. It follows that $\left\|\Phi^1_{X_{m,\mu}} - f_\mu\right\|_{D_0} \le \left\|\Phi^1_{X_{m,\mu}} - \mathrm{id}\right\|_{D_0} + \left\|\mathrm{id} - f_\mu\right\|_{D_0} \le 3\epsilon|\mu|$.

Then, the MMP on $|\mu| \le \mu_m/4$ implies

$$\left\|\Phi^1_{X_m} - F\right\|_{D_0} \le 3\epsilon \left(\frac{4}{\mu_m}\right)^m = 3\epsilon \left(\frac{6\epsilon(m-1)}{\delta}\right)^m.$$

For $m = M_{\epsilon/\delta}$ we obtain the exponentially small upper bound.

Dynamics near a resonant equilibrium - hidden symmetries of IVFs

If $f(0) = 0$ is fully resonant (i.e. the Jacobian is a root of unity), there is $n \geq 1$ s.t. f^n is tangent to identity: $(f^n)'(0) = \left(f'(0)\right)^n = Id$.

Then, there is a unique formal vector field X such that the formal series for the time-n flow defined by X coincide with f^n, i.e. $\Phi_X^n = f^n$.

If f is symplectic, then X is Hamiltonian and $h_X \circ f^n = h_X \circ \Phi_X^n = h_X$.

Lemma. Let f be a formal diffeomorphism with $f(0) = 0$ such that f^n is tangent to the identity for some $n \in \mathbb{N}$, and let X be the formal vector field such that $f^n = \Phi_X^n$. Then the vector field X is f-invariant, i.e.,

$$X \circ f = D(f)X, \qquad \text{where } D(f) \text{ is the Jacobian matrix.}$$

Moreover, if f is symplectic, then X is Hamiltonian with a formal Hamiltonian h_X and

$$h_X \circ f = h_X.$$

Comment: This was observed for the Hénon map:
the IVF computed for H_c^4 gives an adiabatic invariant for H_c.

V.Gelfreich and AV, Discrete averaging for discrete time dynamical systems. arXiv:2603.10737

On the interpolation scheme to compute IVFs

Any interpolation scheme can be used to compute IVFs but...

The Newton forward interpolation scheme is a convenient tool for analytical proofs as it avoids inversion of the original map.

For example, we used it to proof the optimal Nekhoroshev estimates for a family of real-analytic symplectic maps of the form

$$F_\varepsilon : \begin{cases} \bar{I} = I + \varepsilon a(I, \varphi), \\ \bar{\varphi} = \varphi + \omega(I) + \varepsilon b(I, \varphi) \pmod 1, \end{cases}$$

depending on $\varepsilon \geq 0$, $I \in \mathbb{R}^d$, a, b are periodic in $\varphi \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, under the assumption that $\omega = h_0'$ with h_0 strongly convex:

Nekhoroshev estimates. There are $c_1, c_2, c_3 > 0$ such that for every initial condition (I_0, φ_0) one has

$$|I_k - I_0| < c_1 \varepsilon^{1/2(d+1)} \quad \text{for} \quad 0 \leq k \leq T_\varepsilon = c_2 \exp\left(c_3 \varepsilon^{-1/2(d+1)}\right).$$

V.Gelfreich and AV. Nekhoroshev theory and discrete averaging. DCDS-A, 2025.

Nekhoroshev theorem - where IVFs are used?

Our *"Lochak-Neishtadt"* direct proof of the exponentially long stability times relies on the following observations.

- We construct a covering of the action space by open neighbourhoods of a finite number (depending on ϵ) of unperturbed tori bearing periodic motions (maximum resonances).
- For every $n \ll \varepsilon^{-1/2}$, the covering identifies domains where F_ε^n is close to the identity. Each of these domains is centered at $I = I_*$ such that $n\omega(I_*) \in \mathbb{Z}^d$, corresponding to a torus invariant by F_0 foliated by invariant n-periodic orbits.
- In these domains, we use IVFs to show that F_ε^n is exponentially close to the time-one map of an autonomous Hamiltonian flow.
  Since the Hamiltonian flow preserves its Hamiltonian function H_n, we conclude that $H_n \circ F_\varepsilon^n - H_n$ is exponentially small.
- The leading order approximation for H_n can be obtained from analysis of the integrable map F_0^n. The convexity assumption can be used to show that there is a small neighbourhood of $I = I_*$ which can be represented as a union of level sets of H_n. For an initial condition that is not too close to its boundary, an exponentially large number of iterates of F_ε^n is needed to achieve a change in H_n sufficiently large to leave the neighbourhood.

