NSDS'05

First Steps in the Study of Weakly Dissipative Two-Dimensional Maps

C. SIMÓ & A. VIEIRO

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Spain

Goal

To understand the effect of adding a weakly dissipation on a conservative map.

- Clarify the global behaviour around an elliptic fixed point.
- Describe the structure of the resonances / Geometry of the invariant manifolds.
- Analize transport properties / Probability of capture in a resonance.
- Identify the regions where the topology of the resonances gives rise to different dynamics (critical radii).
- Illustrate some “limit” properties when the dissipation goes to zero.

The analytical tools

To study a generic APM F we use Birkhoff normal form approximation around the elliptic fixed point with multiplier $\lambda = e^{2\pi i q/m+\delta}$,

$$\mathrm{NF}(F) : z \longmapsto R_{2\pi\frac{q}{m}}\left(e^{2\pi i\gamma(r)}z + c\bar{z}^{m-1}\right) + \mathcal{R}_m(z,\bar{z}),$$

where $b_i \in \mathbb{C}$, $R_{2\pi\frac{q}{m}}$ is the rigid rotation of angle $2\pi q/m$ and

$$\gamma(r) = \delta + b_1 r^2 + b_2 r^4 + \cdots + b_s r^{2s},\ r^2 = |z|^2.$$

Also, when it is justified, interpolation of the normal form by a flow is used. We add the dissipation to the normal form approximation in order to study both quantitatively and qualitatively its effect.

The model

We use for the illustrations a **radially dissipative** version of the classical conservative **Hénon map**. Concretely,

$$H_{\alpha,\epsilon}(x,y) = (1-\epsilon)H_\alpha(x,y), \tag{1}$$

where

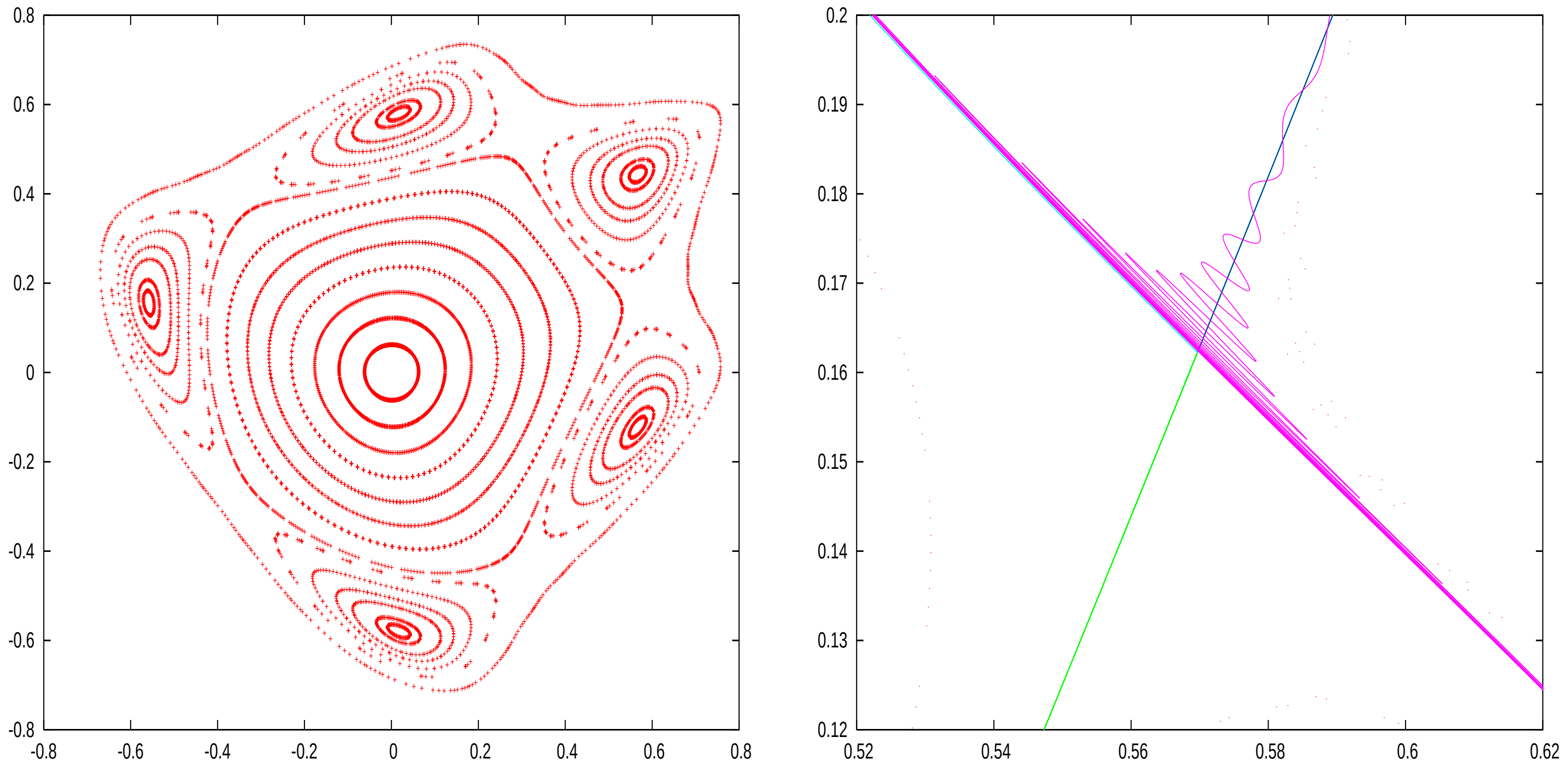
$$H_\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto R_{2\pi\alpha} \begin{pmatrix} x \\ y - x^2 \end{pmatrix},$$

or $(x,y) \mapsto (1 - ax^2 + y, -bx)$ with suitable $a, b \approx 1$.
Motivation:

- “Simplest” planar map. Composition of two simple reversors.
- Appears when modelling a saddle-node bifurcation.

Conservative case ($\epsilon = 0$) (I)

Generically, around an elliptic point it is expected to have an infinite number of resonances. Moreover, unless the map is integrable, the invariant manifolds do not coincide and a splitting of the separatrices is created (homoclinic tangle).



