Study of the effect of conservative and weakly dissipative perturbations on symplectic maps and Hamiltonian systems

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Contents

- **Local (semi-local) study of APMs:** Resonances (including strong resonances), inner/outer splittings of separatrices.
- **Semi-global study:** Dynamics in chaotic zones.
- **Global study:** Evolution of the domain of stability with respect parameters.
- **Weakly dissipative maps:** Coexistence of attractors and probability of capture.

The Hénon map was used (eventually dissipatively perturbed) as a paradigm of APM. One of the formulations is

$$H_c : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} c(1-x^2)+2x+y \\ -x \end{pmatrix}$$

I Local (semi-local) study of APMs

$\longrightarrow$ We want:

- A description of the resonant structures (islands).
- To study the inner and the outer splittings of separatrices.
- To study the strong resonances.

$\longrightarrow$ Steps to follow:

1) Consider BNF (**local** study).
2) Construct a suitable model from BNF (an interpolating Hamiltonian flow).
3) Localise the model around the resonance strip we want to study (**semi-local** study).
4) Use this model to study the properties we want.

BNF + Interpolating Hamiltonian

F_δ one-parameter δ-family of APMs with $F(E_0) = E_0$ elliptic fixed point.
Spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = e^{2\pi i\alpha}$, $\alpha = q/m + \delta$, δ small enough.
(x, y)-Cartesian coord., $(z, \bar{z})$-complex coord. ($z = x + iy$, $\bar{z} = x - iy$).
(I, φ)-Poincaré variables ($z = \sqrt{2I}\exp(i\varphi)$).
$\longrightarrow$ Consider the Birkhoff NF of $F_\delta(x, y)$ to order m around E_0
(say BNF$_m(F_\delta)(z, \bar{z})$) and let $K(z, \bar{z}) =$ BNF$_m^m(F_\delta)(z, \bar{z})$ (near Id).
$\longrightarrow$ Define
$\mathcal{H}_{nr}(I) = \pi \sum_{n=0}^{s} \frac{b_n}{n+1}(2I)^{n+1}$ and $\mathcal{H}_r(I, \varphi) = \frac{1}{m}(2I)^{\frac{m}{2}} \cos(m\varphi)$.
The time-1 map generated by the flow defined by the Hamiltonian

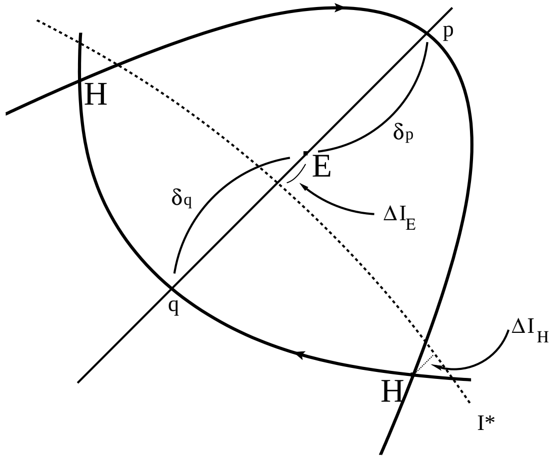
$$\mathcal{H}(I, \varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I, \varphi)$$

interpolates K with an error of order $m + 1$ with respect to the $(z, \bar{z})$-coordinates, in a suitable annulus containing the resonant m-island.

Description of resonances

Generic case: $\alpha = q/m + \delta$, $m > 5$, δ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then F **has** a resonant island of order m.
- The resonant zone is determined by **two periodic orbits** of period m located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.



A model around a generic resonance

For a generic APM such that $\delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the m-resonance strip $(m \geq 5)$ can be modeled, after suitable scaling ($J \sim \delta^{-m/4}(I - I_*)$), by the time-$\log(\lambda)$ map of the flow generated by

$$\mathcal{H}(J,\psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos(\psi),$$

where $c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1})$. Bounding the errors it is shown that it gives a "good" enough approximation of the dynamics in an annulus containing the m-islands.

$\longrightarrow$ Then, we have the following... [a]

[a] The details of the proof (singularities, suitable Hamiltonian,...) can be found in:
Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps. Nonlinearity 22, 5:1191–1245, 2009.

Main result: comments on the hypothesis

- **A1.** $b_1(\delta)$ is non-zero for $\delta = 0$.
- **A2.** $W^u = G(W^s)$, G periodic (between homo p and $F(p)$),

  s scaled variable s.t. $G(s) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\,2\pi s)$.

  We assume: The maximum of the norms of the functions $c_{\pm 1}\exp(\pm i2\pi s)$ is bounded away from zero, when δ tends to zero, on suitable lines whose imaginary part tend to $\tau_\pm$ when $\delta \to 0$.
- **A3.** There exists a fixed $\alpha > 0$ s.t.

  $$\sigma_\pm = \exp\left(-\frac{2\pi\,\mathrm{Im}\,\tau_\pm - \eta_\pm}{\log(\lambda(\delta))}\right)\left(\cos\left(\frac{2\pi\,\mathrm{Re}\,\tau_\pm}{\log(\lambda(\delta))} - \phi_\pm\right) + o(1)\right),$$

  where $|\eta_\pm| < \log(\lambda(\delta))^{1-\alpha}$ for δ sufficiently small.
- **A4.** F maybe meromorphic but the singularity remains at a finite distance when δ goes to 0.

Main result

Theorem. Let F be an APM. Assume that it has an m-order resonance strip, $m > 4$, located at an average distance $I = I_* = \mathcal{O}(\delta)$ from the elliptic fixed point and δ is sufficiently small. Under the assumptions **A1**, **A2**, **A3** and **A4**, the following conclusions hold.

a) The outer splitting is larger than the inner one being the difference between the position of the corresponding nearest singularities $\mathcal{O}(\delta^{m/4-1})$.

b) Neither the inner nor the outer splittings oscillate.

