Some quantitative global aspects of (perturbations of) area preserving maps.

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Carles Simó and Arturo Vieiro

Before starting...

Our philosophy:

We are interested in **mathematical proofs** but...

we do not want qualitative studies or abstract/theoretical reasonings because ...

...we want our results to be useful in **real applications**.

Contents

- **Local (semi-local) study of APMs:** Resonances (including strong resonances), inner/outer splittings of separatrices.
- **Semi-global study:** Dynamics in chaotic zones.
- **Global study:** Evolution of the domain of stability with respect parameters.
- **Weakly dissipative maps:** Coexistence of attractors and probability of capture.

Along the presentation the Hénon map will be used as a paradigm of APM.

One of the formulations is

$$H_c : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} c(1-x^2)+2x+y \\ -x \end{pmatrix}$$

Local (semi-local) study of APMs

$\longrightarrow$ We want:

- A description of the resonant structures (islands).
- To study the inner and the outer splittings of separatrices.
- To study the strong resonances.

$\longrightarrow$ Steps to follow:

1) Consider BNF (**local** study).
2) Construct a suitable model from BNF (an interpolating Hamiltonian flow).
3) Localise the model around the resonance strip we want to study (**semi-local** study).
4) Use this model to study the properties we want.

BNF + Interpolating Hamiltonian

F_δ one-parameter δ-family of APMs with $F(E_0) = E_0$ elliptic fixed point.
Spec $DF(E_0) = \{\mu, \bar{\mu}\}$, $\mu = e^{2\pi i\alpha}$, $\alpha = q/m + \delta$, δ small enough.
(x, y)-Cartesian coord., $(z, \bar{z})$-complex coord. ($z = x + iy$, $\bar{z} = x - iy$).
(I, φ)-Poincaré variables ($z = \sqrt{2I}\exp(i\varphi)$).
$\longrightarrow$ Consider the Birkhoff NF of $F_\delta(x, y)$ to order m around E_0
(say BNF$_m(F_\delta)(z, \bar{z})$) and let $K(z, \bar{z}) =$ BNF$_m^m(F_\delta)(z, \bar{z})$ (near Id).
$\longrightarrow$ Define
$\mathcal{H}_{nr}(I) = \pi \sum_{n=0}^{s} \frac{b_n}{n+1}(2I)^{n+1}$ and $\mathcal{H}_r(I, \varphi) = \frac{1}{m}(2I)^{\frac{m}{2}} \cos(m\varphi)$.
The time-1 map generated by the flow defined by the Hamiltonian

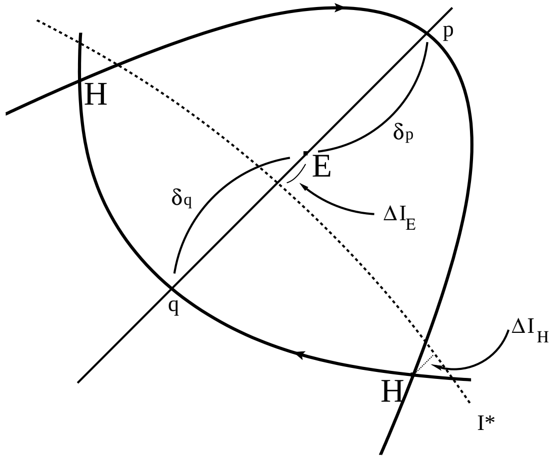
$$\mathcal{H}(I, \varphi) = \mathcal{H}_{nr}(I) + \mathcal{H}_r(I, \varphi)$$

interpolates K with an error of order $m + 1$ with respect to the $(z, \bar{z})$-coordinates, in a suitable annulus containing the resonant m-island.

Description of resonances

Generic case: $\alpha = q/m + \delta$, $m > 5$, δ sufficiently small, $b_1 \neq 0$.

- If $b_1\delta < 0$ then F **has** a resonant island of order m.
- The resonant zone is determined by **two periodic orbits** of period m located near two concentric circumferences (in the BNF variables). The closest orbit to the external circumference is elliptic while the one located close to the inner circumference is hyperbolic.
- The **width** of the resonant island is $\mathcal{O}(I_*^{m/4})$, $I_* = -\delta/2b_1$.



A model around a generic resonance

For a generic APM such that $\delta < 0,\ b_1 > 0,\ b_2 \neq 0$, the dynamics around an island of the m-resonance strip $(m \geq 5)$ can be modeled, after suitable scaling ($J \sim \delta^{-m/4}(I - I_*)$), by the time-$\log(\lambda)$ map of the flow generated by

$$\mathcal{H}(J, \psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos(\psi),$$

where $c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1})$. Bounding the errors it is shown that it gives a “good” enough approximation of the dynamics in an annulus containing the m-islands.

$\longrightarrow$ Then, we have the following... [a]

[a] The details of the proof (singularities, suitable Hamiltonian,...) can be found in:
Resonant zones, inner and outer splittings in generic and low order resonances of area preserving maps. Nonlinearity 22, 5:1191–1245, 2009.

Main result: comments on the hypothesis

- **A1.** $b_1(\delta)$ is non-zero for $\delta = 0$.
- **A2.** $W^u = G(W^s)$, G periodic (between homo p and $F(p)$),

  s scaled variable s.t. $G(s) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\, 2\pi s)$.

  We assume: The maximum of the norms of the functions $c_{\pm 1} \exp(\pm i 2\pi s)$ is bounded away from zero, when δ tends to zero, on suitable lines whose imaginary part tend to $\tau_\pm$ when $\delta \to 0$.
- **A3.** There exists a fixed $\alpha > 0$ s.t.

$$\sigma_\pm = \exp\left(-\frac{2\pi \,\mathrm{Im}\, \tau_\pm - \eta_\pm}{\log(\lambda(\epsilon))}\right)\left(\cos\left(\frac{2\pi \,\mathrm{Re}\, \tau_\pm}{\log(\lambda(\epsilon))} - \phi_\pm\right) + o(1)\right),$$

  where $|\eta_\pm| < \log(\lambda(\epsilon))^{1-\alpha}$ for ϵ sufficiently small.
- **A4.** F maybe meromorphic but the singularity remains at a finite distance when δ goes to 0.

Main result

Theorem. Let F be an APM. Assume that it has an m-order resonance strip, $m > 4$, located at an average distance $I = I_* = \mathcal{O}(\delta)$ from the elliptic fixed point and δ is sufficiently small. Under the assumptions **A1**, **A2**, **A3** and **A4**, the following conclusions hold.

a) The outer splitting is larger than the inner one being the difference between the position of the corresponding nearest singularities $\mathcal{O}(\delta^{m/4-1})$.

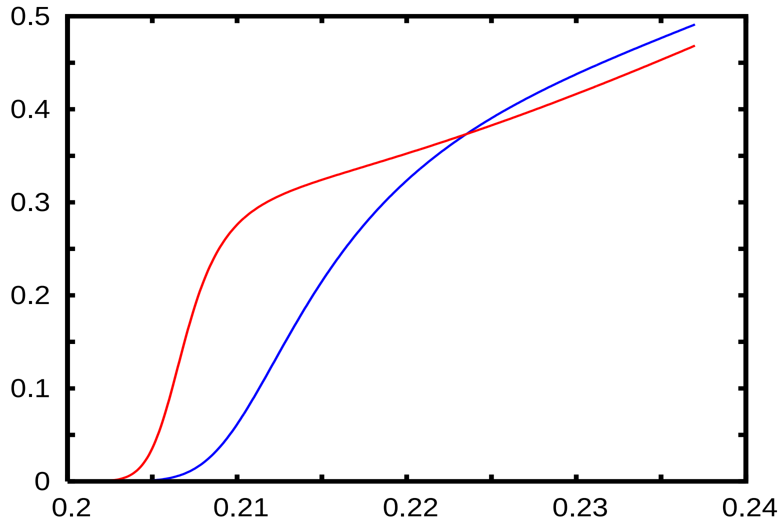
b) Neither the inner nor the outer splittings oscillate.