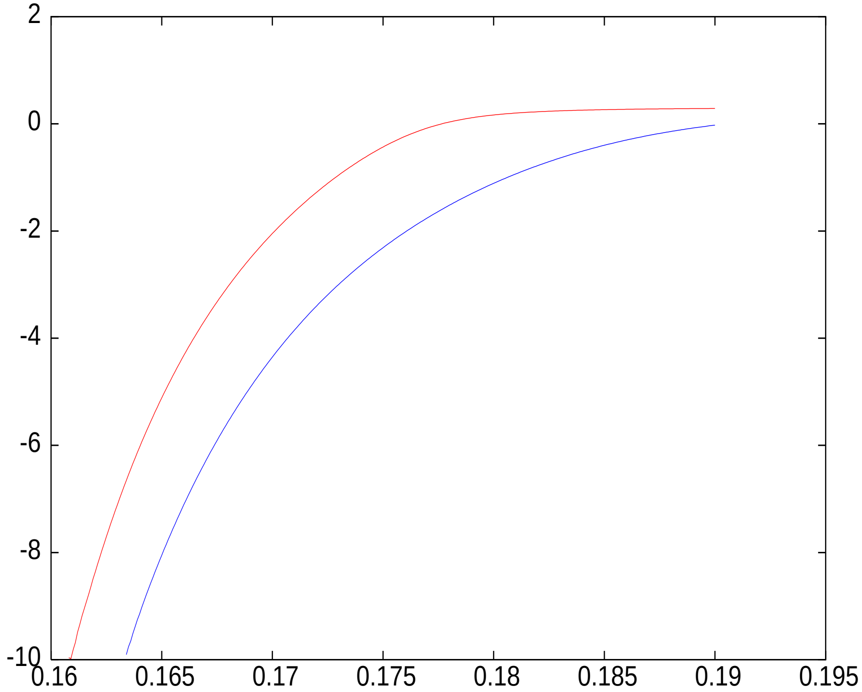
Conservative case

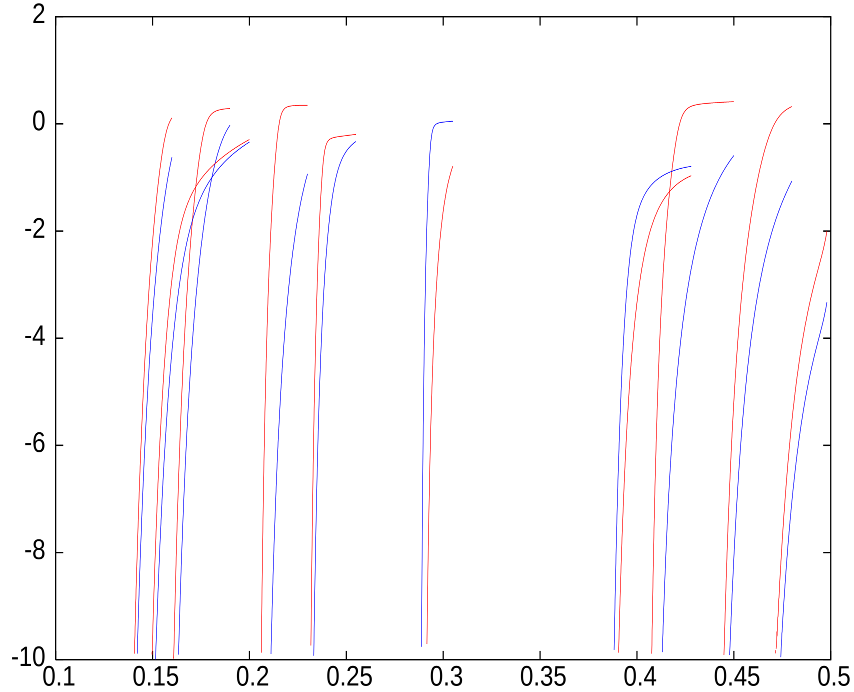
Remarks:

- **The size of the perturbation from integrable depends on the domain.**
  The splitting of separatrices has different order depending on the resonance considered.
- **For a generic APM, the inner and the outer splittings of the same island are generically different.**
  It depends on the derivatives (mainly on the sign of the first two derivatives) of the rotation number with respect to the radial coordinate (that is, on the torsion coefficient and its first derivative).

Conservative case

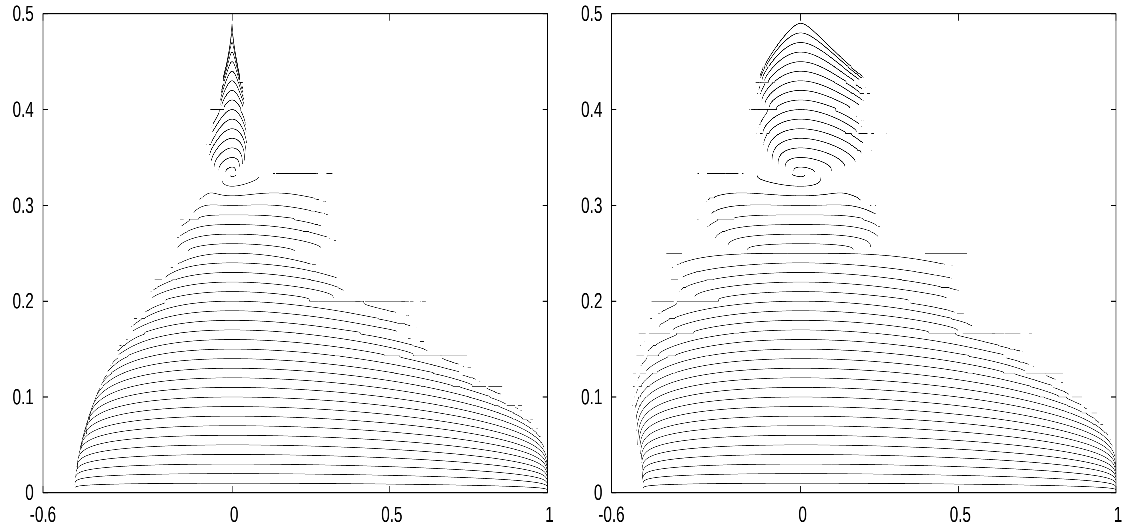
In the plots it is depicted $\log_{10}(\text{splitting angle})$ vs α. Red (blue) line represents the outer (inner) splitting of the resonance measured on the reversibility axis $y = \tan(\pi\alpha)x$ and $y = x^2/2$. Left picture is for the resonance (1:7) of the Hénon map. Right picture represents different resonances of the map.





Conservative case

Generically, in a neighbourhood of an elliptic fixed point a twist condition holds, that is, the rotation number is a monotone function of the action. Nevertheless, far from that point the twist condition can be violated. Close to the value of the radius where the twist condition does not hold the rotational invariant curves give rise to meandering curves.



Weak Dissipation

Previous considerations:

- For ϵ “big” enough the dynamics collapses to the origin and no resonances outlast the dissipation. Hence, the periodic points which configure a concrete resonance should be destroyed, when ϵ increases, as a result of saddle-node bifurcation.
- For ϵ “small” some resonances survive and their topological shape changes.
- The destruction of a resonance depends on the width and order it has (a priori).