$\rightarrow$ **Question:** Consequences in the width of the chaotic zones of this fact?

II Semi-global study of APMs: Chaotic regions

We focus now on the chaotic regions created by the invariant manifolds emanating from a fixed/hyperbolic point h. We do not assume the region of interest to be close to the origin but we require the system to be not too far from integrable in the selected domain to be studied.

$\longrightarrow$ We want:

- To study the resonant islands far from the elliptic point
- To study the dynamics in the chaotic zones

$\longrightarrow$ How?:

Using suitable return maps.

Chaotic regions considered

For each of the following cases we use a concrete return map model to study the dynamics.

- Open case (fish like) — Separatrix map
- Figure eight case (pendulum like) — Double separatrix map
- Large regions of instability (e.g. Birkhoff z.i.) — Biseparatrix map

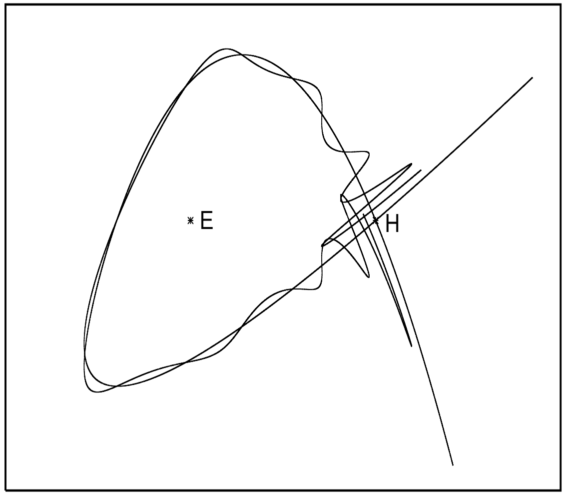
$\longrightarrow$ We look for **quantitative** information on the dynamics within the chaotic zones. However, the biseparatrix model only gives us a topological description of the dynamical behaviour. [a]

[a] The following results can be found in:

Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.

Submitted to Physica D.

Open case



$$SM : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b\log|y'| \\ y + \sin(2\pi x) \end{pmatrix}$$

where $b = 1/\log(\lambda)$, λ the dominant eigenvalue of $DF(h)$ and a is a “shift”.

The y-vble. is scaled by the amplitude of the splitting.

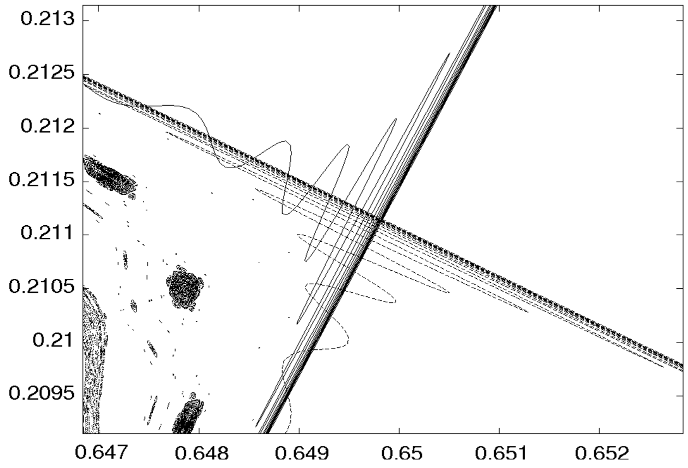
Open case: results

$\longrightarrow$ Distance to r.i.c. from the separatrix: $d_c \sim |b|/k^*$, $k^* \approx 0.97/2\pi$ Greene

$\longrightarrow$ Using interp. Ham. flow we obtain the distance from the hyperbolic point.

$\longrightarrow$ Estimate # chains of islands before r.i.c.: $\#\{islands\} \approx 1.415 \times b$

Example: Hénon map $H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2)$, $\alpha = 0.1$

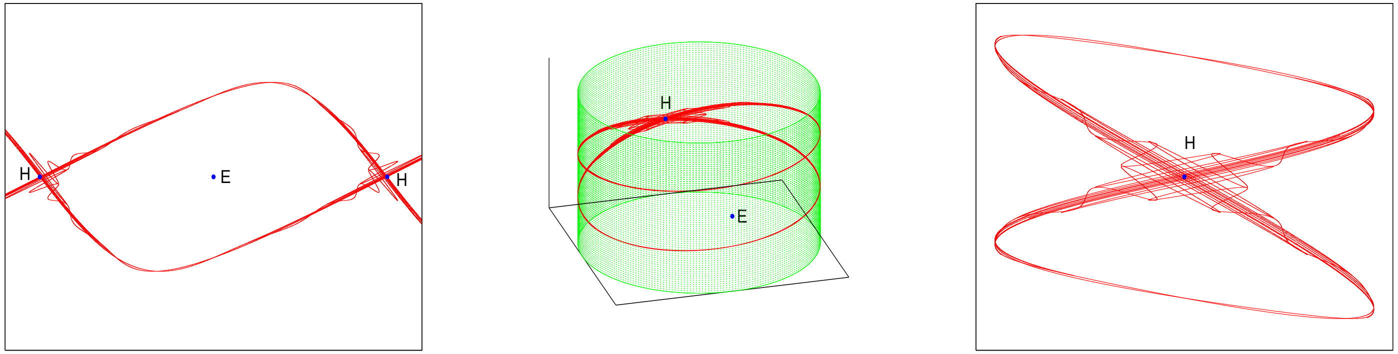


Experimental values: $(D_c^H)_e \approx 2.94 \times 10^{-3}$, $(D_i^H)_e \approx 2.08 \times 10^{-3}$

