The discrete Hamiltonian-Hopf bifurcation for 4D symplectic maps

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Motivation: A paradigmatic Froeschlé-like map

Consider the map $T : (\psi_1, \psi_2, J_1, J_2) \mapsto (\bar{\psi}_1, \bar{\psi}_2, \bar{J}_1, \bar{J}_2)$ given by

$$\begin{aligned} \bar{\psi}_1 &= \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), & \bar{\psi}_2 &= \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2), \\ \bar{J}_1 &= J_1 + \delta \sin(\psi_1), & \bar{J}_2 &= J_2 + \delta \epsilon \sin(\psi_2). \end{aligned}$$

- T is related to the time-δ map of the flow associated to the Hamiltonian

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + \cos\psi_1 + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} + \epsilon \cos(\psi_2),$$

- **4 fixed points:** For $\epsilon d > 0$, $d = a_3 - a_2^2$, $|\epsilon| \ll 1$ and $\delta \lesssim 2$

  $p_1 = (0, 0, 0, 0)$ HH, $p_2 = (\pi, 0, 0, 0)$ EH, $p_3 = (0, \pi, 0, 0)$ HE, $p_4 = (\pi, \pi, 0, 0)$ EE.

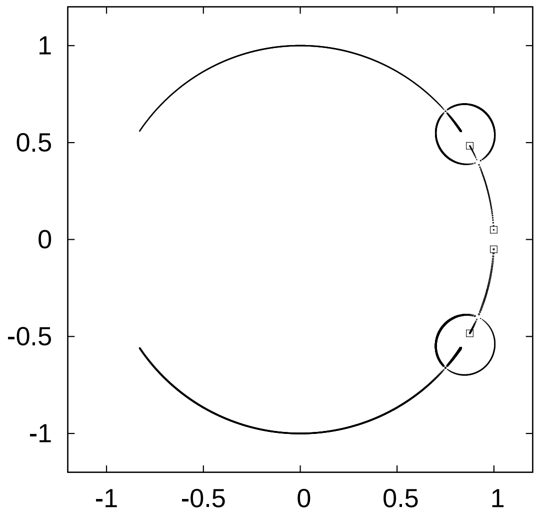
$\rightarrow T$ models the dynamics at a double resonance, it was derived from BNF around an EE point of a symplectic map in V. Gelfreich, C. Simó & AV, *Dynamics of 4D symplectic maps near a double resonance*, Phys D 243(1), 2013.

Motivation: Transition to complex unstable

$\rightarrow$ If $d>0$ (definite case) the EE point remains EE for all ϵ and δ.
$\rightarrow$ If $d<0$ (non-definite case) the point p_4 suffers a Krein collision at

$$\epsilon = \left(-(2a_3-4d) \pm \sqrt{(2a_3-4d)^2-4a_3^2}\right)/(2a_3^2),$$

and becomes a **complex-unstable** point (**Hamiltonian-Hopf bifurcation**).



Eigenvalues of $DT(p_4)$ for

$\delta = 0.5$, $a_2 = 0.5$, $a_3 = -0.75$ (hence $d = -1$)

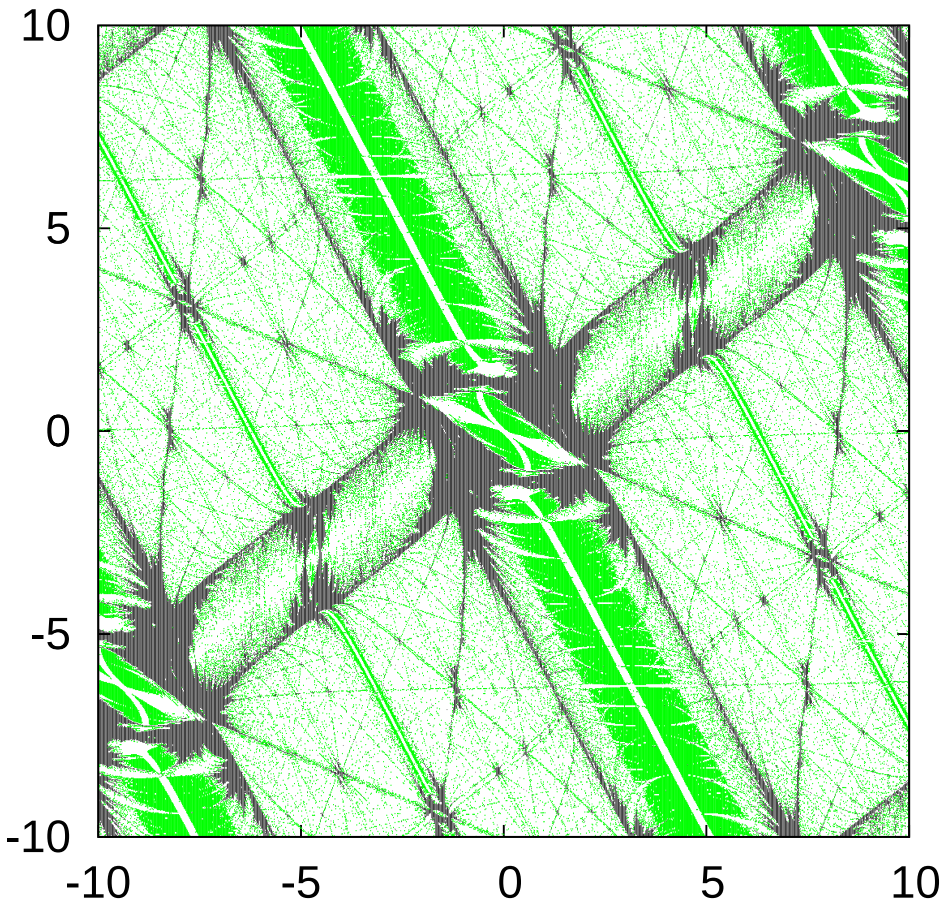
and ϵ from -0.01 (squares) to -20. The (first) Krein collision takes place at $\epsilon_c = -4/9$ at a collision angle $\hat{\theta}_0 = \arctan(\sqrt{23}/11) \in \mathbb{R}\setminus\mathbb{Q}$.