$\longrightarrow$ **Question:** Consequences in the width of the chaotic zones of this fact?

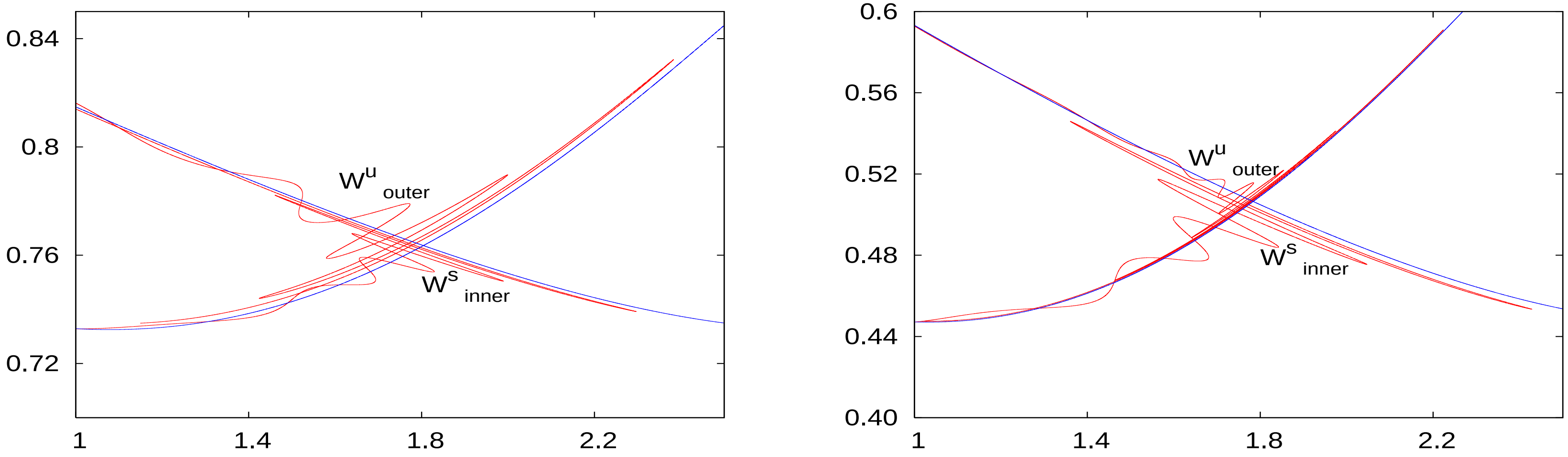
Before some comments...

Some comments: Far from the elliptic point

2:11 Hénon



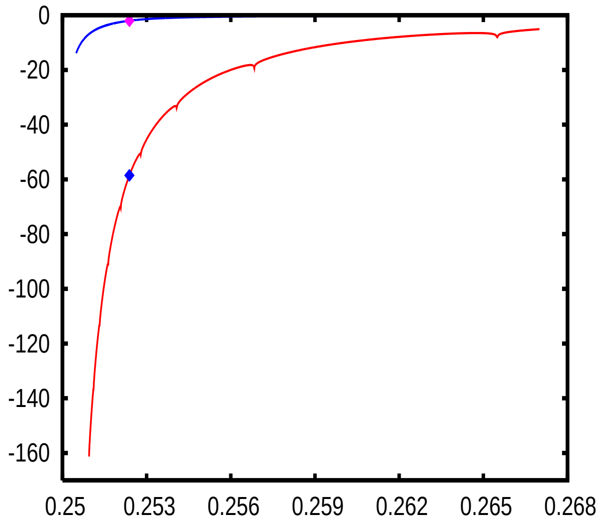
twist: $(I, \theta) \to (I + 0.14\cos(\theta + \alpha(I)), \theta + \alpha(I)), \alpha(I) = b_1 I + b_2 I^2$.

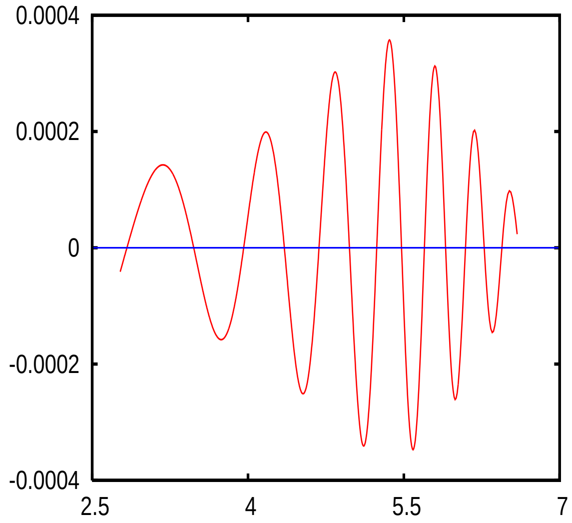


Some comments: Strong resonances

- The description of the resonant structure by means of the interpolating Hamiltonian does not hold if $m \leq 4$.
- We have studied in detail the generic cases for the resonances (1:3) and (1:4), computing the Hamiltonian and the singularities, and also some non-generic cases:

Hénon map 1:4 resonance **Non-generic!!**





Semi-global study of APMs: Chaotic regions

We focus now on the chaotic regions created by the invariant manifolds emanating from a fixed/hyperbolic point h. We do not assume the region of interest to be close to the origin but we require the system to be not too far from integrable in the selected domain to be studied.

$\longrightarrow$ We want:

- To study the resonant islands far from the elliptic point
- To study the dynamics in the chaotic zones

$\longrightarrow$ How?:

Using suitable return maps.

Chaotic regions considered

For each of the following cases we use a concrete return map model to study the dynamics.

- Open case (fish like) — Separatrix map
- Figure eight case (pendulum like) — Double separatrix map
- Large regions of instability (e.g. Birkhoff z.i.) — Biseparatrix map

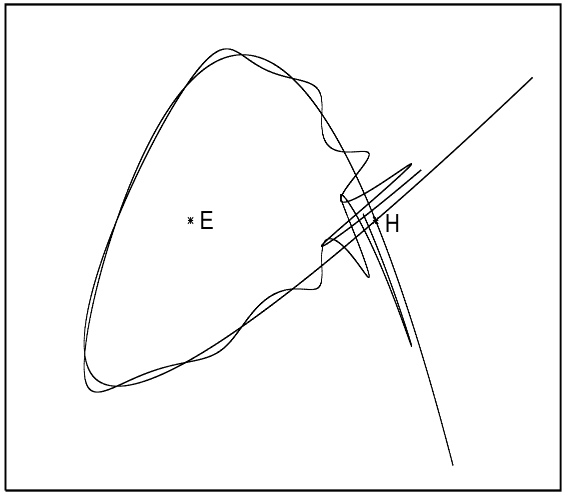
$\longrightarrow$ We look for **quantitative** information on the dynamics within the chaotic zones. However, the biseparatrix model only gives us a topological description of the dynamical behaviour. [a]

[a] The following results can be found in:

Dynamics in chaotic zones of area preserving maps: close to separatrix and global instability zones.

Submitted to Physica D.