P.Lochak, A.I.Neishtadt. Estimates of stability time for nearly integrable systems with a quasiconvex hamiltonian. Chaos, 2:495, 1992

In practice, use any interpolation scheme to compute IVFs but...

The Stirling-Newton central interpolation scheme is often a better tool for numerical simulations as it provides more accurate approximations. For example, we used it for visualization of 4D near-id dynamics.

"Poincaré" sections: Let $\Sigma = \{x \in \mathbb{R}^m : g(x) = 0\}$ be a smooth codim-1 hyper-surface. Take x_0 and iterate $x_{k+1} = f_\epsilon(x_k)$. If $g(x_k)g(x_{k+1}) \leq 0$ and X_1 is (locally) transversal to Σ then, for ϵ small enough, $\exists!\ t_k \in [0, \epsilon]$ s.t. $g(\Phi_{X_n}^{t_k}(x_k)) = 0$.

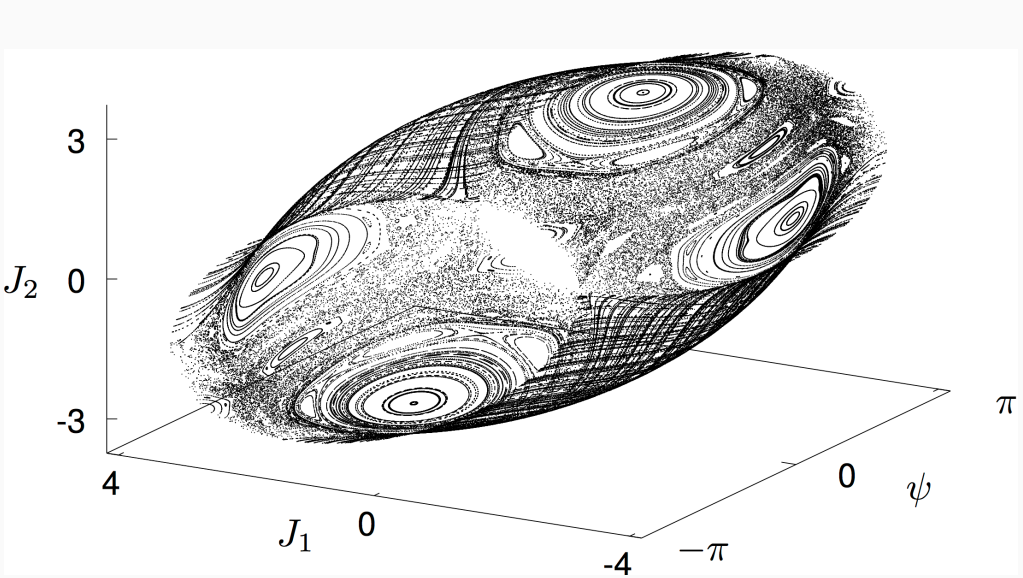
$\longrightarrow$ Plot $y_k = \Phi_{X_n}^{t_k}(x_k)$ instead of (any other projection of) x_k.

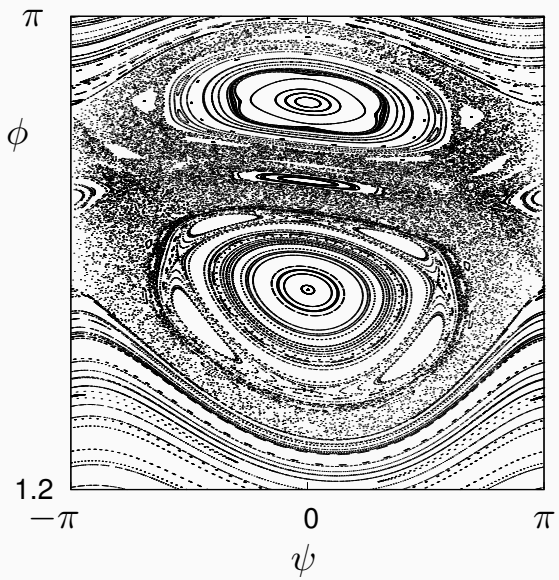
V.Gelfreich and AV. Interpolating vector fields for near identity maps and averaging Nonlinearity 31(9), 2018.