First critical radius

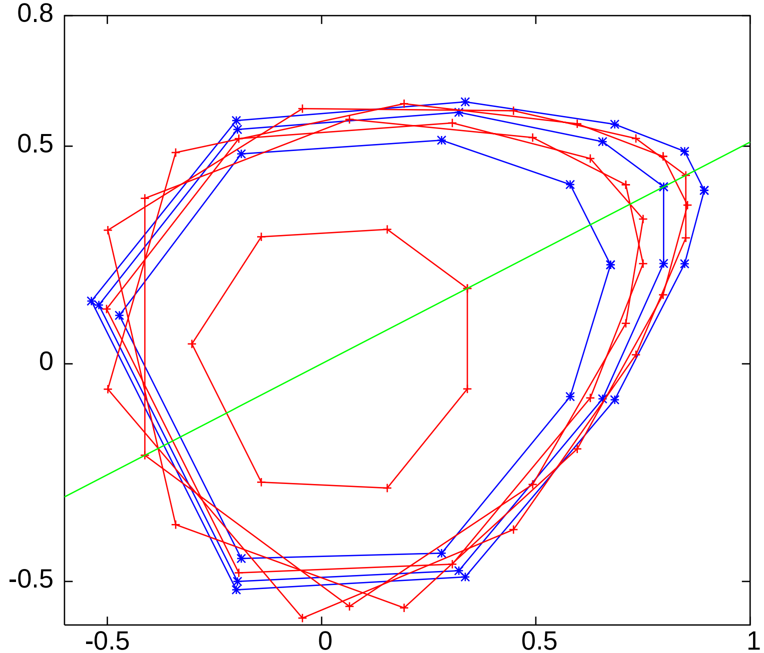
- We expect the small resonances to be destroyed by the dissipation.
- Due to the twist condition, the resonances are arranged by rotation number.
- The width of an m-resonance is of the order $\mathcal{O}(I_*^{m/4})$, where $I_* = -\delta/2b_1$ being $\alpha = q/m + \delta$ and b_1 the first Birkhoff coefficient.

Conclusion:

Close to the origin we can expect a neighbourhood where no resonance survives.

First critical radius: example

$\log_{10}(\epsilon)$	**Res. destroyed**
-6	Inside $B_0(0.27)$
-4.569	(2:19)
-4.625	(1:7)
-3.456	(1:8)
-3.297	(1:9)



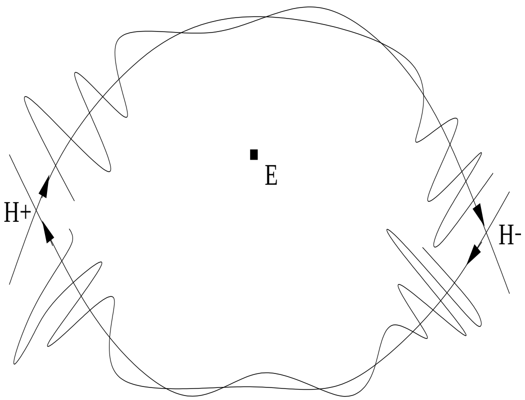
Resonances: (1:7), (1:8), (2:17), (1:9), (2:19) and (1:10) ($\alpha = 0.15$).

Second critical radius

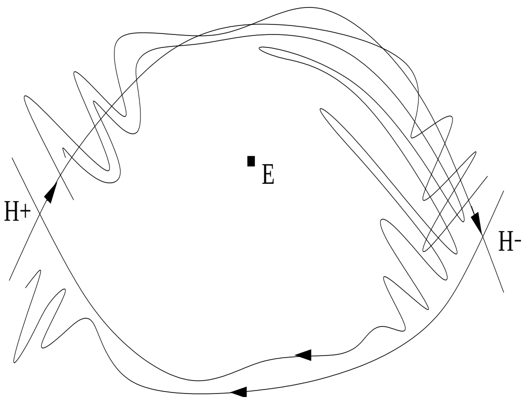
- Outside the first critical radius we find some resonances that have survived the dissipation.
- The structure of these resonances has changed:
  - The elliptic points become attractor foci.
  - The position of the invariant manifolds allows to pass more points through the resonance.

We can determine **two different “types” of resonances** depending on the existence or destruction of homoclinic orbits. We need first to understand the **evolution of a resonance under the dissipation effect** better.

A possible scenario



1.



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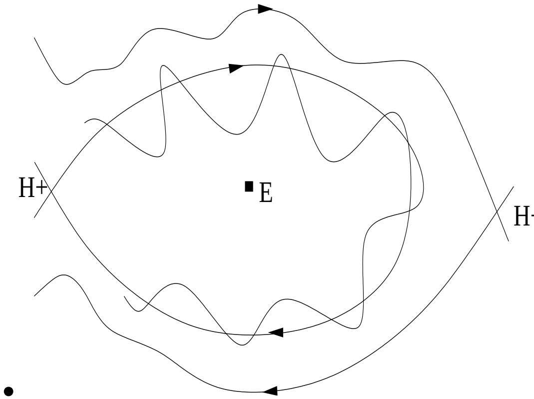

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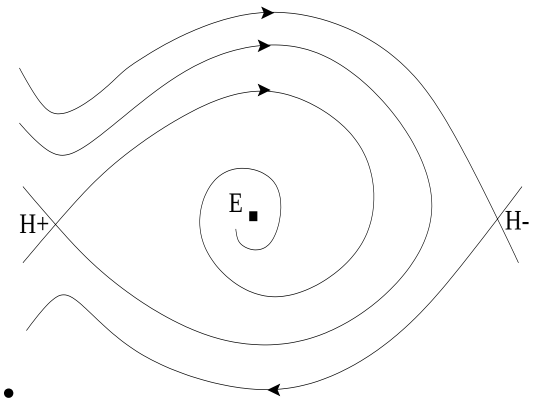