"Fish" interpolating Hamiltonian: $D_c^H \approx 2.47 \times 10^{-3}$, $D_i^H \approx 1.85 \times 10^{-3}$

5-order interp. Hamiltonian: $D_c^H \approx 2.731 \times 10^{-3}$, $D_i^H \approx 2.050 \times 10^{-3}$

Figure eight case



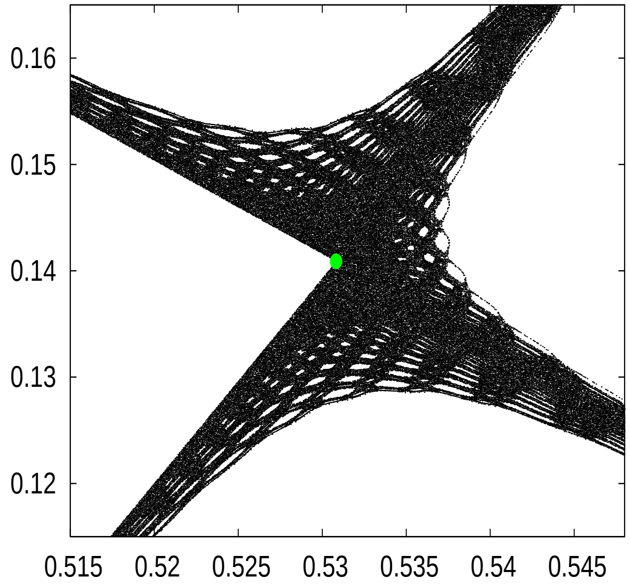
$$DSM : \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{\bar{s}} + b \log |\bar{y}| \pmod 1 \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix},$$

where $\nu_1 = 1$, $\nu_{-1} = A_{-1}/A_1$, (A_1,A_{-1} amplitudes of the outer/inner splittings).

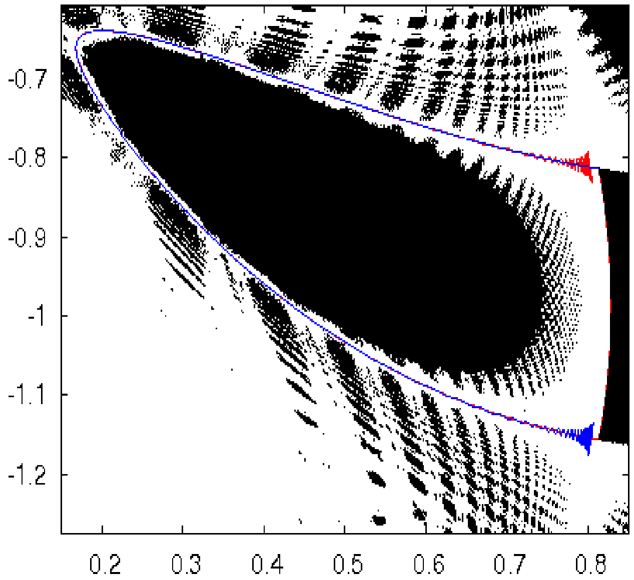
Theorem. Assume $b_1\delta < 0$, δ small, $m \geq 5$ and that the hypothesis of the theorem on the difference of the inner and outer splittings hold. Then,

- The width of the outer chaotic zone is larger than the width of the inner chaotic one if, and only if, sign $b_1 \cdot$ sign $b_2 < 0$.
- Both amplitudes of the stochastic layer are of the order of magnitude of the outer splitting (the largest one).

Pendulum-like islands: comments



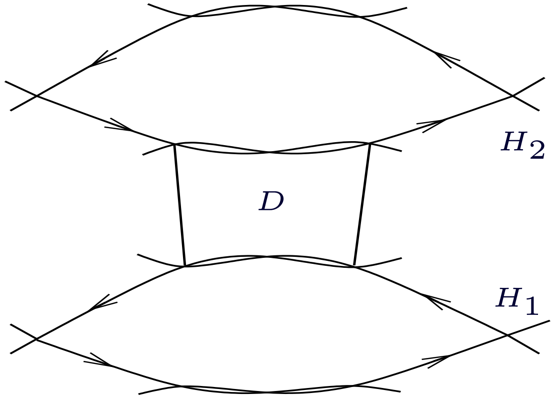
$\alpha = 0.21$,
$\sigma_- = \mathcal{O}(10^{-12})$,
$\sigma_+ = \mathcal{O}(10^{-3})$.

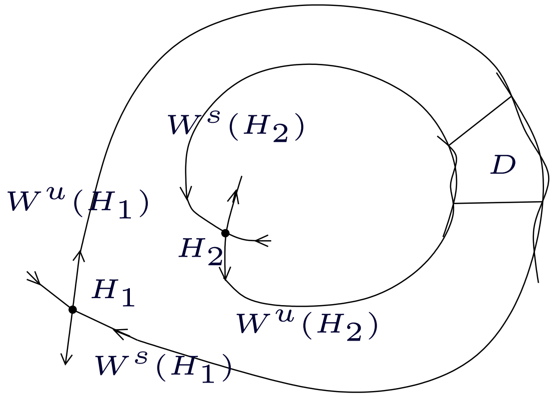


$c = 1.015$,
$\sigma_+ = \mathcal{O}(10^{-54})$, $\sigma_- = \mathcal{O}(10^{-1})$.

Non-generic!! The inner splitting oscillates!!