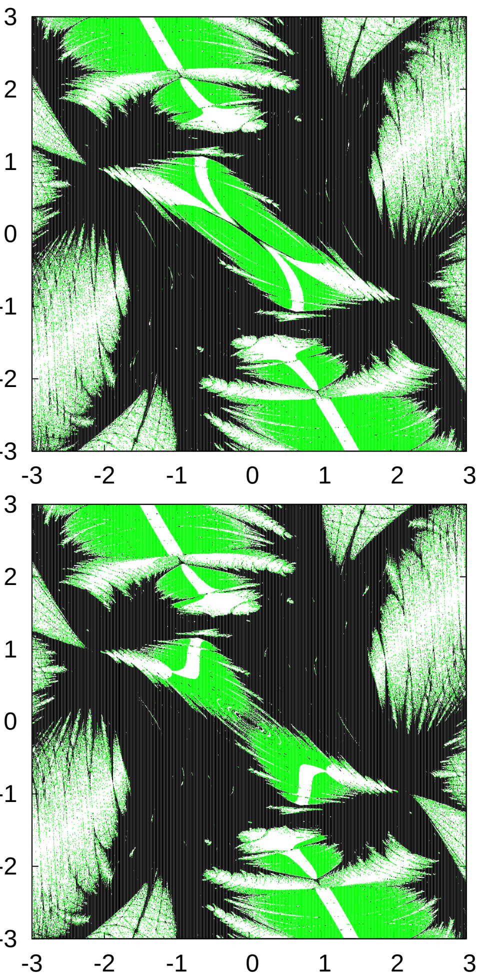
$\rightarrow$ The CS point has 2D stable/unstable invariant manifolds. $\rightarrow$ Next plots show their role!

$\rightarrow$ The previous considerations also hold for H: the eigenvalues collide at the imaginary axis and the 2-dof analogous Hamiltonian-Hopf bifurcation takes place. **Later:** differences discrete/continuous cases in the splitting of the 2D inv. manifolds.

Motivation: Dynamical consequences

Lyapunov exp. MEGNO, i.c. on $\psi_1 = \psi_2 = 0$: white $\rightarrow$ regular, green $\rightarrow$ mild chaos, black $\rightarrow$ chaos.





Left: $\epsilon = -0.4$. Right: top $\epsilon = -0.44$, bottom: $\epsilon = -0.45$. (Rec: $\epsilon_c = -4/9$.)

$\rightarrow$ Lyapunov inv. curves families, local character of the bifurcation, evolution to global connection,...

Goal of this work

We want...

1. Analysis of the Hamiltonian-Hopf bifurcation for 4D maps.
2. Geometry of the 2D invariant manifolds: behaviour of the splitting for the 4D map.

→ But, **previously,** we review the 2-d.o.f. analogous Hamiltonian-Hopf case.

1. Sokolskii NF.
2. Splitting of the invariant manifolds: Reduction to a 2D near-the identity area-preserving map.

→ **Important:** How are both cases related?

1. Main idea: Takens NF + interpolating Hamiltonian
2. Differences in the behaviour of the splitting: energy function

2-dof Hamiltonian Hopf (HH): Sokolskii NF

2-dof HH codim 1: Consider a 1-param. family of 2-dof Hamiltonians H_ν undergoing a HH bifurcation (at the origin).

Concretely: for $\nu > 0$ elliptic-elliptic, $\nu < 0$ complex-saddle.

Analysis of the HH bifurcation $\longrightarrow$ Reduction to **Sokolskii NF**:

1. Taylor expansion at $\mathbf{0}$: $H_\nu = \sum_{k\geq 2} \sum_{j\geq 0} \nu^j H_{k,j}$, where $H_{k,j} \in \mathbb{P}_k$ homogeneous of order k.
2. Williamson NF (double purely imaginary eigenvalues):
   $H_{2,0} = -\omega(x_2 y_1 - x_1 y_2) + \frac{1}{2}(x_1^2 + x_2^2)$.
3. Use Lie series to order-by-order simplify $H_{2,j}, j > 1$ and $H_{k,j}, k > 2, j > 0$.
   But: **non-semisimple** linear part!
   Then, at each order (k, j), one looks for $G \in \mathbb{P}_k$ s.t.

$$H_{k,j} + \mathrm{ad}_{H_2}(G) \in \mathrm{Ker}\, \mathrm{ad}_{H_2}^\top .$$

2-dof HH: Sokolskii NF

4. Introducing the Sokolskii coordinates ($dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dR \wedge dr + d\Theta \wedge d\theta$)

$$y_1 = r\cos(\theta),\ y_2 = r\sin(\theta),\ R = (x_1 y_1 + x_2 y_2)/r,\ \Theta = x_2 y_1 - x_1 y_2,$$

one has $H_2^\top = -\omega\Theta + \frac{1}{2}r^2$ and

$$\text{NF}(H_\nu) = -\omega\Gamma_1 + \Gamma_2 + \sum_{\substack{k,l,j\geq 0\\ k+l\geq 2}} a_{k,l,j}\,\Gamma_1^k\,\Gamma_3^l\,\nu^j.$$

where

$$\Gamma_1 = x_2 y_1 - x_1 y_2,\ \Gamma_2 = (x_1^2 + x_2^2)/2 \text{ and } \Gamma_3 = (y_1^2 + y_2^2)/2.$$