Visualizing the dynamics of 4D near-id symplectic maps

Example (Froeschlé-like map). We consider the 4D map T_ε of $\mathbb{T}^2 \times \mathbb{R}^2$ given by $(\psi, J) \mapsto (\bar{\psi}, \bar{J}) =$
$\left(\psi_1 + \varepsilon(\bar{J}_1 + a_2\bar{J}_2), \psi_2 + \varepsilon(a_2\bar{J}_1 + a_3\bar{J}_2), J_1 - \varepsilon \sin\psi_1, J_2 - \varepsilon\delta \sin\psi_2\right)$ and $\Sigma = \{\psi_1 = \psi_2\}$. On a moderate time scale, the iterates of x_0 remain close to the "energy surface" $\{x : h_n(x) = h_n(x_0)\}$.

Plot: $T_{0.35}$, 400 i.c. on $\Sigma \cap \{h_{10} = 4\}$, 500 iterates

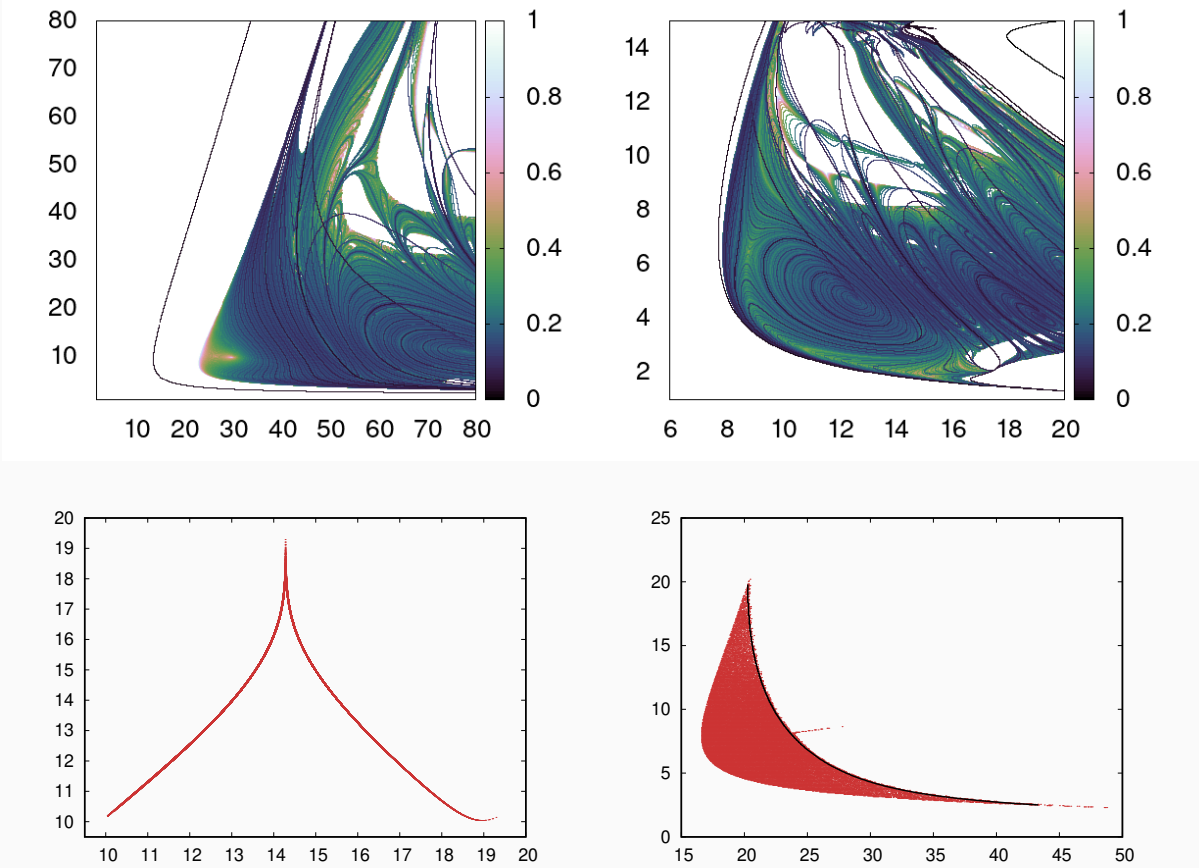




Comment: Allows to rigorously explore Arnold diffusion both qualitative and quantitative (with error estimates). It works better closer to double resonances!