Large regions of stability: biseparatrix map





A simple *qualitative* model on the domain $0 < v < d$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}.$$

where $\beta_1 = 1/\log(\lambda)$, $\beta_2 = 1/\log(\mu)$, and $|\lambda| > 1$ (resp. $\mu| > 1$) eigenvalue of the hyperbolic point of the bottom (resp. top) separatrix.

- The model has rotational invariant curves for $d > (\sqrt{b_1} + \sqrt{b_2})^2/k^*$ ($k^* =$ Greene's value).
- It remains to generalise it to different order for the top/bottom resonances and to make it quantitative.

III Global study of APMs.

We consider the full domain of stability around the elliptic fixed point.

$\longrightarrow$ We want: To describe the evolution of the domain of stability when changing parameters.

$\longrightarrow$ How?

- We perform numerical simulations.
- We try to explain what is observed in computations by using different theoretical frameworks.

No “new” theoretical results!!

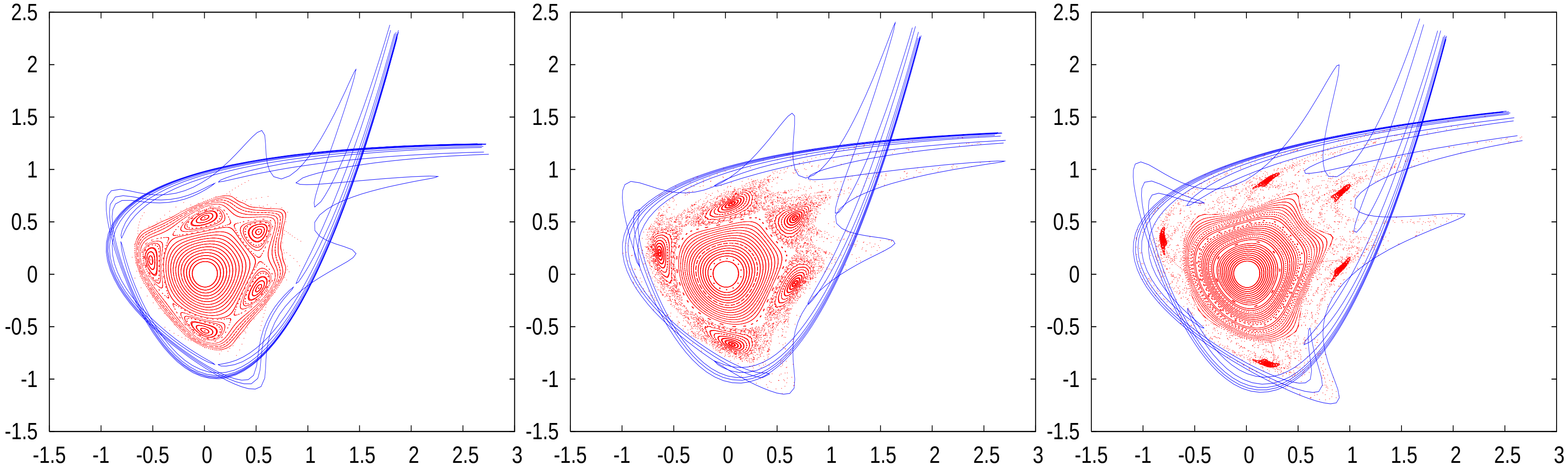
Just a description of what is observed in simulations!!

Useful to understand/predict for a large variety of APM.

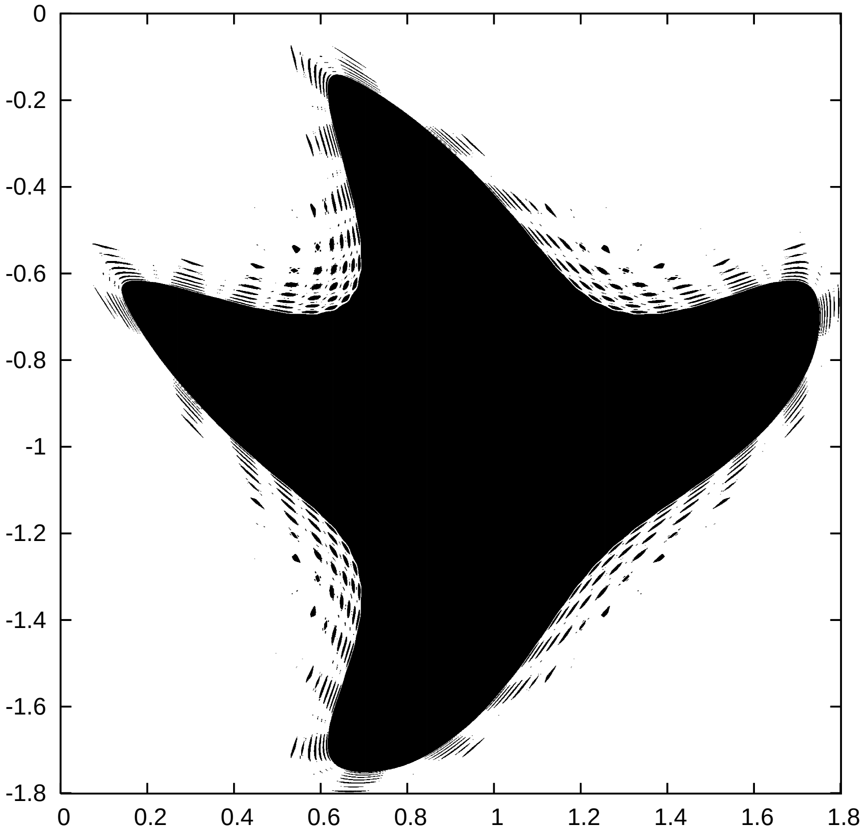
The domain of stability (DS)