5. Introducing $\nu = -\delta^2$, and rescaling $x_i = \delta^2 \tilde{x}_i$, $\omega y_i = \delta\,\tilde{y}_i$, $i = 1, 2$, $\omega t = \tilde{t}$, one has

$$\boxed{\text{NF}(\tilde{H}_{\delta_t}) = -\tilde{\Gamma}_1 + \delta\left(\tilde{\Gamma}_2 + a\tilde{\Gamma}_3 + \eta\tilde{\Gamma}_3^2\right) + \mathcal{O}(\delta^2).}$$

The $\tilde{\Gamma}_i$ written in terms of the Sokolskii coordinates are given by

$$\tilde{\Gamma}_1 = \Theta,\ \tilde{\Gamma}_2 = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right),\ \text{and } \tilde{\Gamma}_3 = \frac{r^2}{2}.$$

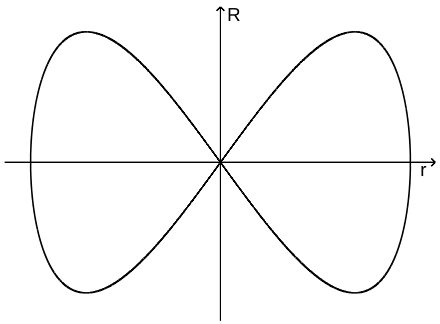
2-dof HH: invariant manifolds

For $\nu < 0$ the origin has stable/unstable inv. manifolds $W^{s/u}(\mathbf{0})$. Note that

- $W^{s/u}(\mathbf{0})$ are contained in the zero energy level of NF$(\tilde{H}_\delta)$.
- $\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\} = \{\tilde{\Gamma}_1, \tilde{\Gamma}_3\} = 0 \Rightarrow \tilde{\Gamma}_1$ is a formal first integral of NF$(\tilde{H}_\delta)$. Hence $\tilde{\Gamma}_1 = 0$ on $W^{s/u}(\mathbf{0})$.

Then, **ignoring** $\mathcal{O}(\delta^2)$ terms, $W^{s/u}(\mathbf{0})$ are given by $R^2 + ar^2 + \eta r^4/2 = 0$, which is the zero energy level of a **Duffing Hamiltonian system**.

$\Rightarrow$ $W^{u/s}(\mathbf{0})|_{(R,r)\text{-plane}}$ form a **figure-eight** (for $a<0$, $\eta>0$; unbounded otherwise! but only $r>0$ has sense!).



The 2D $W^{s/u}(\mathbf{0})$ are rotated around the origin (on $W^{s/u}(\mathbf{0})$ one has $\Theta=0$, $\dot{\theta}=1$).

For the truncated NF (i.e. ignoring $\mathcal{O}(\delta^p)$-terms, $p > 1$) the 2D stable/unstable inv. manifolds coincide. **But:** For the complete 2-dof Hamiltonian **they split!**

2-dof HH: splitting of inv. manifolds

The **asymptotic expansion** of this splitting has been obtained in
J.P.Gaivao, V.Gelfreich, *Splitting of separatrices for the Hamiltonian-Hopf bifurcation with the Swift-Hohenberg equation as an example*, Nonlinearity 24(3), 2011.

$$\alpha \sim A\delta^B \exp\left(\frac{-\pi}{\sqrt{-a\delta}}\right) \sim A|\mathrm{Re}\,\lambda|^B \exp\left(\frac{-\pi\,|\mathrm{Im}\,\lambda|}{|\mathrm{Re}\,\lambda|}\right)$$

Main idea: The exponential part of this formula can be obtained by reducing to a near Id family of analytic APMs + Fontich-Simó thm. (upper bounds are generic!).

Consider $\Sigma = \{\theta = 0\}$ (but in Cartesian coord. to avoid singularities) and

$T_\delta : \Sigma \to \Sigma$ (Poincaré map of the full 2-dof Hamiltonian) $\rightsquigarrow$ separatrices split,

$T_\delta^0 : \Sigma \to \Sigma$ (Poincaré map of the truncated 2-dof Hamiltonian, ignoring $\mathcal{O}(\delta^2)$) $\rightsquigarrow$ Homoclinic loop.

Then, $T_\delta^0(R, r, \Theta, \theta) = (\phi_{2\pi}^X, \Theta, \theta \bmod 2\pi)$, being X the vector field

$$\dot{R} = \delta\left(ar + \eta r^3\right), \quad \dot{r} = -\delta R, \qquad \longleftarrow \text{Duffing!}$$