Other settings to use IVFs: Discrete Lorenz attractors

Lorenz map: $\bar{x} = x + \delta(\sigma(y - x)),\ \bar{y} = y + \delta(\bar{x}(\rho - z) - y),\ \bar{z} = z + \delta(\bar{x}y - 8z/3).$



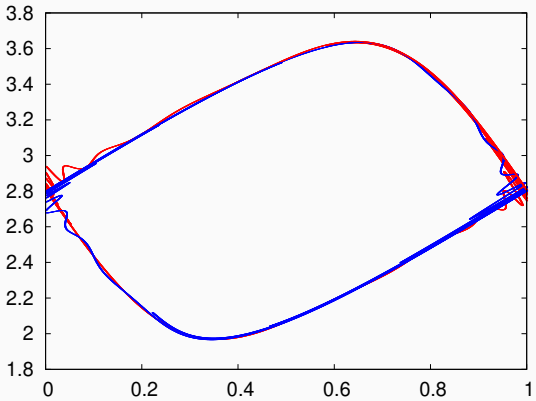
For δ small, we use IVFs to compute kneading diagram, (ρ, σ)-parameter space (top right $\delta = 0.001$, top left $\delta = 0.06$), reduce dynamics to 1D-"Poincaré maps" (bottom left, $\delta = 0.001$), and determine the region with pseudohyperbolic discrete Lorenz-like attractors (bottom night, $\delta = 0.01$).

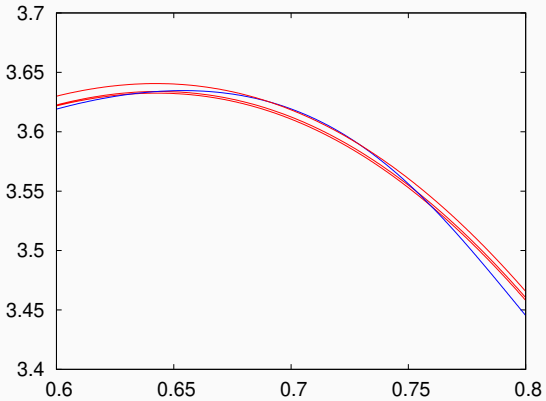
To do: Use IVFs to convert numerical evidence into a direct proof for the existence of discrete Lorenz attractors.

A.Kazakov, A.Murillo, AV, K.Zaichikov, "Numerical study of discrete Lorenz-like attractors." Reg. Chaotic Dyn. 29(1), 2024.

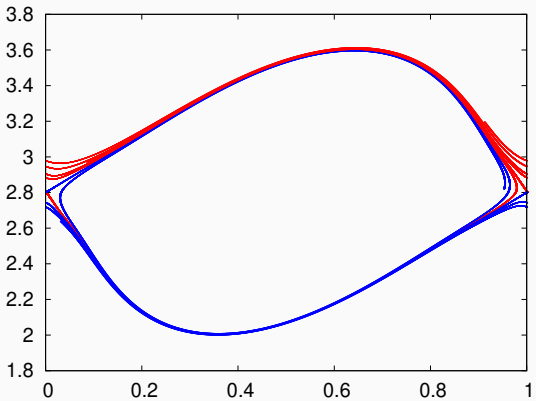
Other settings to use IVFs: probability of capture (weak dissipation)

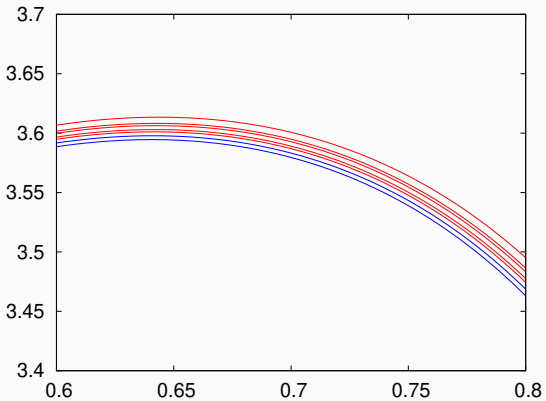
Dissipative standard map (top) vs IVF (bottom)





Transversal intersection of $W^{u/s}$





Flow-like pendulum without intersections

Preliminary numerical explorations indicates that the probability of capture by the attracting focus can be defined as the ratio between the entrance/exit strips (one can avoid homoclinics). Note that this implies $\lim_{\epsilon \to 0} P_{\text{capture}} > 0$.

To do: IVFs as a numerical tool ok! But theoretical justification is missed!

With R. Calleja (UNAM).

Thank you for your attention