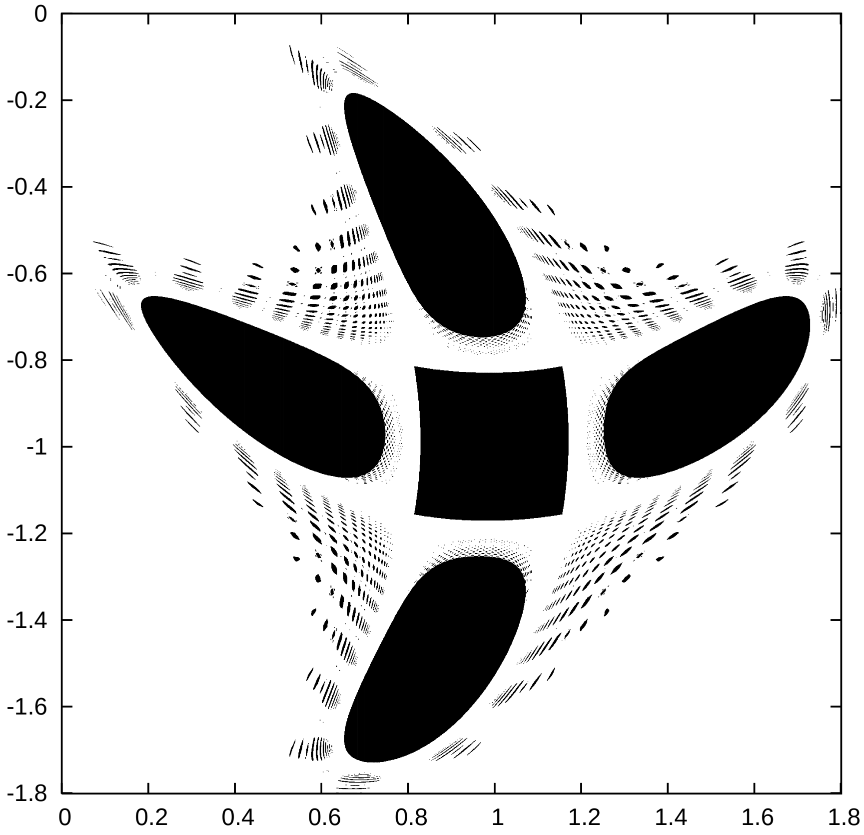
DS for APMs are related with elliptic fixed/periodic points (locally KAM thm. assures the existence of rotational inv. curves under some conditions).

For a given map $F : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, and given a compact set $K \subset U$, the *stability domain of F relative to K* is the largest F-invariant subset of K (a chaotic orbit can be stable, in the sense that it does not escape)