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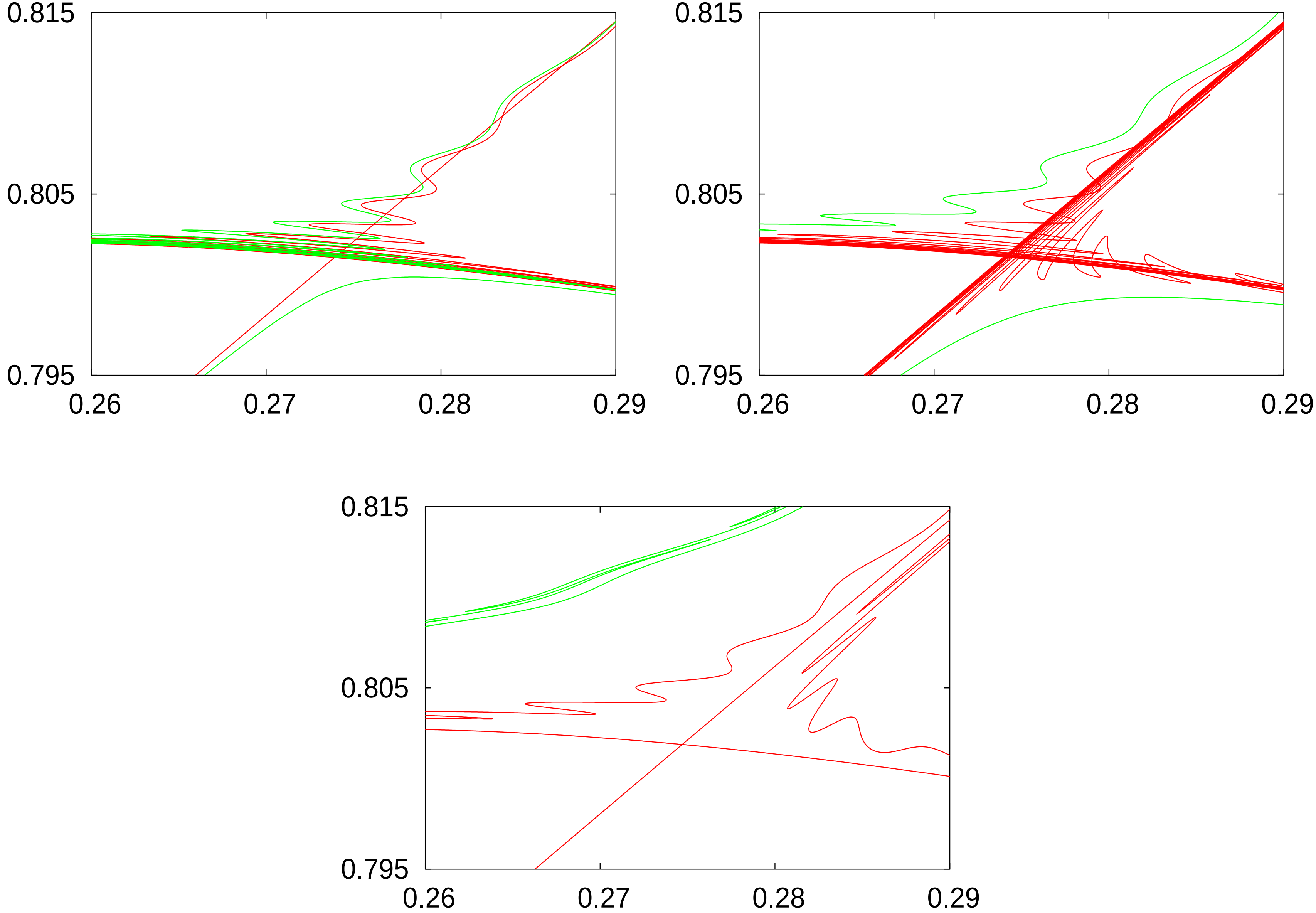
Evolution: $\alpha = 0.17$; $\log_{10}(\epsilon) = -6, -5.4, -4$, res.(1:7)



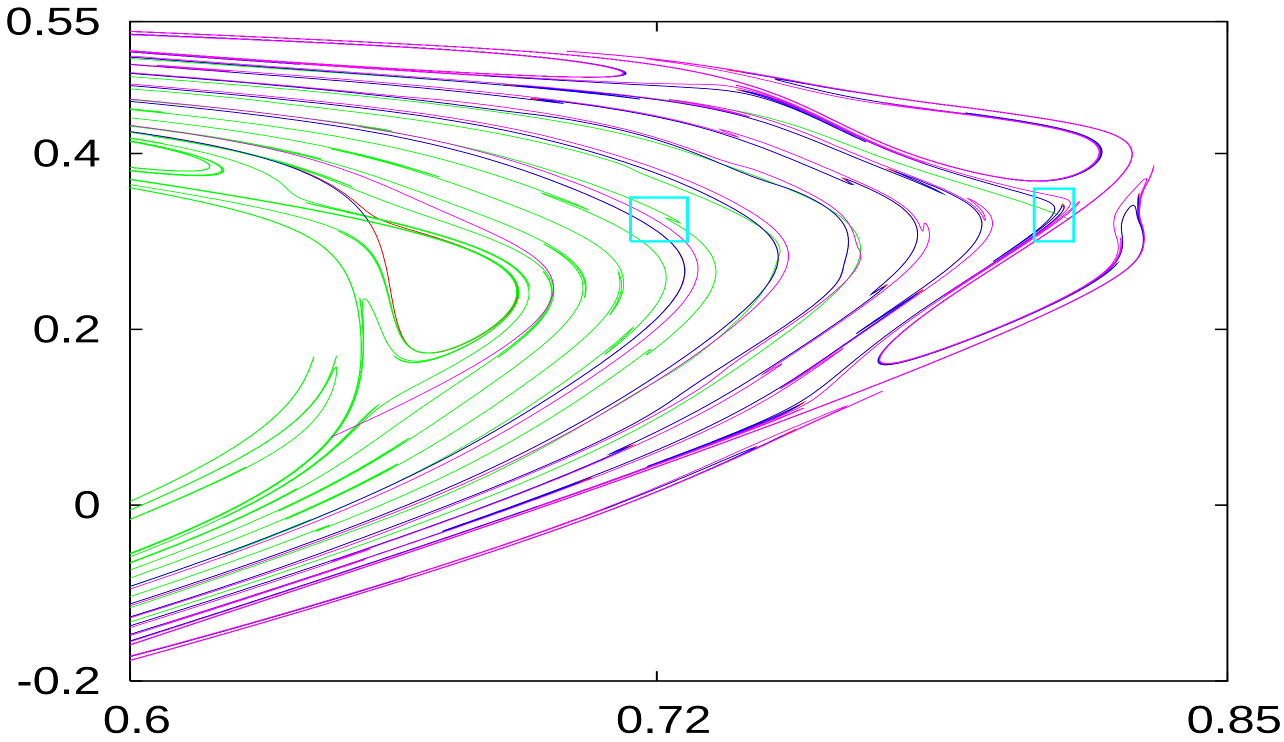
Flow type resonances

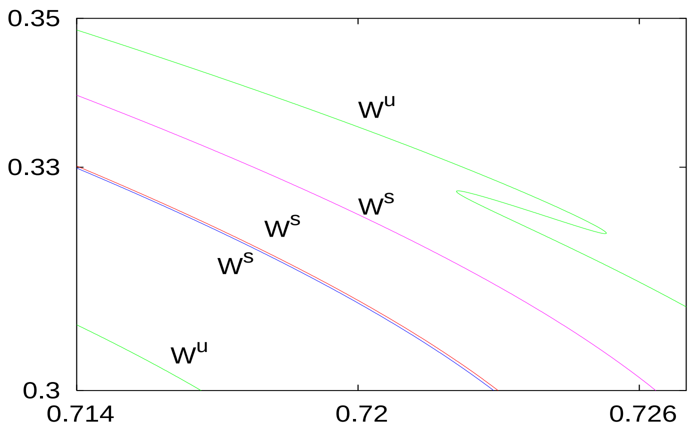
- When all the homoclinic orbits are destroyed by the dissipation (and the system is not very close to homoclinic tangency) the dynamics in a resonance can be approximated by a flow.
- The probability of capture in a flow type resonance depends on the strips that determine the invariant manifolds of the hyperbolic point.