Open case



$$SM : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + a + b\log|y'| \\ y + \sin(2\pi x) \end{pmatrix}$$

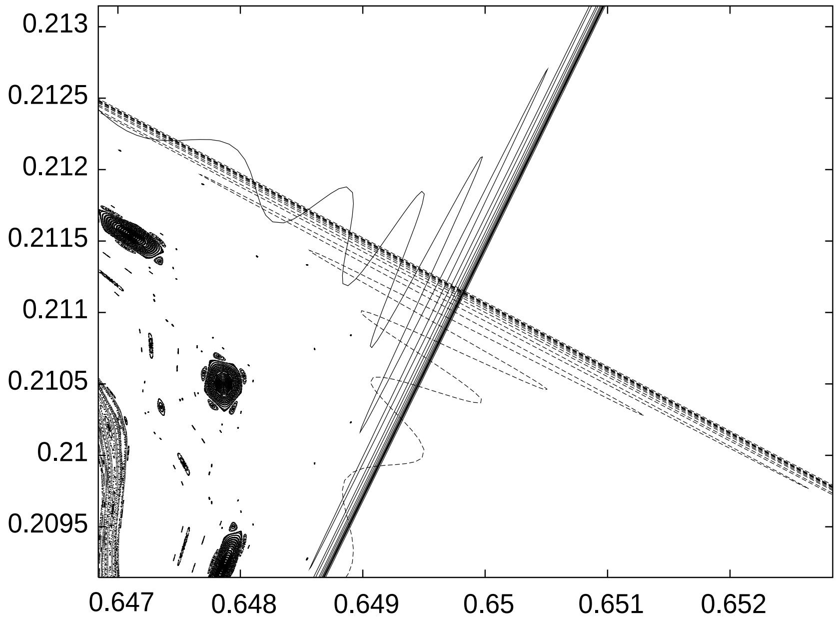
where $b = 1/\log(\lambda)$, λ the dominant eigenvalue of $DF(h)$ and a is a “shift”.

The y-vble. is scaled by the amplitude of the splitting.

Open case: results

- Distance to invariant curves from the separatrix: $d_c \sim |b|/k^*$ (SM is approximated by STM, $k^* \approx 0.97/(2\pi)$ Greene value).
  - When coming back to the original variables: $D_c \sim \sigma\ell/(2\pi k^* \log(\lambda))$,
  - If measured from the hyperbolic point, assuming the map close to the time-ϵ flow of $H(x,y) = y^2/2 - \alpha x^3 - \beta x^2$, one has: $D_c^h \approx (3LD_c/2)^{1/2}$, where L is the distance between the hyperbolic and the elliptic point inside the "fish". This result can be improved using higher order interpolating Hamiltonians.
- Distance to islands from the separatrix: $d_i \sim |b|/\tilde{k}$, $\tilde{k} = 2/\pi$.
- Expected number of "central" islands before the r.i.c. $\#\{islands\} \approx 1.415 \times b$.

Hénon map $H_\alpha(x,y) = R_{2\pi\alpha}(x, y - x^2)$



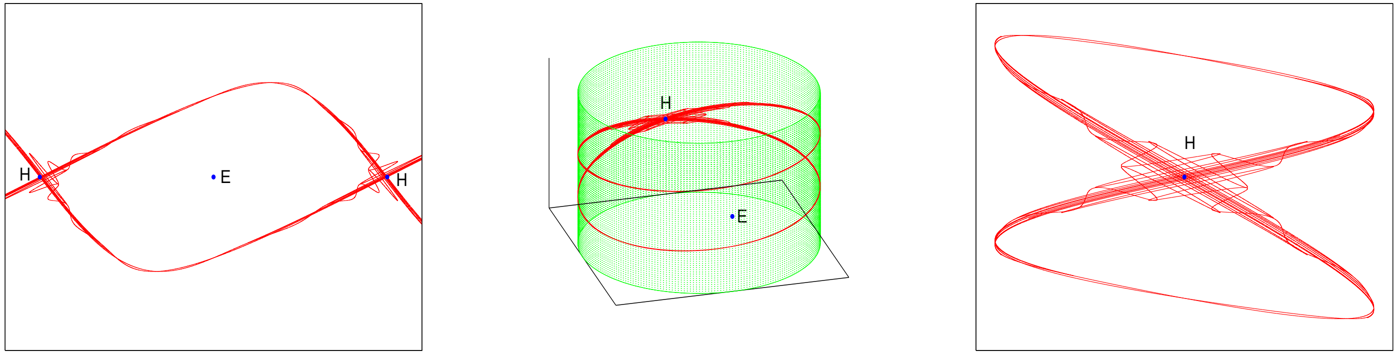
$\alpha = 0.1$

Experimental values: $(D_c^H)_e \approx 2.94\times10^{-3}$, $(D_i^H)_e \approx 2.08\times10^{-3}$

“Fish” interpolating Hamiltonian: $D_c^H \approx 2.47\times10^{-3}$, $D_i^H \approx 1.85\times10^{-3}$

5-order interp. Hamiltonian: $D_c^H \approx 2.731\times10^{-3}$, $D_i^H \approx 2.050\times10^{-3}$

Figure eight case



$$DSM : \begin{pmatrix} x \\ y \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} x + a_{\bar{s}} + b \log|\bar{y}| \pmod 1 \\ y + \nu_{\bar{s}} \sin 2\pi x \\ \text{sign}(y)\, s \end{pmatrix},$$

where ν is such that $\nu_1 = 1$ and $\nu_{-1} = A_{-1}/A_1$, being A_1 and A_{-1} the amplitudes of the outer and inner splittings, respectively, of the resonant island.

Comments:

- It is defined on a domain $\mathcal{W} = \mathcal{U} \cup \mathcal{D}$ (upper and lower domains around the outer and inner separatrices of the resonance).
- $y > 0$ means we are outside the stable manifold (either in $\mathcal{U}$ or $\mathcal{D}$).

Figure eight: results

Generic resonances close to the origin. Assume $b_1\delta < 0$ and that the hypothesis of the theorem concerning the difference of the inner and outer splittings hold. Then,

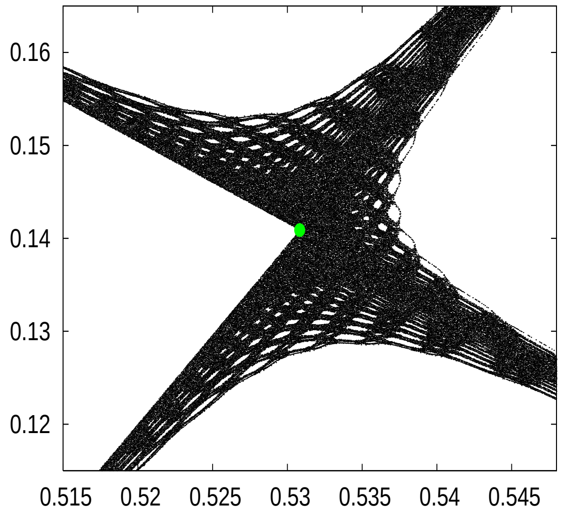
- The width of the outer chaotic zone is larger than the width of the inner chaotic one if, and only if, sign $b_1 \cdot$ sign $b_2 < 0$.
- Both amplitudes of the stochastic layer are of the order of magnitude of the outer splitting (the largest one).