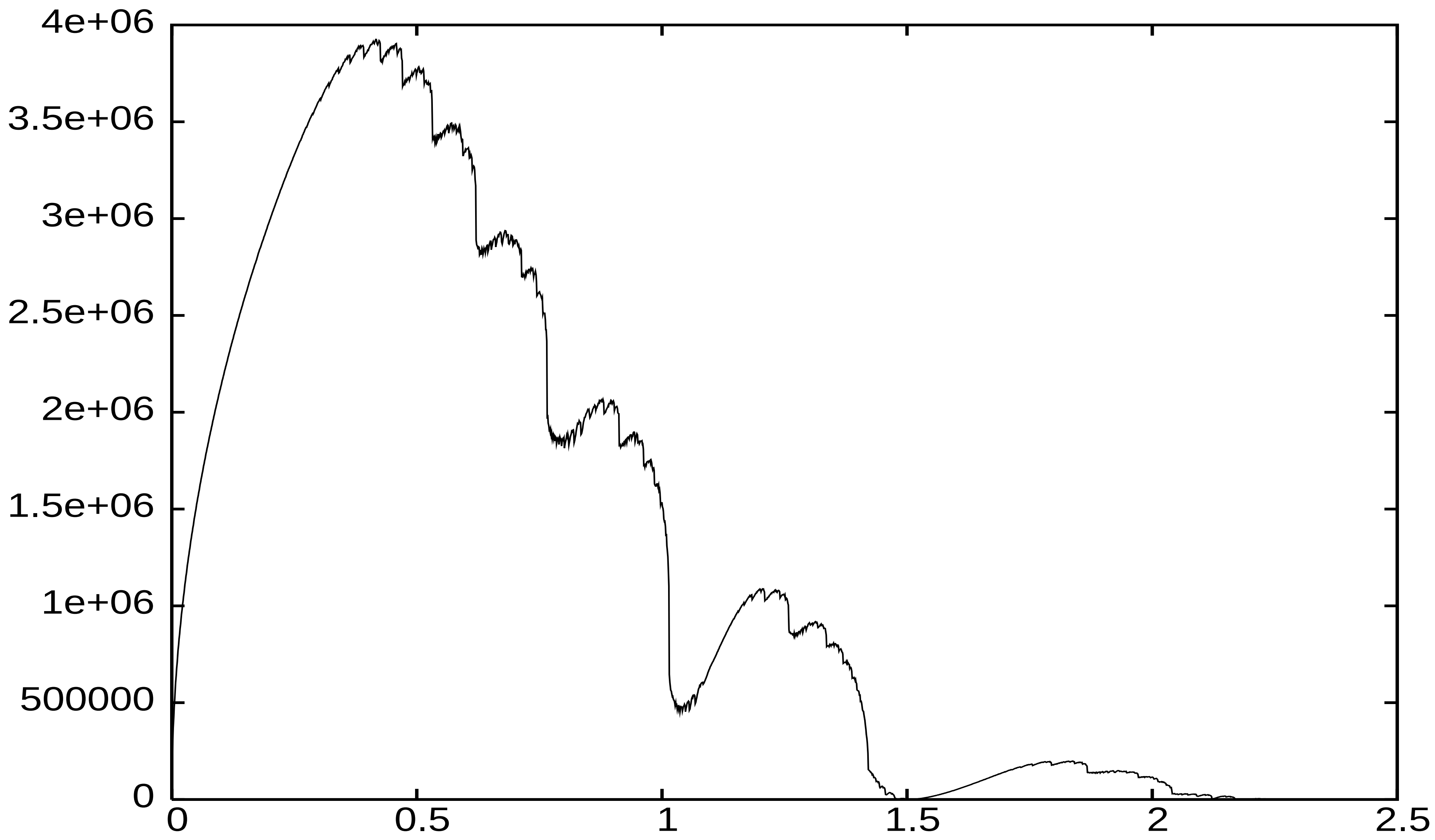
Drastic changes in the stability domain





LEFT: $c = 1.014$, RIGHT: $c = 1.015$.

The evolution of the size of the stability domain



Orbits method: $N = 3100$, $R = [-1, 2.1] \times [-2.1, 1]$, $n_{it} = 10^7$ and $\tilde{n}_{it} = 10^4$.

IV Weakly dissipative maps

We study the effect of dissipation on the family of maps F_δ.

Interest in:

- Evoltution of geometrical structures from the conservative case
- Study the probability of capture

We consider a *radially dissipative perturbation*: [a]

$F_\delta(x, y)$ – the family of APMs s.t. $F_\delta(0) = 0$ is an elliptic fixed point

ϵ – the dissipation parameter

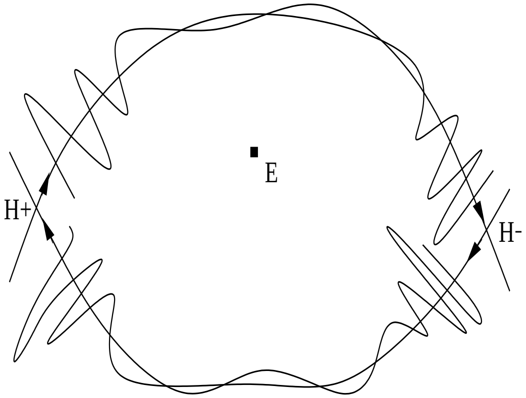
$\longrightarrow$ the dissipative perturbation is of the form:

$$F_{\delta,\epsilon}(x, y) = (1 - \epsilon)F_\delta(x, y)$$

[a] *Planar Radial Weakly-Dissipative Diffeomorphisms.* To appear in Chaos.

Dissipation effect on the conservative islands

We recall that generically it is expected, for conservative resonant islands close to the origin, to have the outer splitting larger than the inner one. The effect of the dissipation gives rise to the following structures:

1.


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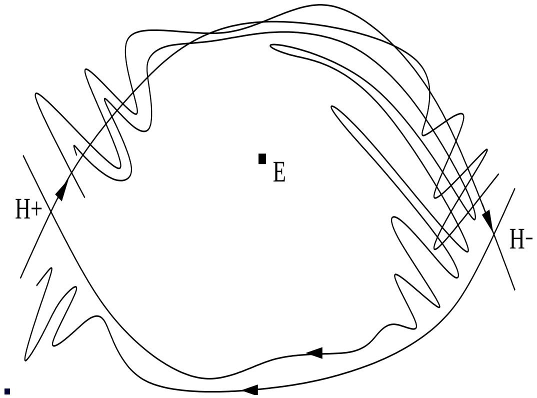

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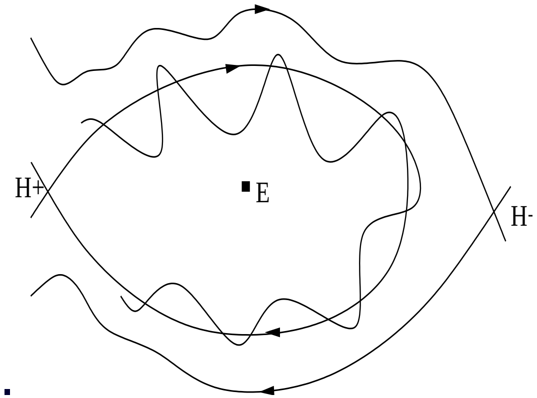