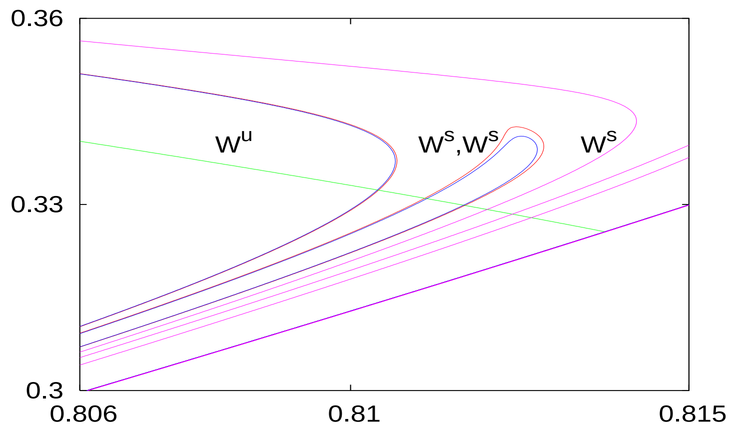
We want...

- ...to describe how the strips travel along the phase space.
- ...to determine how the probability of capture changes depending on ϵ and on the map we have. In particular, which is the limit of this probability when $\epsilon \searrow 0$?