Pendulum-like islands: comments

The idea is to construct an interpolating Hamiltonian of the map (in a domain containing the resonance) and to use preservation of energy to see how the distance to the rotational invariant curves changes when measuring from the upper $\mathcal{U}$ and the lower $\mathcal{D}$ domains. This can be done computing the ratio

$$f = \nabla\mathcal{H}(J_M)/\nabla\mathcal{H}(J_m)$$

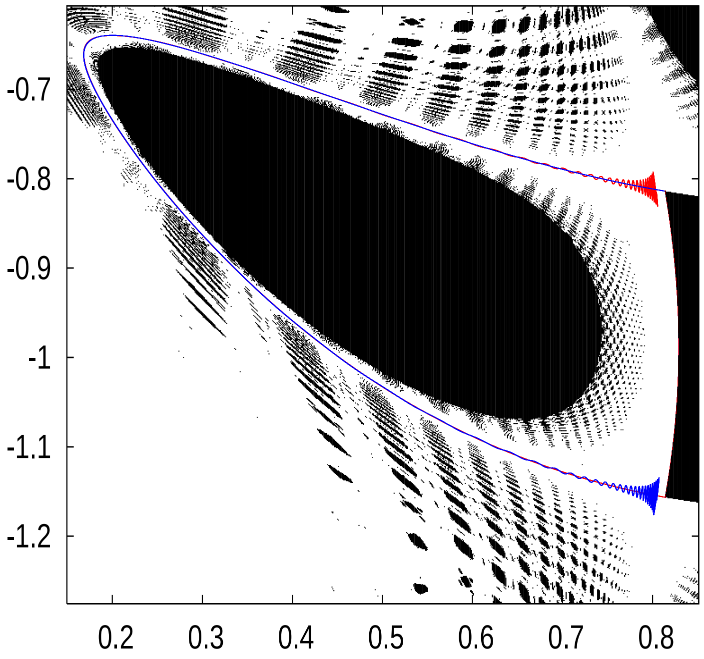
where J_M and J_m are the maximum (minimum) of the outer (inner) separatrix of the Hamiltonian. For close to the origin resonances $f = 1 + \mathcal{O}(\delta^{m/4})$.



$$\alpha = 0.21,$$
$$\sigma_- = \mathcal{O}(10^{-12}),$$
$$\sigma_+ = \mathcal{O}(10^{-3}).$$

Pendulum-like islands: comments II

The same idea applies to resonances far from the origin as well as for strong resonances but, for each case, a suitable interpolating Hamiltonian must be considered. In these cases the chaotic zone width measured in both domains can be of different order of magnitude:



$c = 1.015$,

$\sigma_+ = \mathcal{O}(10^{-54})$, $\sigma_- = \mathcal{O}(10^{-1})$.

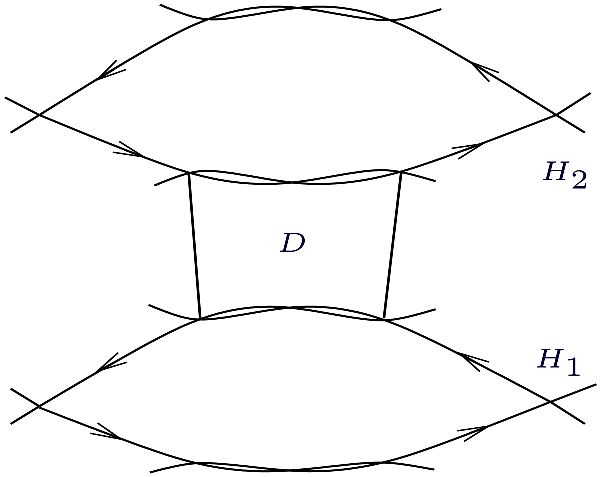
Experimentally, $f \approx -5$. Using interp. Ham. up to order $\delta \approx c - 1$ we obtain $f \approx -5.64$.

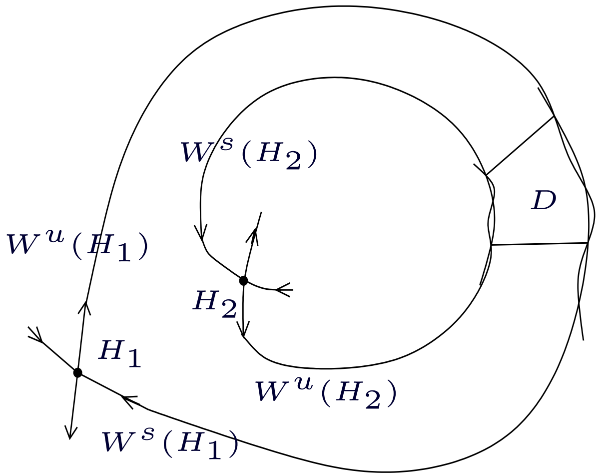
But $\delta = 0.015$ is too large. For δ small we obtain better results (even we can predict # tiny islands).

Large regions of stability

Due to the interaction of resonances large chaotic zones of instability appear. These are regions without rotational invariant curves (e.g. Birkhoff zones of instability). We have considered the **biseparatrix model** and we have studied different situations (twist and non-twist case). On the other hand, it helps to study the phenomena taking place at the border of the stability domain.

Geometrical situation:





The biseparatrix model

Between two concentric chains of islands, the simplest *qualitative* model on the domain $0 < v < d$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} u + \alpha + \beta_1 \log(v') - \beta_2 \log(d - v') \\ v + \sin(2\pi u) \end{pmatrix}.$$

where $\beta_1 = 1/\log(\lambda)$, $\beta_2 = 1/\log(\mu)$, λ being the eigenvalue of modulus greater than one of the hyperbolic point of the bottom separatrix and μ the corresponding one of the top separatrix.

- For this model it is theoretically expected to have rotational invariant curves provided $d > (\sqrt{b_1} + \sqrt{b_2})^2/k^*$ ($k^* =$ Greene's value).
- Changing - for + in the 1st. row it is a model for non-twist Birkhoff zones.
- It remains to generalise it to different order for the top/bottom resonances and to make it quantitative.

Global study of APMs.

We consider the full domain of stability around the elliptic fixed point.

$\longrightarrow$ We want: To describe the evolution of the domain of stability when changing parameters.

$\longrightarrow$ How?

- We perform numerical simulations.
- We try to explain what is observed in computations by using different theoretical frameworks.

No “new” theoretical results!!

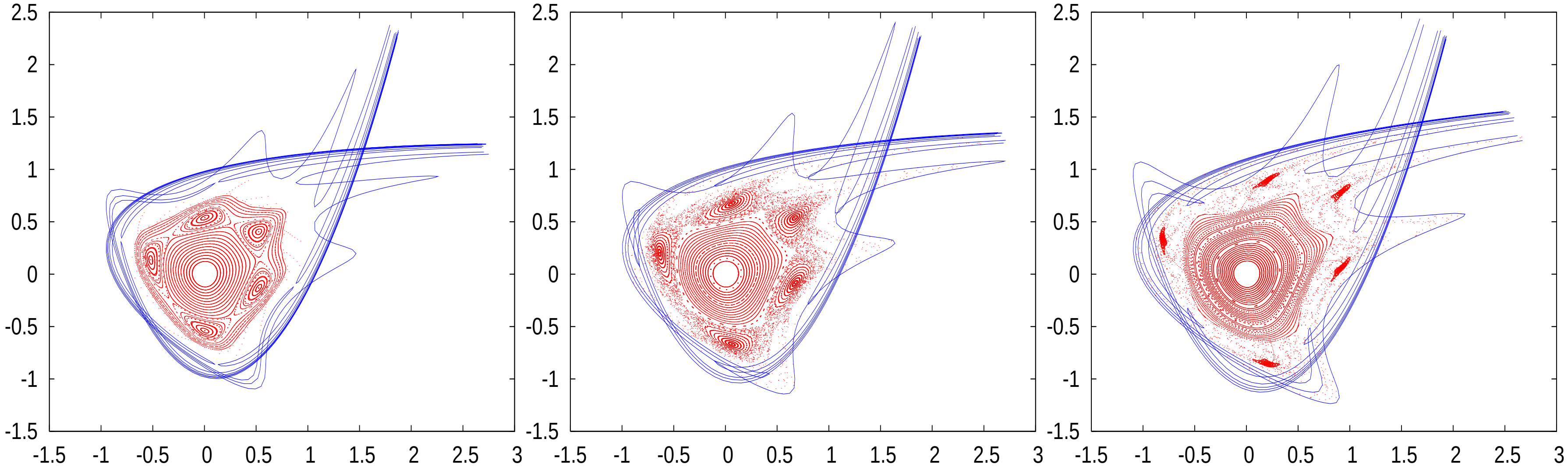
Just a description of what is observed in simulations!!