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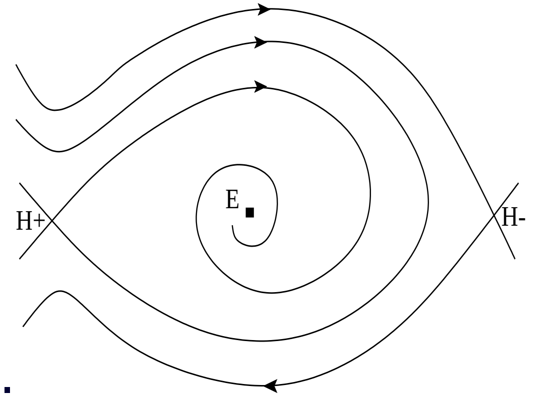
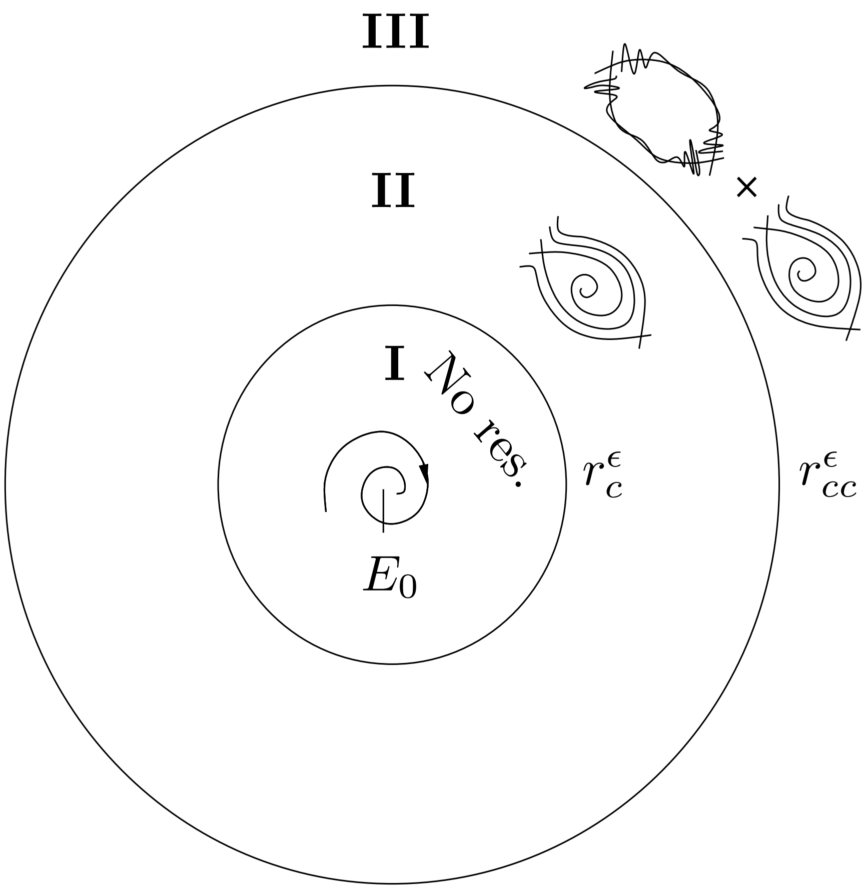


General picture: 1st. and 2nd. critical radius



In region I there are not resonances of the conservative case surviving the dissipative effect. This region, in normal form coordinates, is bounded by the first critical radius r_c. In region 2, we expect only flow type resonances without homoclinic points. This region is bounded by the critical radius r_c and r_{cc}. Finally, region III contains resonances which have homoclinic points despite the dissipation.

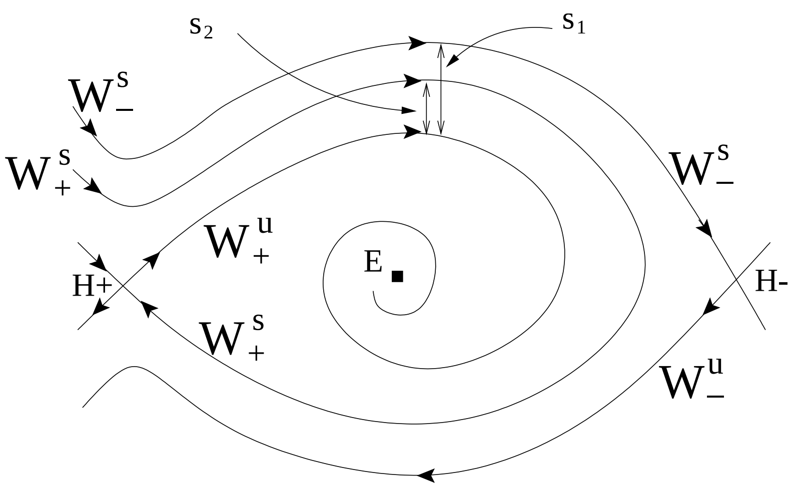
Flow type resonances: probability of capture

Model: Interp. Ham. flow of BNF + dissipation

$$X_{\hat\epsilon}: \begin{cases} \dot J = -(1+dJ)\sin\psi - \hat\epsilon - k\delta^{\frac{m}{4}-1}\hat\epsilon J, \\ \dot\psi = J + cJ^2 - d\cos\psi, \end{cases} \quad \text{where} \quad \begin{array}{ll} c = \mathcal{O}(\delta^{\frac{m}{4}}), & d = \mathcal{O}(\delta^{\frac{m}{4}-1}), \\ k = \mathcal{O}(1), & \hat\epsilon = \mathcal{O}(\delta^{1-\frac{m}{2}}\,\epsilon). \end{array}$$

Thm: Assume that no homoclinic points appear as $\epsilon \searrow 0$ in the m-resonance. Then, the probability of capture by the stable focus of an island of the m-order resonance behaves, when ϵ goes to zero, as

$$P_{capture} = \frac{16|b_1|^{\frac{2-m}{4}}}{m\pi\sqrt{\pi}}\,|\delta|^{\frac{m}{4}-1} + \mathcal{O}(\delta^{1-\frac{m}{2}}\epsilon, \delta^{\frac{m}{4}}).$$



Resonances with homoclinics: a suitable model

A "dissipative" version of the separatrix map:

$$\begin{pmatrix} t \\ J \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{t} \\ \bar{J} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} t + \omega + \beta \log |J| \pmod 1 \\ J + \nu_{\bar{s}} \sin 2\pi \bar{t} + \tilde{B}_{\bar{s}} \\ \mathrm{sign}(J)\, s \end{pmatrix},$$

where

$\nu_{\bar{s}} = A_{\bar{s}}/A_1$, A_i amplitude of the splitting

$\omega = \beta \log(C_1 A_1/\bar{x}_1^*)$, $\beta = -1/\log(\lambda)$ and $\tilde{B}_{\bar{s}} = B_{\bar{s}}/A_1$.