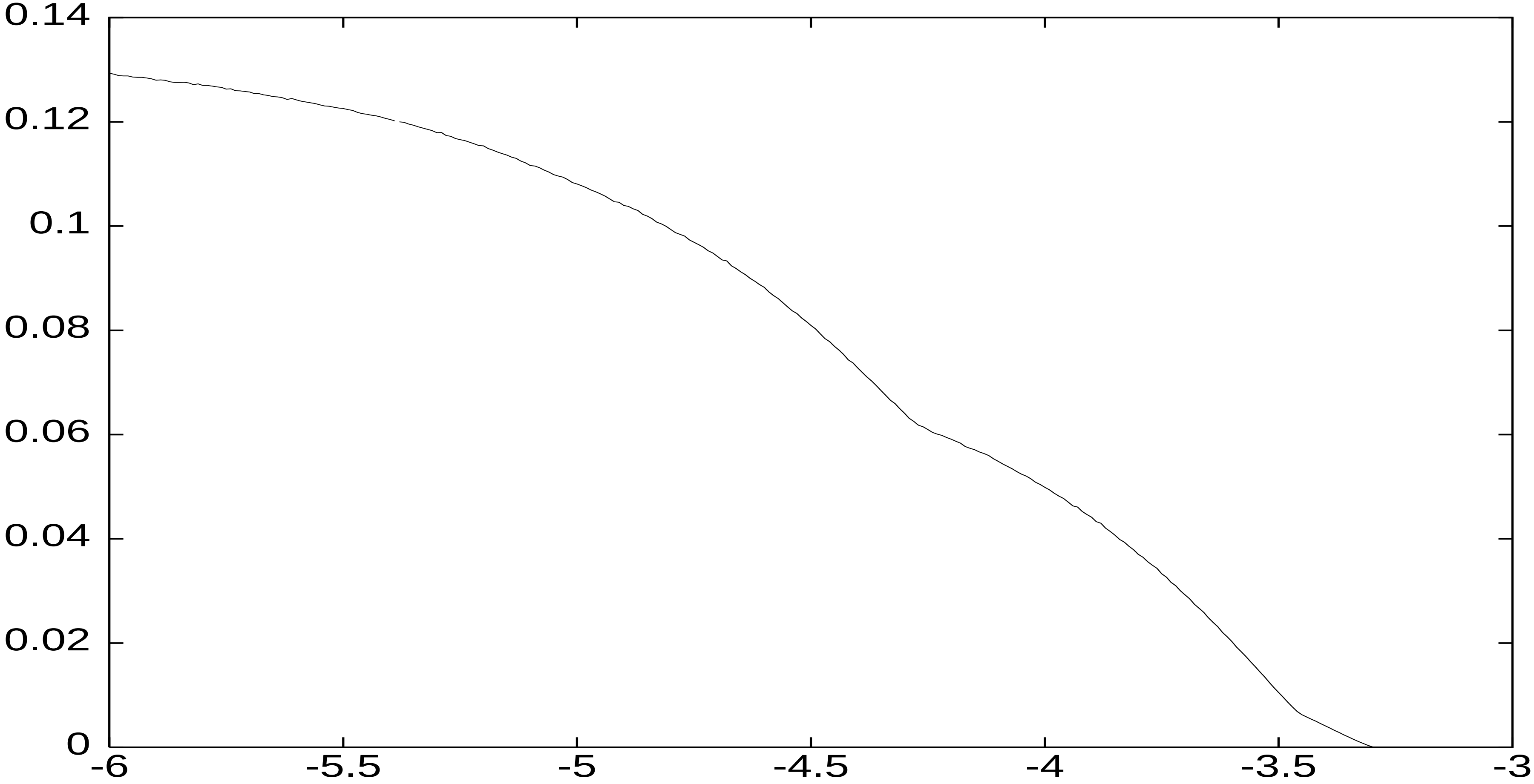
The strips


0.55
0.4
0.2
0
-0.2
0.6
0.72
0.85


0.35
0.33
0.3
0.714
0.72
0.726
W^u
W^s
W^s
W^s
W^u

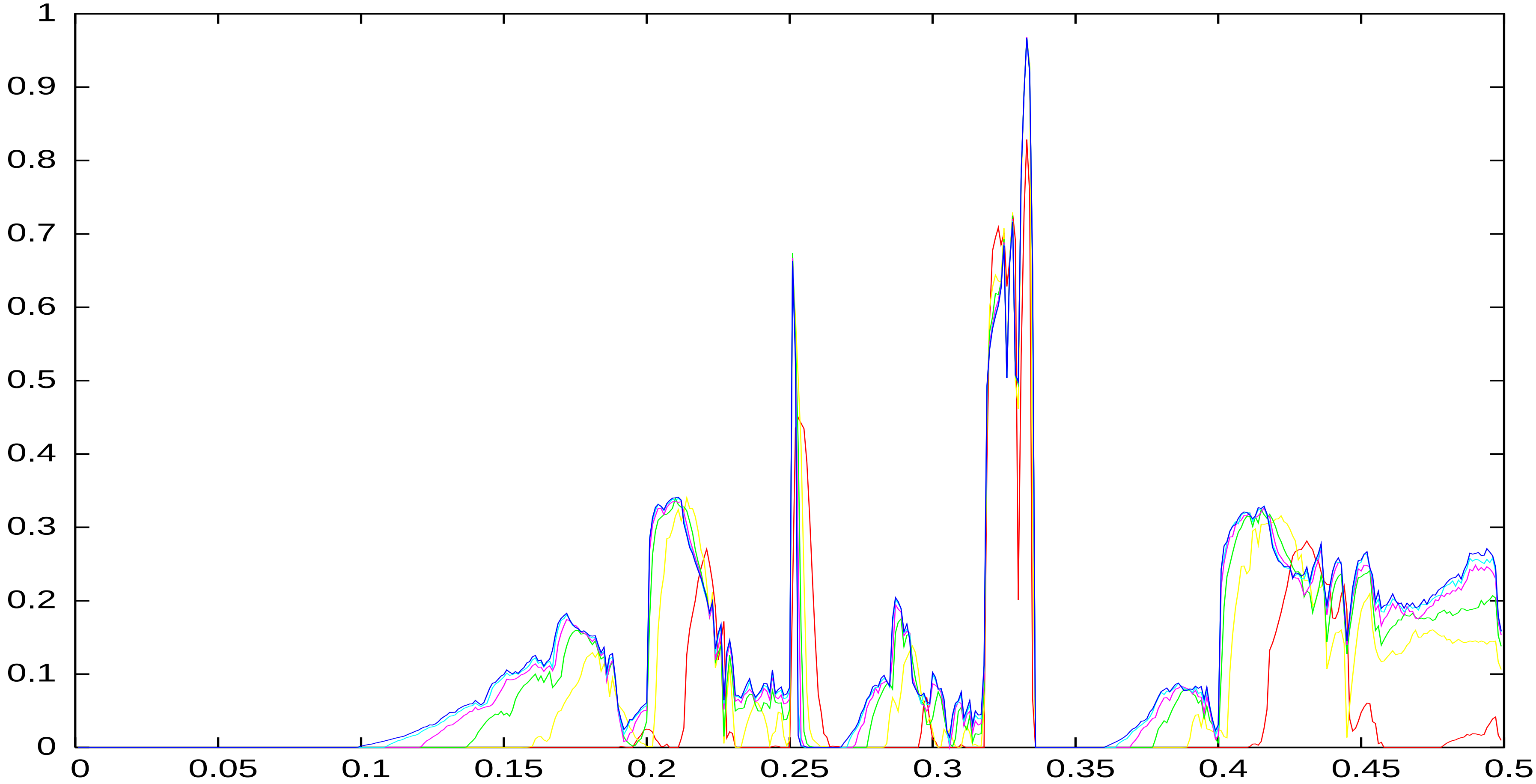

0.36
0.33
0.3
0.806
0.81
0.815
W^u
W^s,W^s
W^s

Points captured by resonances - ϵ



x-axis: $\log_{10}(\epsilon)$ **y-axis:** ratio of number of points captured by the foci of resonances and number of points that do not leave the ball of radius 0.97 $\alpha = 0.15$

Points captured by resonances - α



x-axis: α **y-axis:** ratio of the number of points captured by the foci of the islands and the ones that do not escape;
curves: values of $\epsilon : \; 10^{-k}, \; k = 2$ (red), $\ldots, 7$ (blue).

Behaviour of the capture probability

Assume we aproach the conservative case from the flow domain, that is, when $\epsilon \to 0$ no homoclinic points appear in the resonance. Then,

Proposition: The probability of capture by the perturbed elliptic point (stable focus) of an island of the m-order resonance behaves, when ϵ goes to zero, as $\delta^{\frac{m}{4}-1}$.

$\lambda = e^{2\pi i\alpha}$ is the multiplier of the fixed point, $\alpha = q/m + \delta$, δ small and irrational (diophantine).

Conjecture

Assume that ϵ is small but sufficiently large to do not have homoclinic points in a resonance.

- The measure of the points captured by the resonance when approaching the conservative case, assuming that no homoclinic points are created, is the sum of the measure of the islands and the measure of the strips (both can be approximated using normal forms).

$$\Gamma(F,\epsilon) = \{(x,y) \in \mathcal{A} \text{ (a fixed domain)} \mid \omega(x,y) = E\}\,.$$

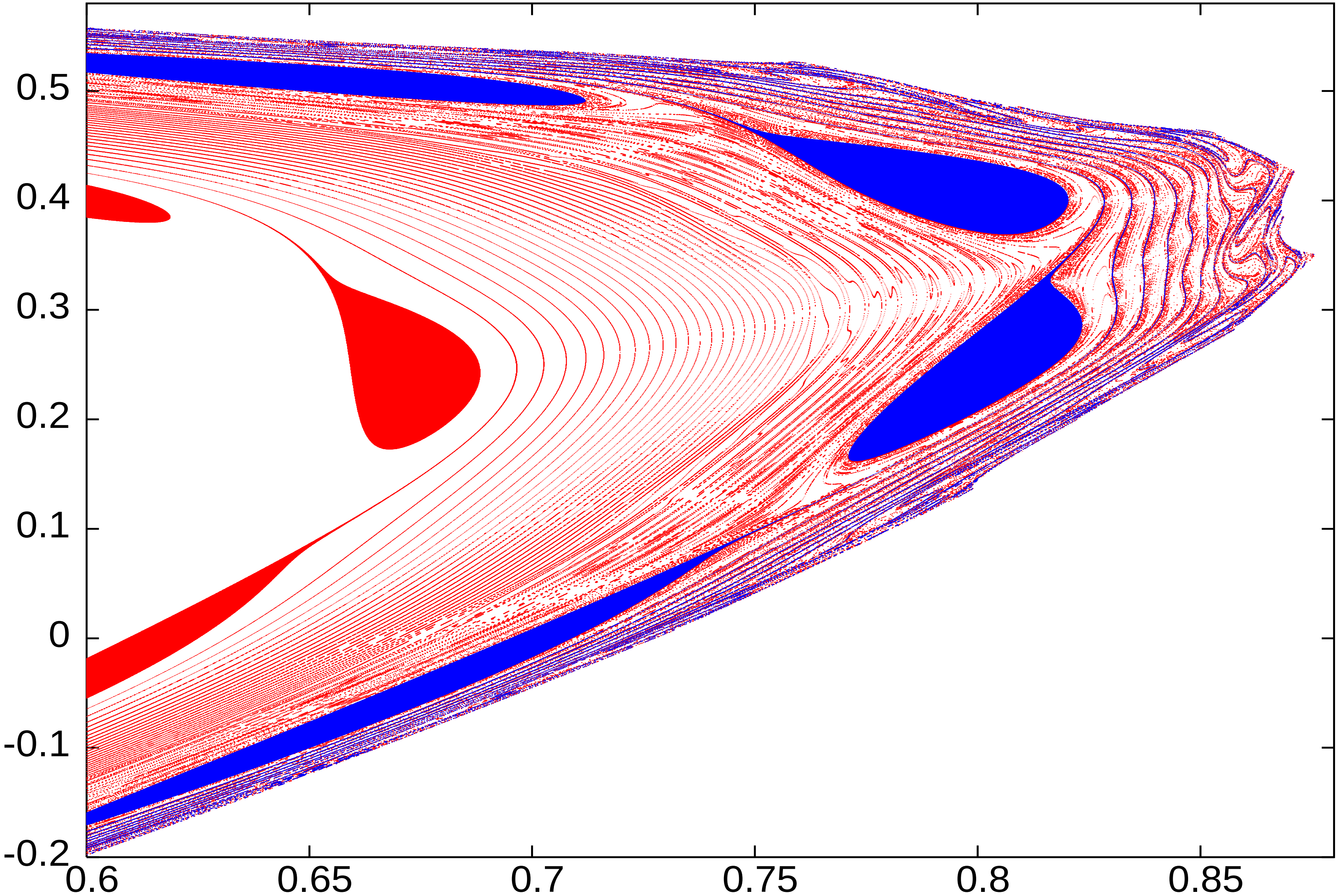
$$\lim_{\epsilon\to 0} \mu(\epsilon) = \lim_{\epsilon\to 0} \left(\frac{mes_L\left(\Gamma(F,\epsilon)\right)}{mes_L\left(\mathcal{A}\right)} \right)^c \overset{conj.}{=} A_{islands} + A_{strips},$$

Open question: what happens when approaching to the conservative case and homoclinic points are allowed?