Useful to understand/predict for a large variety of APM.

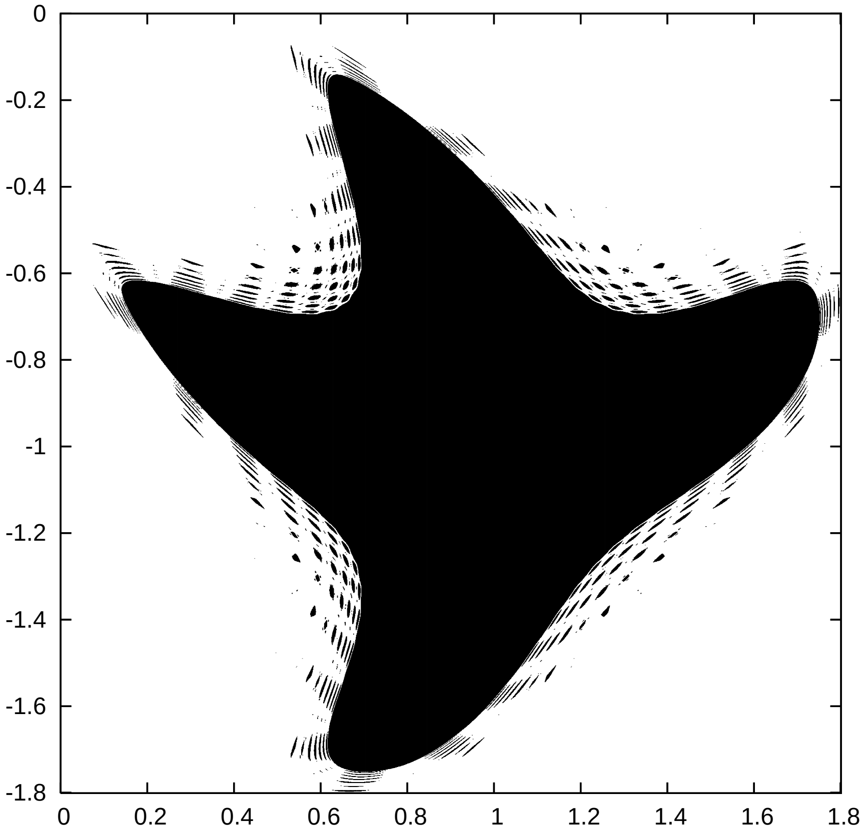
The domain of stability (DS)

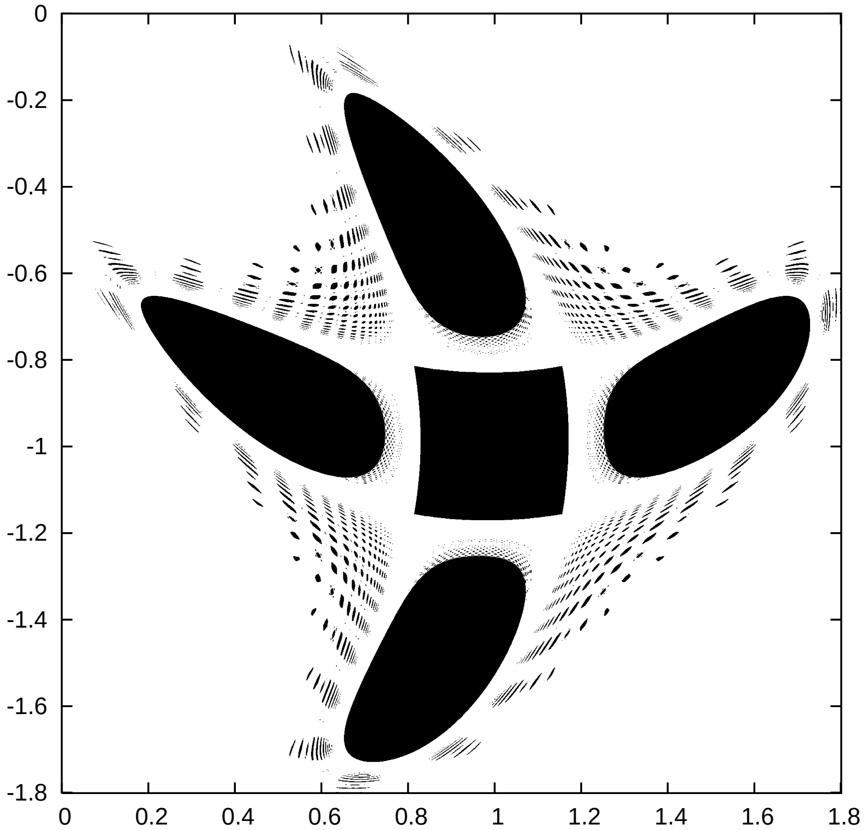
DS for APMs are related with elliptic fixed/periodic points (locally KAM thm. assures the existence of rotational inv. curves under some conditions).

For a given map $F : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$, and given a compact set $K \subset U$, the *stability domain of F relative to K* is the largest F-invariant subset of K (a chaotic orbit can be stable, in the sense that it does not escape)