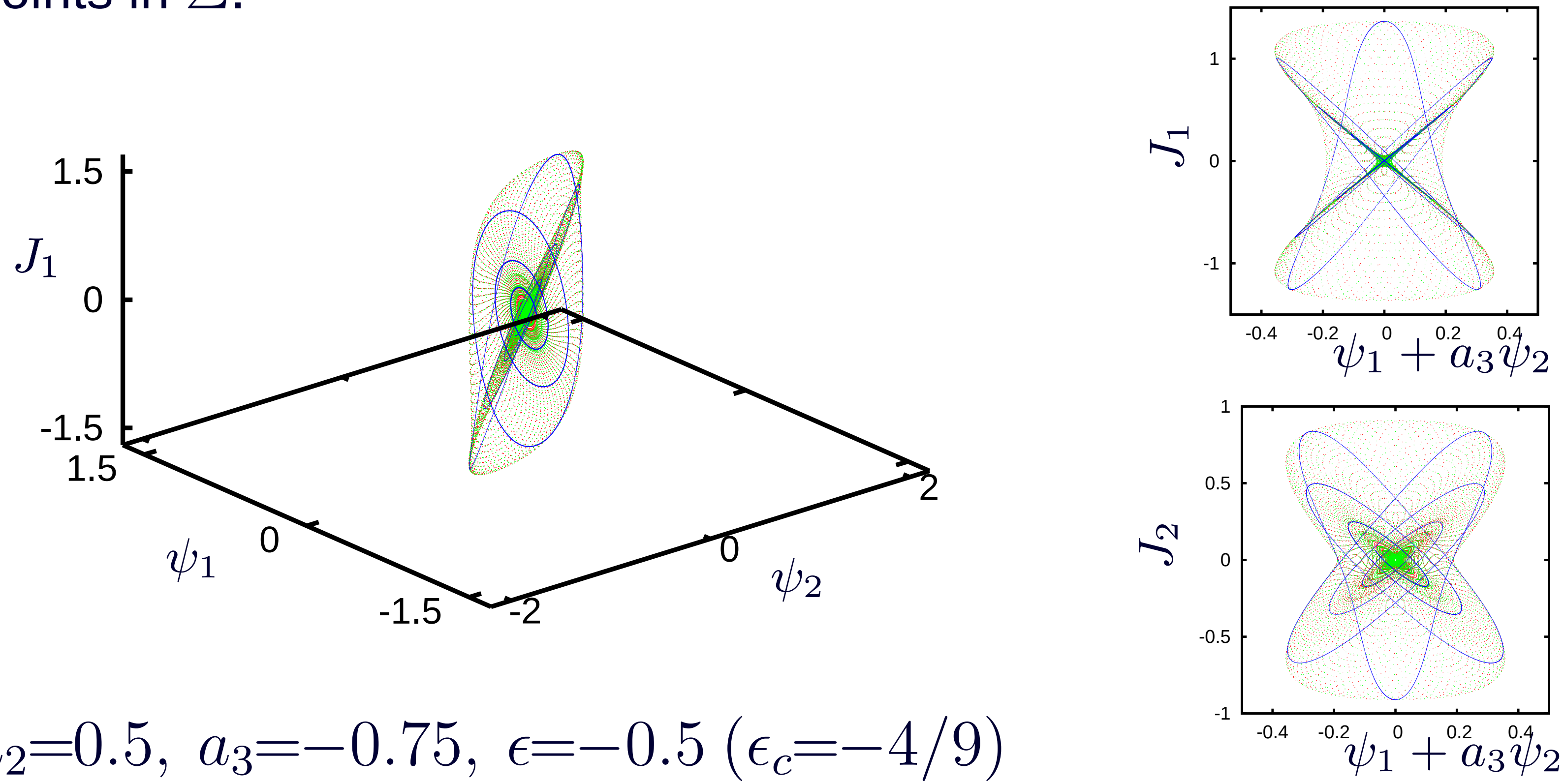
which has a homoclinic solution $\gamma(t)$ with nearest singularity to the real axis $\tau = i\pi/2\sqrt{-a}\delta$ and dominant eigenvalue $\mu = 2\pi\sqrt{-a}\delta$ (then rescale time by $\sqrt{-a}\delta$). But $T_\delta^0 = (\hat{T}_\delta^0)^2$, being $\hat{T}_\delta^0$ close to -Id $\Rightarrow$ use $\mu/2$ instead of μ in the exponential part of the upper bound $C\exp\left(-2\pi(\mathrm{Im}\,\tau - \eta)/\mu\right)$, η small fixed quantity.

2-dof HH: the example

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$$

Reversibility: $(\psi_1, \psi_2, J_1, J_2) \in W^u(\mathbf{0})$ then $(-\psi_1, -\psi_2, J_1, J_2) \in W^s(\mathbf{0})$.

This suggests to consider $\Sigma = \{\psi_1 = 0,\, \psi_2 = 0\}$ and to look for homoclinic points in Σ.



$a_2 = 0.5,\ a_3 = -0.75,\ \epsilon = -0.5\ (\epsilon_c = -4/9)$

2-dof HH: Computing numerically the splitting

$\rightarrow$ One can represent W^u as a series $\mathcal{G}(s_1, s_2)$, where s_1, s_2 are local parameters in a fundamental domain (parameterisation method) and propagate the manifolds (e.g. using Taylor integrator).

$\rightarrow$ Then, one can compute (s_1^h, s_2^h) giving a homoclinic point p_h on Σ (at the first intersection!). $\rightsquigarrow$ The homoclinic orbit was shown in the last plot!

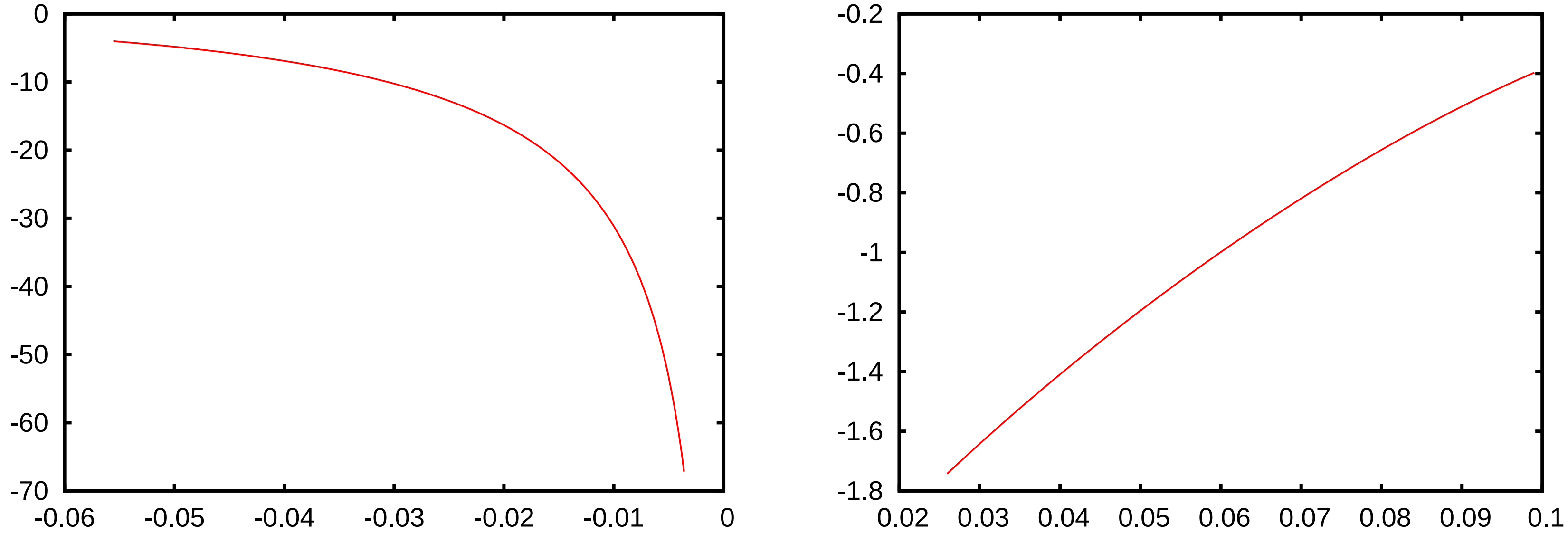
To measure the splitting angle α:

1. **Compute a basis of** $T_{X_h^0}(W^u_{loc}(\mathbf{0})) \rightsquigarrow v_t^0 = \frac{\partial G}{\partial s_1}(s_1^h, s_2^h)$, $v^0_{\texttt{vf}}$
2. **Transport the vectors to** $\Sigma \rightsquigarrow v_t^\Sigma$, $v^\Sigma_{\texttt{vf}}$ (integrating variational eqs.) These vectors form a basis of $T_{p_h}(W^u(\mathbf{0}))$.
3. **Compute an orthogonal basis of** $T_{p_h}(W^u(\mathbf{0})) \rightsquigarrow u_1$, $v^\Sigma_{\texttt{vf}}$
4. **Compute the splitting angle.** By reversibility, from $u_1 \in T_{p_h}(W^u(\mathbf{0}))$ we obtain a vector $u_2 \in T_{p_h}(W^s(\mathbf{0}))$. Then,

$$\alpha = \text{angle}(u_1, u_2)$$

2-dof HH: Checking the behaviour of α

$$H(\psi_1, \psi_2, J_1, J_2) = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$$



Left: $\log(\alpha)$ vs. $\epsilon - \epsilon_c$. Right: Re$(\lambda)\log(\alpha)$ vs. Re(λ).

Recall: $\alpha \sim \tilde{A}(|\text{Re}\ \lambda|)^B \exp\left(\frac{C}{|\text{Re}\ \lambda|}\right)$, where $C = -\pi\,|\text{Im}\ \lambda|$.

For $a_2 = 0.5$, $a_3 = -0.75$ one gets $C = -\sqrt{2}\pi/3 + \mathcal{O}(\nu)$ (Sokolskii NF).

Fitting function (right plot): $f(x) = A\,x + B\,x\,\log(x) + C$.

$\rightsquigarrow$ It perfectly fits the behaviour!

Up to this point: **2-dof Hamiltonian-Hopf** bifurcation.

1. Everything was "more or less" well-known: Sokolskii NF, geometry of the invariant manifolds, the splitting α,...
2. α behaves as expected for a near-the-identity family of 2D APM.

Guiding example: $H = \frac{J_1^2}{2} + a_2 J_1 J_2 + a_3 \frac{J_2^2}{2} - \cos(\psi_1) - \epsilon \cos(\psi_2)$.

Now: 4D discrete Hamiltonian-Hopf!

Guiding example: the 4D symplectic map T given by

$$\begin{aligned} \bar{\psi}_1 &= \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), & \bar{\psi}_2 &= \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2), \\ \bar{J}_1 &= J_1 + \delta \sin(\psi_1), & \bar{J}_2 &= J_2 + \delta\epsilon \sin(\psi_2). \end{aligned}$$

The origin undergoes a HH bif. and 2D stable/unstable manifolds are born.

Question: Behaviour of the splitting of the 2D inv. manifolds?

Planning:

First: Numerical exploration of T.

$\longrightarrow$ Computation of the invariant manifolds.

$\longrightarrow$ Behaviour of the splitting.