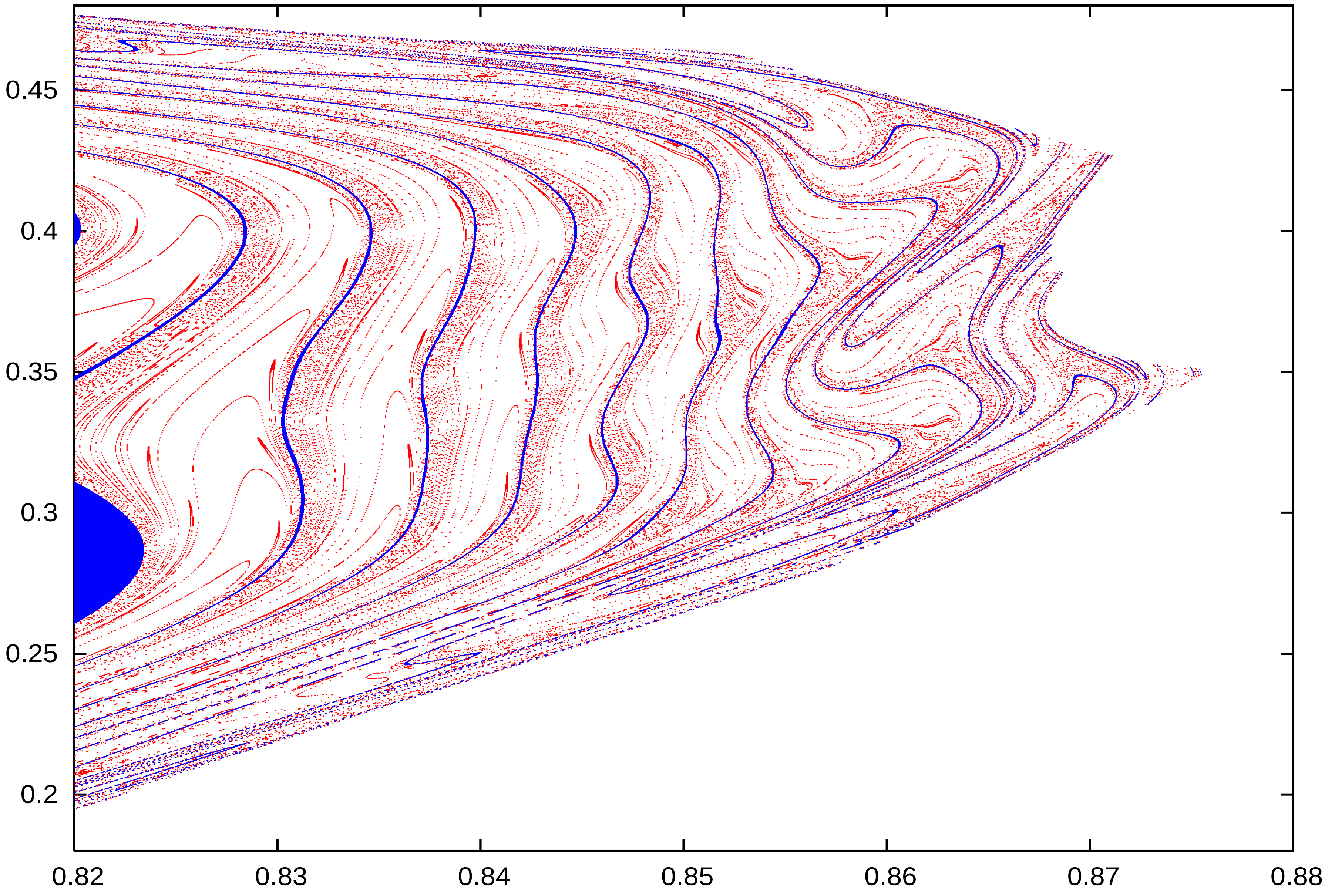
Future

- It is necessary to develop models to understand the dynamics around each type of resonances:
  - Flow type: NF + Approx. by flow
    C.Simó and A.V. (in progress).
  - Homoclinic type: Deal with the diffeomorphism directly. It requires suitable models (in progress).
- To prove the conjecture stated before.
- Generalisations to higher dimensions.

A final picture...