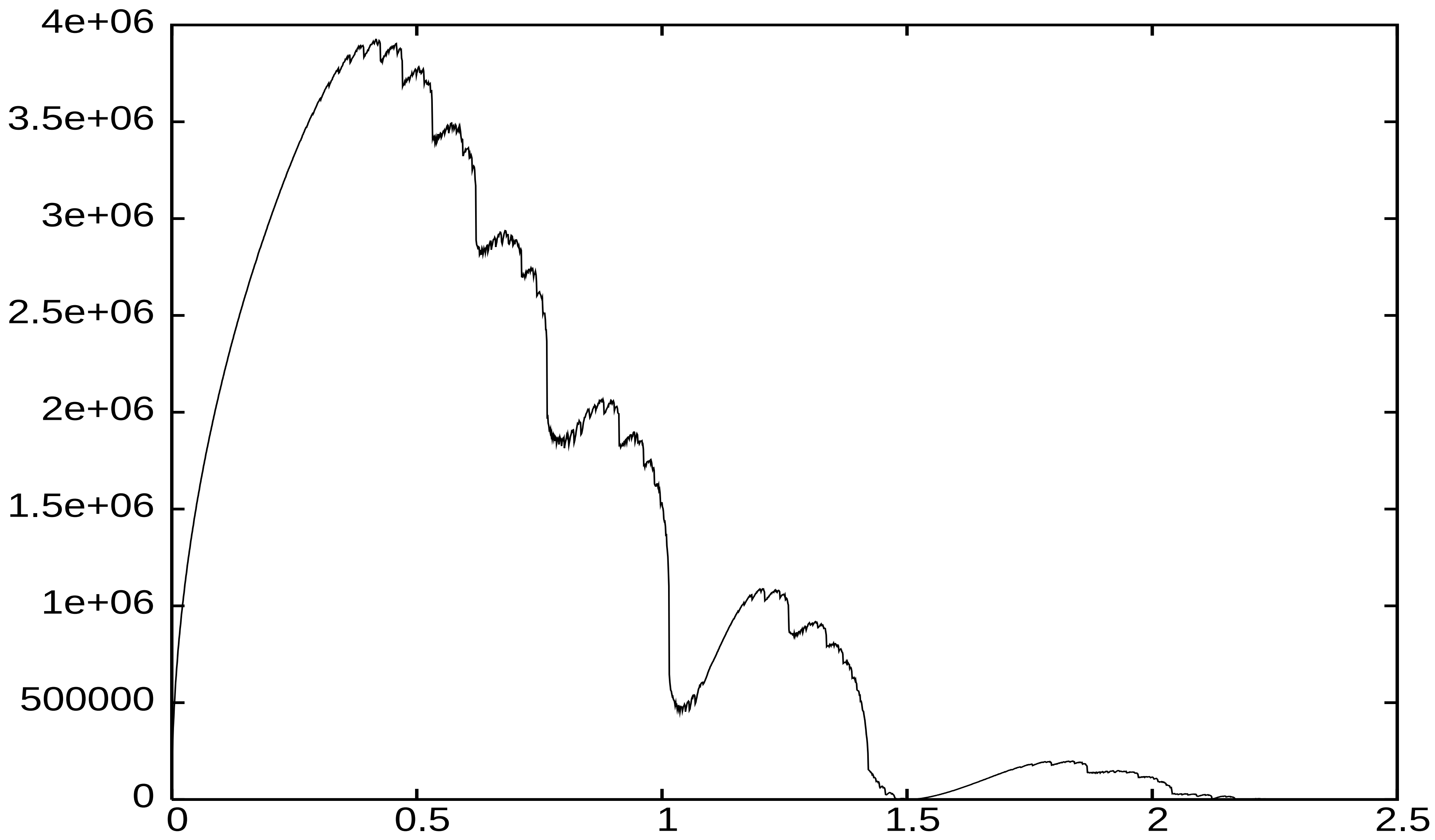
Drastic changes in the stability domain





LEFT: $c = 1.014$, RIGHT: $c = 1.015$.

The evolution of the size of the stability domain



Orbits method: $N = 3100$, $R = [-1, 2.1] \times [-2.1, 1]$, $n_{it} = 10^7$ and $\tilde{n}_{it} = 10^4$.

Weakly dissipative maps

We study the effect of dissipation on the family of maps.

Interest in: [a]

- Describe the geometrical structures and how they arise from the conservative case
- Study the probability of capture

[a] *Planar Radial Weakly-Dissipative Diffeomorphisms.* Work in progress.

The type of dissipative perturbation considered

We consider a *radially dissipative perturbation*:

$F_\delta(x, y)$ – the family of APMs s.t. $F_\delta(0) = 0$ is an elliptic fixed point

ϵ – the dissipation parameter

$\longrightarrow$ the dissipative perturbation is of the form:

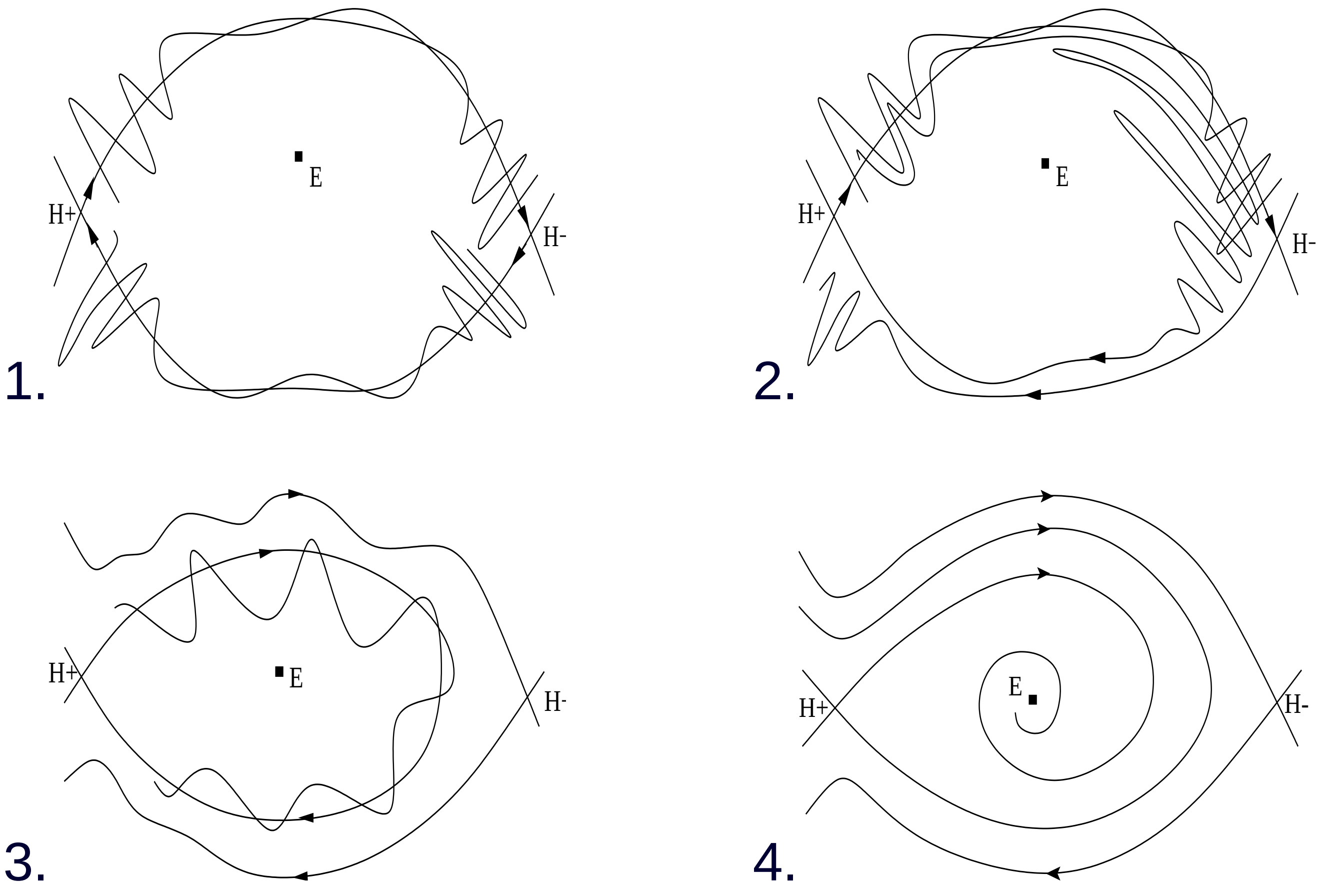
$$F_{\delta,\epsilon}(x, y) = (1 - \epsilon)F_\delta(x, y)$$

Note that with this dissipation most of the orbits will end up at the origin (which becomes a focus under dissipation). In particular, there are **no rotational invariant curves**. Nevertheless, different periodic attractors can coexist if the dissipation is small enough.

Main question: How the geometry of the conservative islands changes when adding a radial dissipative perturbation and how this changes can be related with the probability of capture into resonances?