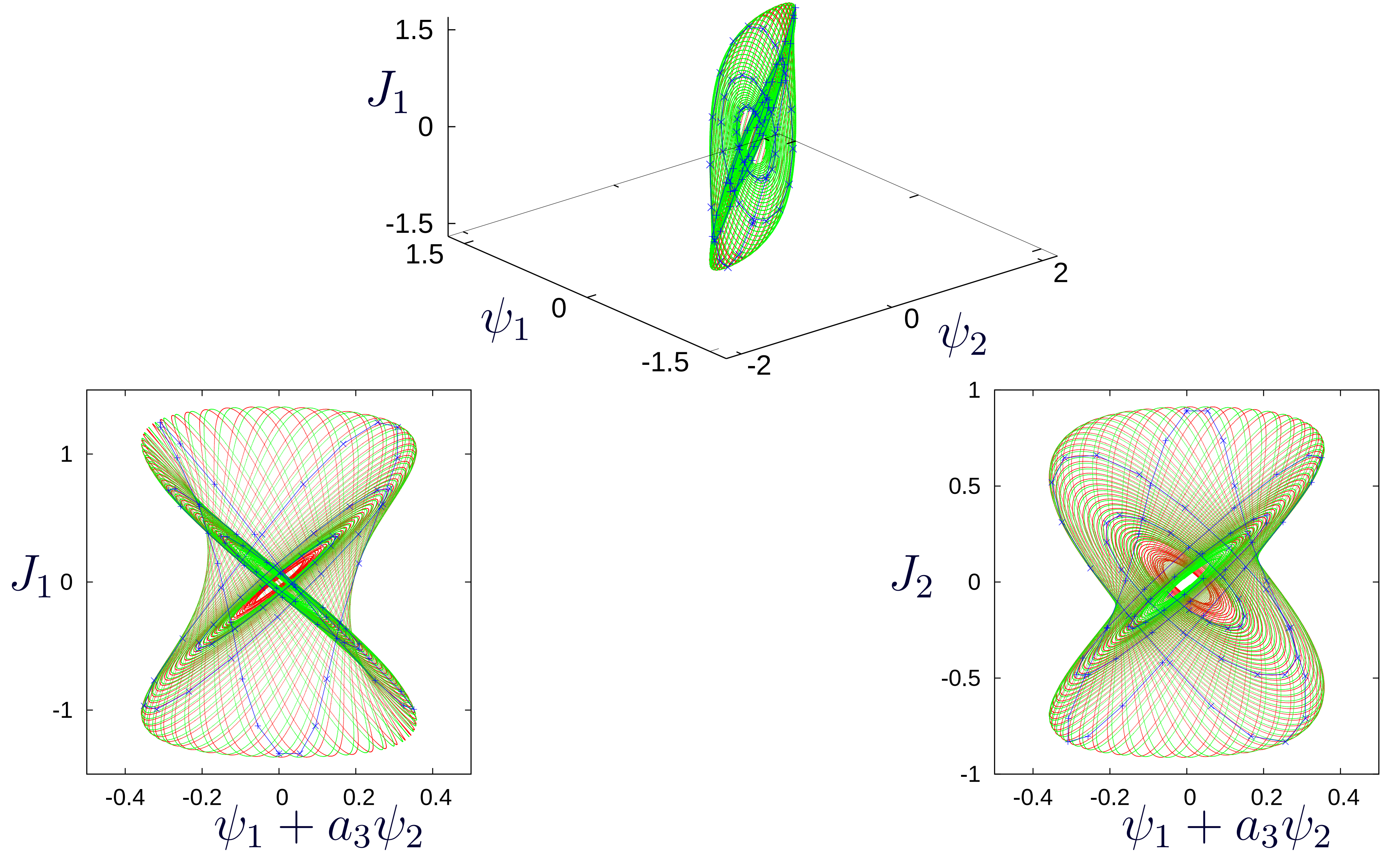
$\longrightarrow$ A naive justification of the behaviour observed.

After: General theoretical results on splitting of inv. manifolds for the 4D HH.

$\longrightarrow$ Upper bounds from a suitable energy function.

T: Invariant manifolds

One can compute $W^{u/s}(\mathbf{0})$ and $p_h \in \Sigma = \{\psi_1 = \psi_2 = 0\}$ (similarly to the 2-dof case).



T: Splitting volume V

We compute the volume of a $4D$ parallelotope defined by two pairs of vectors tangent to W^u and W^s:

$G(s_1, s_2)$ - the (local) parameterisation.

1. Consider the vectors (tangent to W^u):

$$\tilde{v}_1 = (\partial G/\partial s_1)(s_1^h, s_2^h), \quad \tilde{v}_2 = (\partial G/\partial s_2)(s_1^h, s_2^h).$$

2. Transport these vectors under T to p_h and consider, by the reversibility,

$$\tilde{v}_3 = R(\tilde{v}_1^{p_h}), \quad \tilde{v}_4 = R(\tilde{v}_2^{p_h}).$$

3. Finally, normalize them $v_j = \tilde{v}_j^{p_h}/\|\tilde{v}_j^{p_h}\|,\ j = 1, \ldots, 4$ and define

$$\boxed{V = \text{det}(v_1, v_2, v_3, v_4)}$$

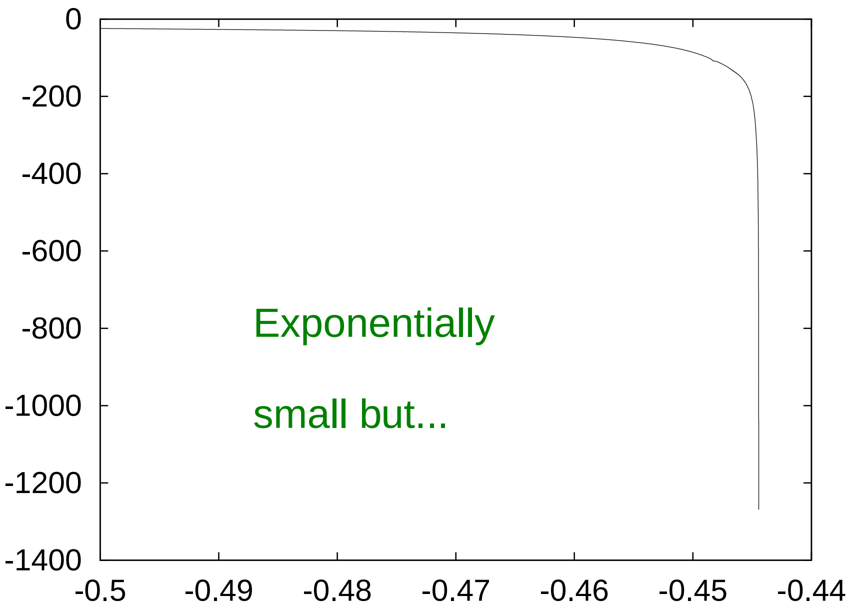
Question: How does V behaves as $\epsilon \to \epsilon_c$?