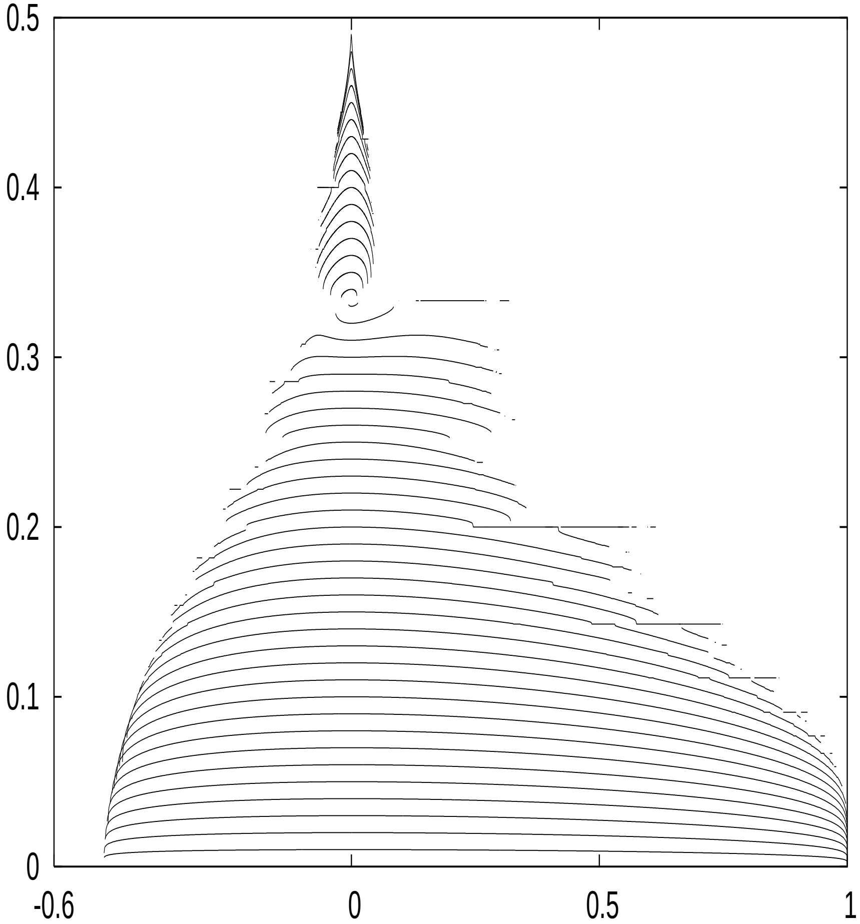


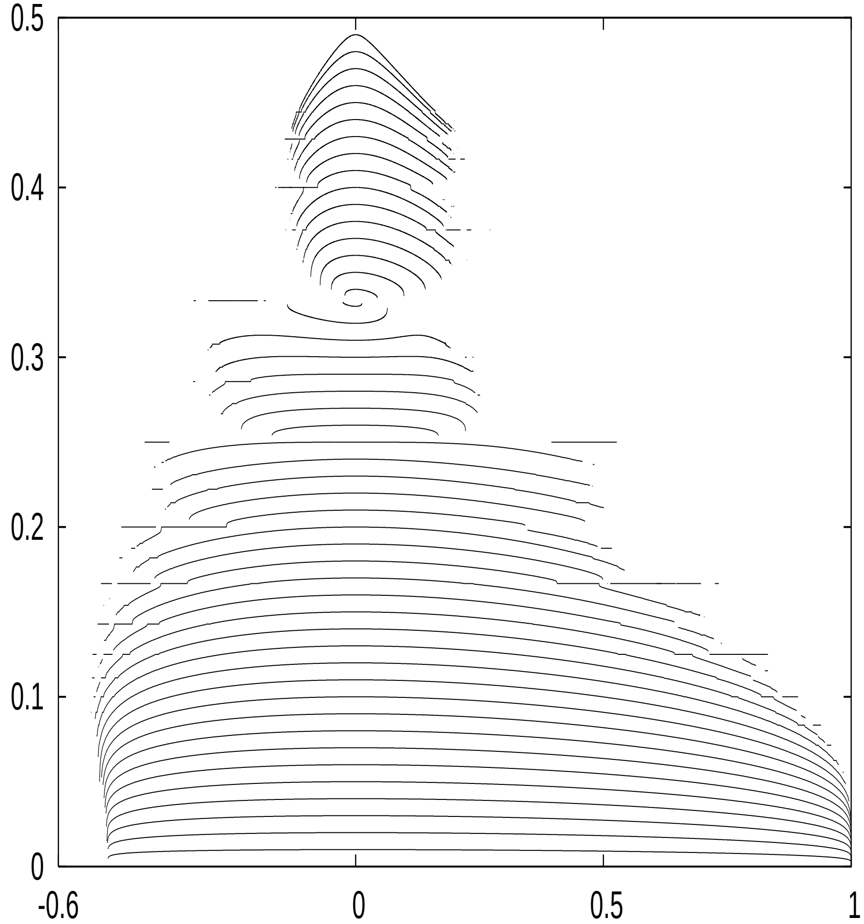
And a magnification



THE END...

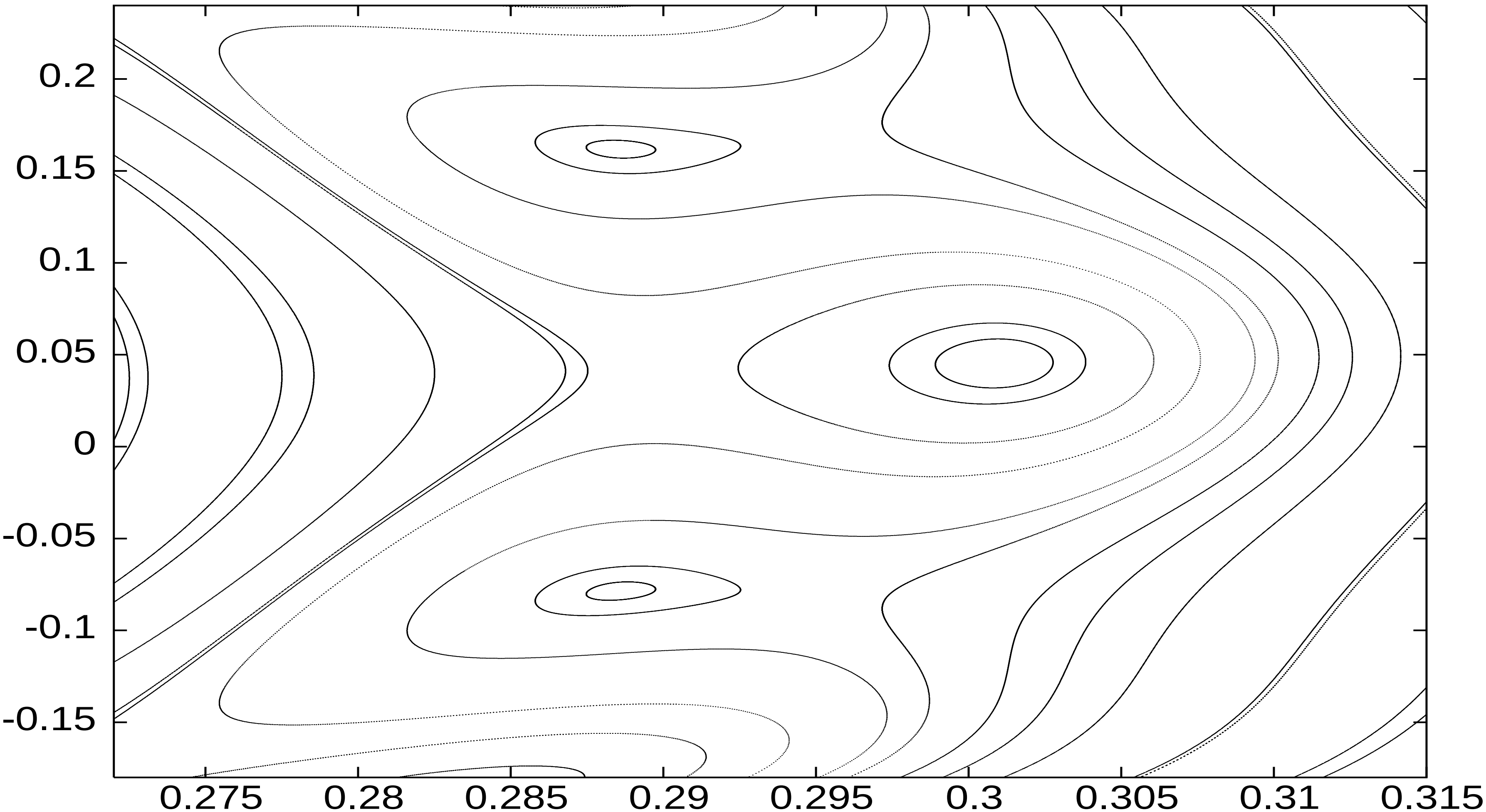
Rotation number





Back to the presentation

Meandering curve



Back to the presentation