Dissipation effect on the conservative islands

We recall that generically it is expected, for conservative resonant islands close to the origin, to have the outer splitting larger than the inner one. The effect of the dissipation gives rise to the following structures:



The 1st. critical radius

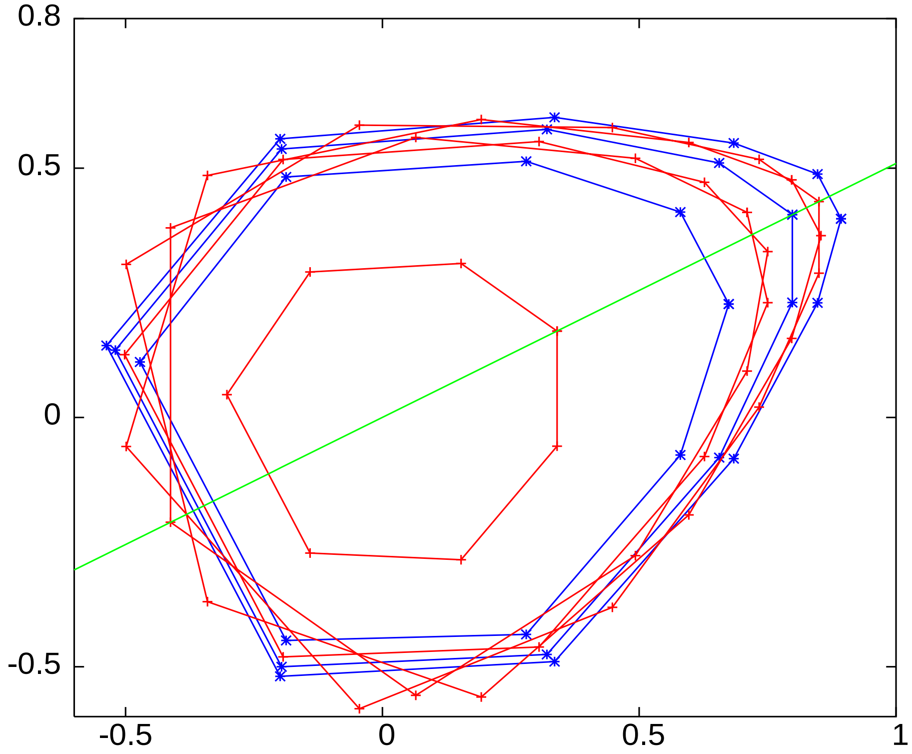
From BNF of the conservative map, adding the dissipation, it is obtained the following model describing the dynamics around the origin (for values of $\epsilon = \mathcal{O}(\delta^{m/2-1})$)

$$\begin{aligned} \dot{I} &= (2I)^{\frac{m}{2}} \sin(m\varphi) - 2\epsilon I, \\ \dot{\varphi} &= 2\pi \sum_{n=0}^{s} b_n (2I)^n + (2I)^{\frac{m}{2}-1} \cos(m\varphi). \end{aligned}$$

In particular, from this model, it is observed that for ϵ large enough (depending on δ, m and b_i) the dissipation will destroy the m-resonance.
Then, for a fixed ϵ the m-resonance is destroyed if δ is small enough. Consequently, there is a *first* critical radius r_c (depending on ϵ and on the twist properties of the map) such that all the conservative resonances inside the disk of radius r_c around the origin are destroyed by the dissipation.

1st. critical radius: illustration

$\log_{10}(\epsilon)$	**Res. destroyed**
-6	Inside $B_0(0.27)$
-4.569	(2:19)
-4.625	(1:7)
-3.456	(1:8)
-3.297	(1:9)



Resonances: (1:7), (1:8), (2:17), (1:9), (2:19) and (1:10) ($\alpha = 0.15$).

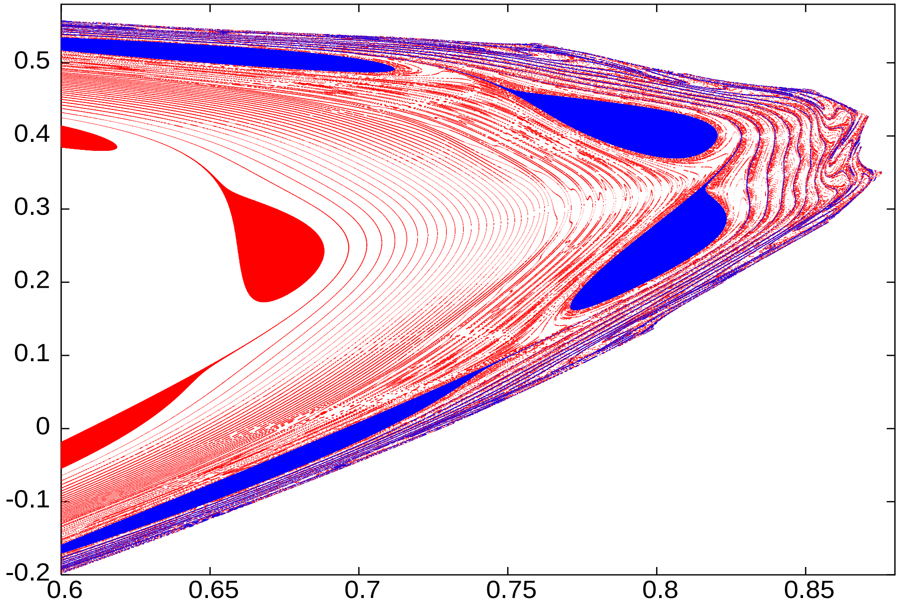
The 2nd. critical radius

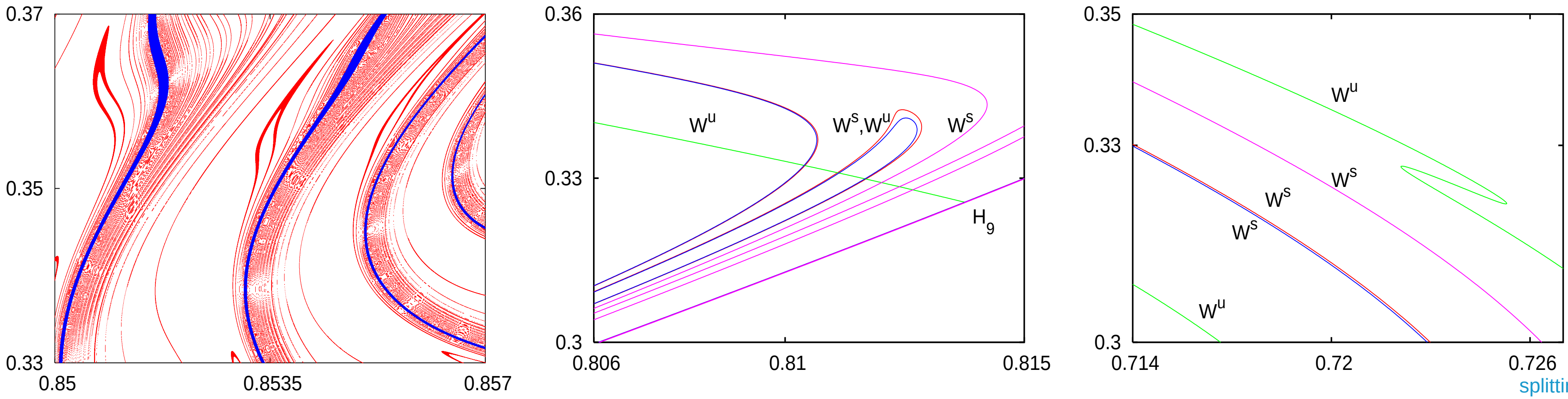
Beyond the first critical radius one expects to have resonances. But there must be an annulus where all the resonances are of flow type.
The condition of having homoclinics in the resonant dissipative structure defines a *second critical radius* r_{cc}.

- For all the resonances in the annulus $r_c < r < r_{cc}$, the dynamics of the map can be well-approximated by a flow.
- This also holds for many of the small dissipative islands surviving with $r > r_{cc}$ but not for the main ones, where a different model (an adapted return map) should be considered to analyse the dynamics.