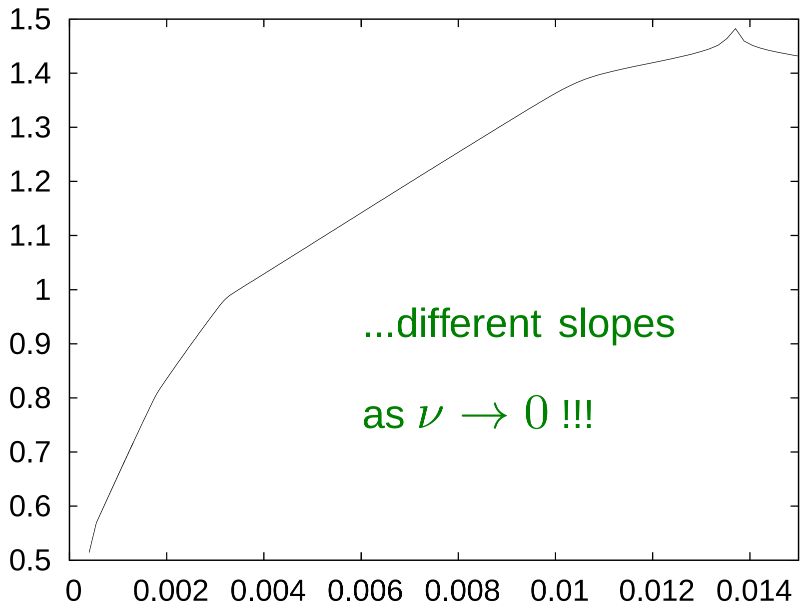
T: Behaviour of V

$$T: \quad \bar{\psi}_1 = \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2),$$
$$\bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta\epsilon \sin(\psi_2).$$

Fixed a_2, a_3 one has $\epsilon_c = \epsilon_c(a_2, a_3)$. The (Krein) collision angle θ_0 depends on δ.

$a_2 = 0.5$, $a_3 = -0.75 \rightsquigarrow \epsilon_c = -4/9$. $\delta = 0.5 \rightsquigarrow \theta_0 = \arctan(\sqrt{23}/11)/2\pi \in \mathbb{R} \setminus \mathbb{Q}$.





Left: $\log V$ vs. ϵ. Right: $h|\log(V)|$ vs. h ($h = \log(\lambda)$).

Naive explanation of the behaviour of V

Consider a (generic) symplectic map F in $\mathbb{R}^4$ undergoing a HH bif.

Discrete HH bif. $\rightsquigarrow$ codim 2 bif $\rightsquigarrow$ Let δ_u, ϵ_u be the unfolding parameters.

δ_u: Collision angle $\hat{\theta}_0 = 2\pi(q/m + \delta_u)$.

ϵ_u: Measures the relative distance to the critical parameter.

Different (naive) important aspects:

1. "Two" exp. small effects: one within the Hamiltonian itself (already studied!), the other measures the "map-Hamiltonian distance".
2. "Two" frequencies: "Duffing" and its $2\pi\theta$-perturbation + **"time" frequency**.
3. The Hamiltonian part is known $\Rightarrow$ only necessary to measure the second effect. **But:** We have a "privilegiated direction" (the time!) $\Rightarrow$ we will use an **energy function** to measure the splitting in that direction (instead of using the splitting potential or the Melnikov vector which measures both effects together).

Towards a sharp upper bound of the splitting (I)

Idea: It is enough to measure the "Hamiltonian-map distance".

Let F_ϵ be a family of symplectic maps s.t. at $\epsilon = 0$ undergoes a HH bifurcation. The inv. manifolds are given by $u(\alpha, t)$ and $s(\alpha, t)$, $(\alpha, t) \in [t_0, t_0 + h) \times S^1$. This defines fundamental domains $\mathcal{D}^{u/s}$.

Main result: Assume that

(H1) There exists an energy function E, i.e. such that $E \circ F_\epsilon = E$, defined in a neighbourhood of the fundamental domain $\mathcal{D}^s$ such that $E(s(\alpha, t)) = 0$. Moreover we assume that E and $s(\alpha, t)$ can be analytically extended to a neighbourhood of $W^u(\mathbf{0})$ within $\mathcal{D}^u$ (by iteration of F_ϵ^{-1}).

We define the *splitting function*:

$$\psi(\alpha, t) = E(u(\alpha, t))$$

Towards a sharp upper bound (II)

(H2) There is a (limit) vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2,$$

such that f is analytic, it possesses a hyperbolic saddle fixed point and a homoclinic orbit $\sigma(t)$ associated to it, and satisfies that compact pieces of the real invariant manifolds of F_ϵ are ϵ-close to an embedding of $\mathbb{S}^1 \times \{\sigma(t),\ t \in \mathbb{R}\}$ for $\epsilon > 0$ small enough.

(H3) F_ϵ can be extended analytically to a neighbourhood of

$$\{\alpha \in \mathbb{C}/2\pi\mathbb{Z}, |\text{Im}\,\alpha| < \rho\} \times \{\sigma(t), \text{Re}\,t \in \mathbb{R}, |\text{Im}\,t| < \tau\}$$

for some $0 < \tau < \tau_0$ and $0 < \rho < \rho_0$.

Towards a sharp upper bound (Result)

Under (H1), (H2) and (H3)...

(i) **Rational Krein collision.** Let $\theta_0 = p/q$, with $(p, q) = 1$. Then, there exists $\epsilon_0 > 0$ s.t. for $\epsilon < \epsilon_0$

$$|\psi(\alpha, t)| \leq K \exp(-C/h), \qquad C, K > 0.$$

(ii) **Irrational Krein collision.** Let $\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$. Then, ψ is bounded by a function that is exponentially small in a parameter γ, s.t. $\gamma \searrow 0$ when $h \searrow 0$. Moreover, the dominant harmonic $k(h)$ of ψ changes infinitely many times as $h \to 0$.