Note that $\nu_1 = 1$.

Constant C_1 can be determined from the variational equations along a suitable interpolating flow (the pendulum flow).

Constant $\bar{x}_1^*$ depends on the properties of the map and it deals with the local radius around the hyperbolic point of the resonance where the BNF holds.

Res. with homoclinics: probability of capture

Under suitable assumptions, dealing with the uniform distribution of the points under iterates of the “dissipative” double separatrix model, the following holds:

- The probability of capture P_{capt} in the m-resonance strip verifies

$$\lim_{\epsilon \to 0} P_{capt} = K(\delta).$$

- If furthermore the m-order resonance is located close enough to the origin E_0, then the constant $K(\delta)$ behaves as

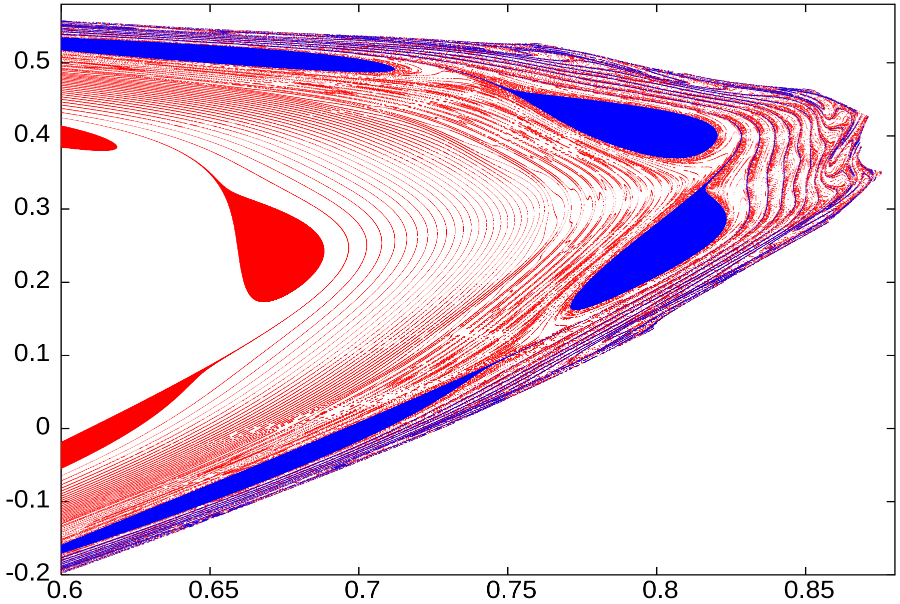
$$K(\delta) \sim \delta^{\frac{m}{4}-1}.$$

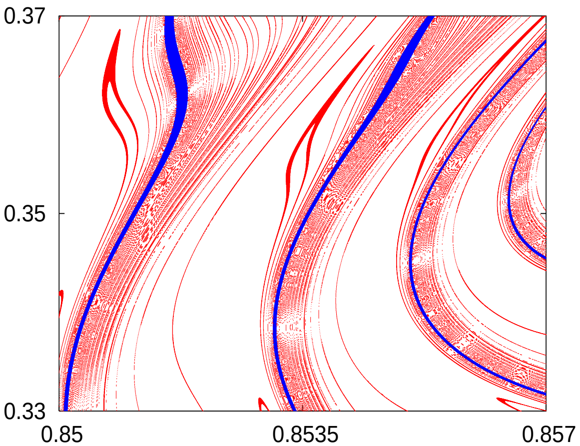
$\rightarrow$ The splitting plays no relevant role when computing a “first” approximation of the probability of capture [a].

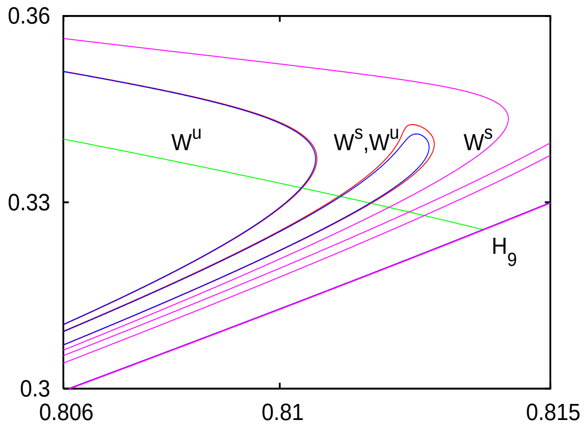
[a] *Planar Radial Weakly-Dissipative Maps: Homoclinic-Type Resonances* in progress

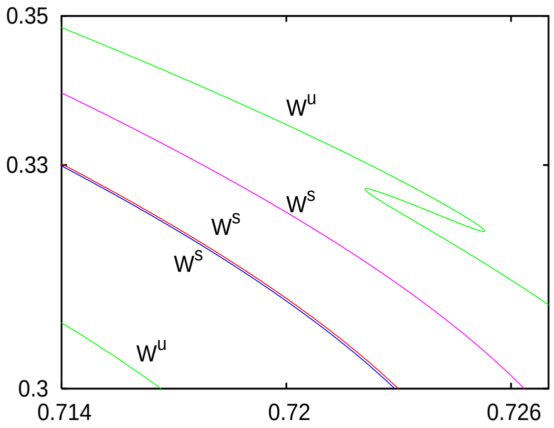
Manifolds of different dissipative resonances

No invariant rotational curves if dissipation $\Rightarrow$ the manifolds of different resonances interact. Easiest case: both resonances of flow type (without homoclinics).









The end...

Thank you for your attention!!