Manifolds of different dissipative resonances

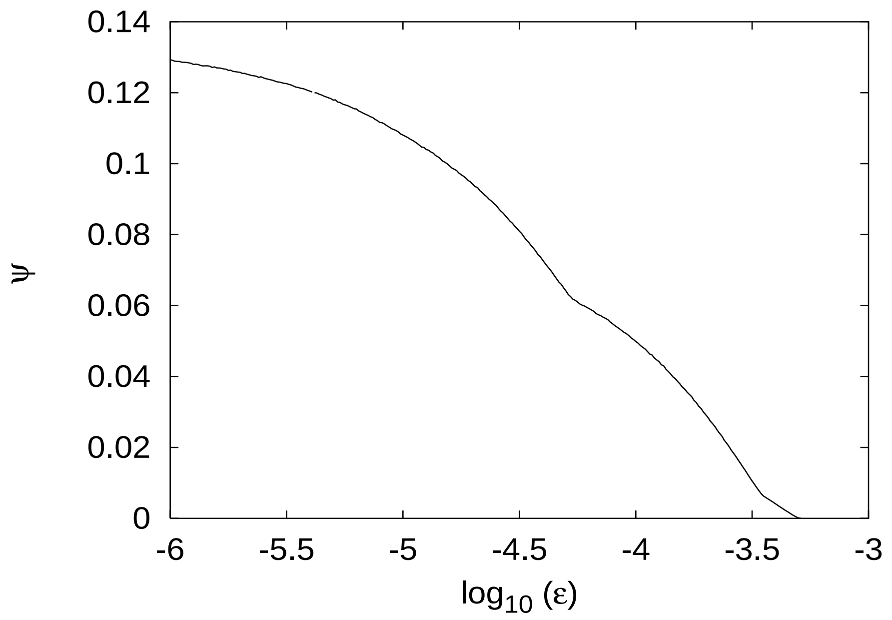
No invariant rotational curves if dissipation $\Rightarrow$ the manifolds of different resonances interact. Easiest case: both resonances of flow type (without homoclinics).





Probability of capture into resonances

We try to compute the probability of capture in resonances. In particular, our interest is on the limit probability of capture when the map approaches to the conservative case.



$\longrightarrow$ According to the previous considerations, the flow type resonances and the resonances having homoclinics must be analysed in a different way.

Flow type resonances: a suitable model

For a radially dissipative planar diffeomorphism the dynamics of the m-order resonance, $m \geq 5$, can be approximated (up to order 2 in ϵ) by the time $\gamma = \hat{\lambda}(1 + \mathcal{O}(\delta))$ map related to the flow generated by the vector field

$$X_{\hat{\epsilon}} : \begin{cases} \dot{J} = -(1 + dJ)\sin\psi - \hat{\epsilon} - k\delta^{\frac{m}{4}-1}\hat{\epsilon}J, \\ \dot{\psi} = J + cJ^2 - d\cos\psi, \end{cases}$$

where $c = \mathcal{O}(\delta^{\frac{m}{4}}),\ d = \mathcal{O}(\delta^{\frac{m}{4}-1}),\ k = \mathcal{O}(1),$ and $\hat{\epsilon} = \mathcal{O}(\delta^{1-\frac{m}{2}}\ \epsilon)$.

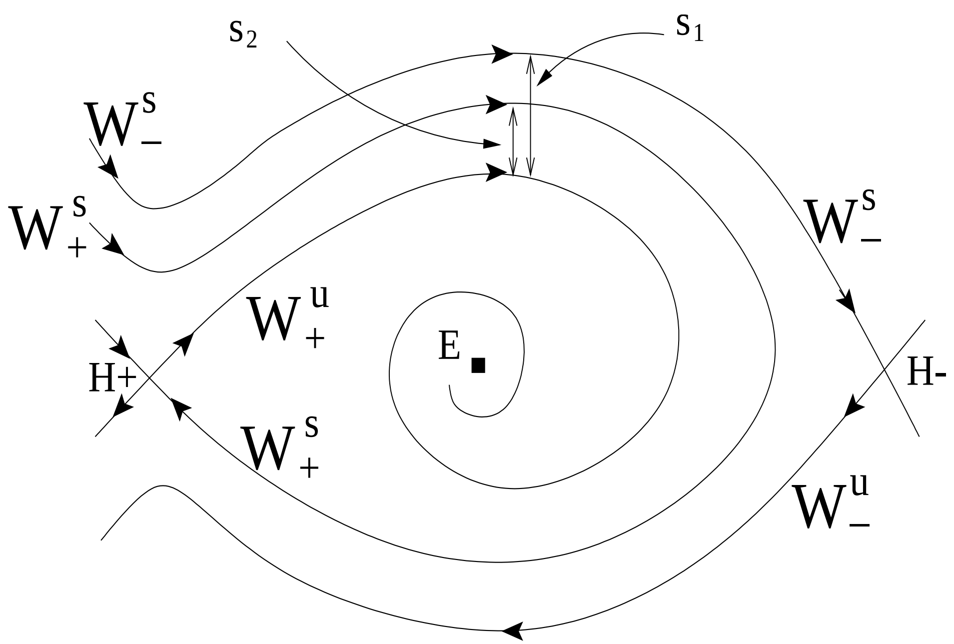
(The constants c, d, k, and $\hat{\epsilon}$ are related to the Birkhoff coefficients of the BNF $\longrightarrow$ **known quantities**)

Hamiltonian part: $\mathcal{H}_{\hat{\epsilon}}(J, \psi) = \frac{1}{2}J^2 + \frac{c}{3}J^3 - (1 + dJ)\cos\psi + \hat{\epsilon}\psi$

Flow type resonances: probability of capture

Assume that no homoclinic points appear as $\epsilon \searrow 0$ in the m-order resonance. Then, the probability of capture by the perturbed elliptic point (stable focus) of an island of the m-order resonance behaves, when ϵ goes to zero, as

$$P_{capture} = \frac{16|b_1|^{\frac{2-m}{4}}}{m\pi\sqrt{\pi}} |\delta|^{\frac{m}{4}-1} + \mathcal{O}(\delta^{1-\frac{m}{2}}\epsilon, \delta^{\frac{m}{4}}).$$



Resonances with homoclinics: a suitable model

A “dissipative” version of the separatrix map:

$$\begin{pmatrix} t \\ J \\ s \end{pmatrix} \longmapsto \begin{pmatrix} \bar{t} \\ \bar{J} \\ \bar{s} \end{pmatrix} = \begin{pmatrix} t + \omega + \beta \log |J| \pmod 1 \\ J + \nu_{\bar{s}} \sin 2\pi \bar{t} + \tilde{B}_{\bar{s}} \\ \text{sign}(J)\, s \end{pmatrix},$$

where

$\nu_{\bar{s}} = A_{\bar{s}}/A_1$, A_i amplitude of the splitting

$\omega = \beta \log(C_1 A_1/\bar{x}_1^*)$, $\beta = -1/\log(\lambda)$ and $\tilde{B}_{\bar{s}} = B_{\bar{s}}/A_1$.

Note that $\nu_1 = 1$.

Constant C_1 can be determined from the variational equations along a suitable interpolating flow (the pendulum flow).

Constant $\bar{x}_1^*$ depends on the properties of the map and it deals with the local radius around the hyperbolic point of the resonance where the BNF holds.

Res. with homoclinics: probability of capture

Under suitable assumptions, dealing with the uniform distribution of the points under iterates of the “dissipative” double separatrix model, the following holds:

- The probability of capture P_{capt} in the m-resonance strip verifies

$$\lim_{\epsilon \longrightarrow 0} P_{capt} = K(\delta).$$

- If furthermore the m-order resonance is located close enough to the origin E_0, then the constant $K(\delta)$ behaves as

$$K(\delta) \sim \delta^{\frac{m}{4}-1}.$$

$\longrightarrow$ The splitting plays no relevant role when computing a “first” approximation of the probability of capture.

The end...

We are working in 4D symplectic maps trying to generalise these studies. [a]

Thank you for your attention!!

[a] Some (numerical) preliminary results can be found in *Some properties of the global behaviour of conservative low dimensional systems*, in Foundations of Computational Mathematics: Hong Kong 2008, F. Cucker et al., editors, London Math. Soc. Lecture Notes Series **363**, Cambridge Univ. Press, 2009.