Idea: Bounding the Fourier coefficients of ψ, one gets

$$|\psi(\alpha, t)| < K \sum_{(k,n)\in\mathbb{Z}^2_*} \exp(\underbrace{-2\pi|n - \theta_0 k|\tau/h - |k|\rho}_{\beta_k}).$$

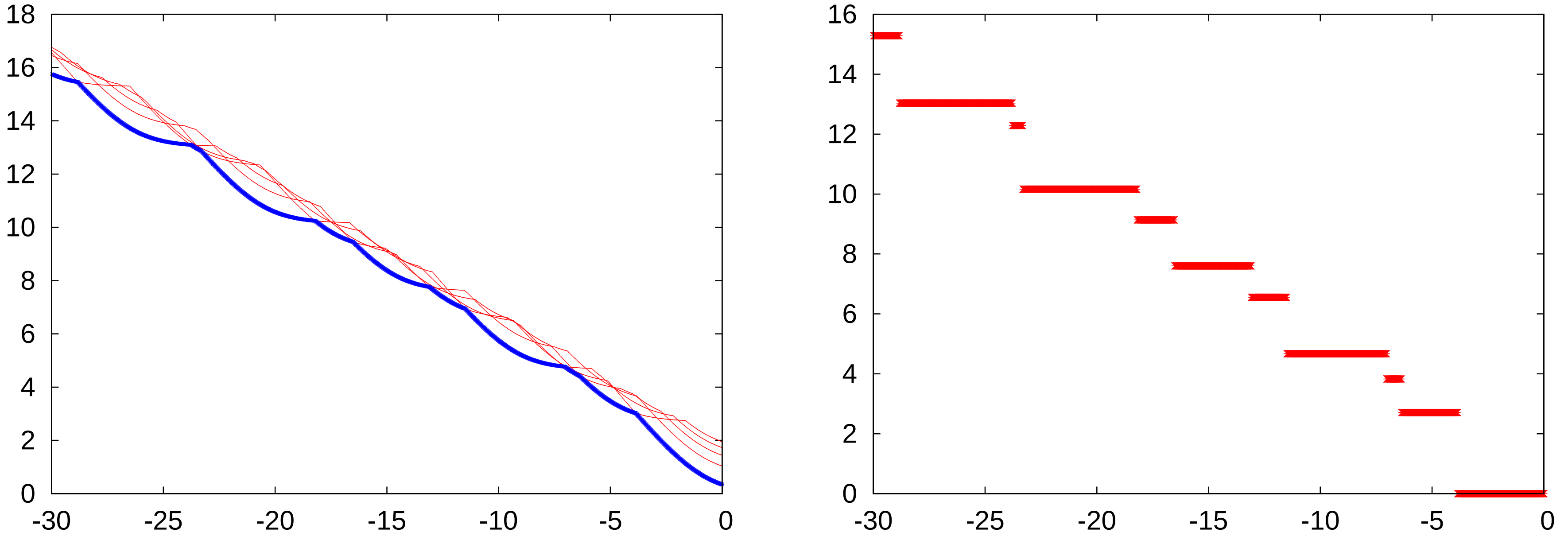
Then we look for $k = k_*(h) > 0$ s.t. the dominant coefficient β_{k_*} in the exponential bound is minimum (different cases according to the properties of θ_0).

Map T: fit of the volume V (I)

$$T: \quad \bar{\psi}_1 = \psi_1 + \delta(\bar{J}_1 + a_2 \bar{J}_2), \quad \bar{\psi}_2 = \psi_2 + \delta(a_2 \bar{J}_1 + a_3 \bar{J}_2),$$
$$\bar{J}_1 = J_1 + \delta \sin(\psi_1), \quad \bar{J}_2 = J_2 + \delta \epsilon \sin(\psi_2).$$

$a_2 = 0.5$, $a_3 = -0.75 \rightsquigarrow \epsilon_c = -4/9$.

We look for the dominant coefficients $\beta_{k(h)}$. They depend on θ_0 and $h = \log(\lambda) = \mathcal{O}(\sqrt{|\epsilon - \epsilon_c|})$. We fix $\theta_0 = \arctan(\sqrt{23}/11)/2\pi$.



Left: first five dominant exponents β_k as a function of h. Right: values of k_* corresponding to the minimum exponent β_k. Both in $\log - \log$ scale.

Map T: fit of the volume V (II)

- We have $k_* = 1, 15, 46, 107, 703, 2002, 9307, 25919,$ ... as $h \to 0$.
- The values of k_* are related to the approximants of $\theta_0 \approx 0.0654346230 8$:
  $1/15,\ 3/46,\ 4/61,\ 7/107,\ 39/596,\ 46/703,\ 85/1299,\ 131/2002, \ldots$
- Not all the approximants produce a change of $k_*(h)$ as $h \to 0$, only those that are smaller than θ_0 play a role (except the first one $1/15 > \theta_0$).
- The lenght of the interval in h where $k_*(h)$ dominates depends on the CFE$(\theta_0) = [15,\ 3,\ 1,\ 1,\ 5,\ 1,\ 1,\ 3,\ 1,\ 2, \ldots]$, but also on the constants in front of the exponential terms of V (terms with larger β_k can dominate for finite $h > 0$!!!)

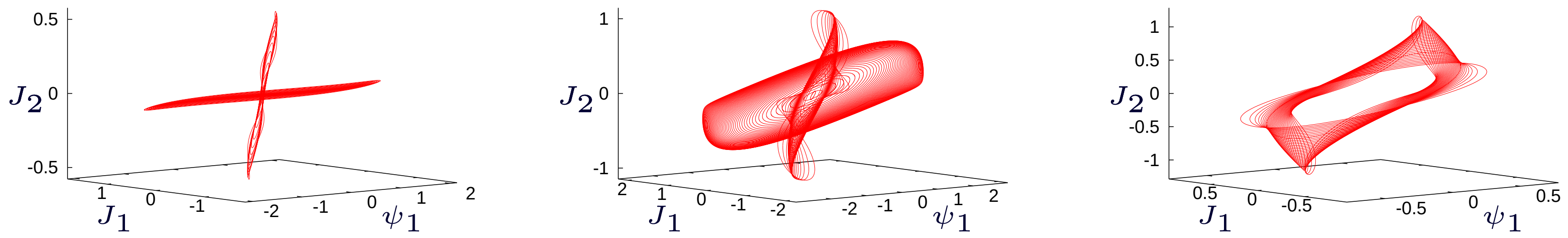
Conclusion: The numerical fit data show that the different slopes observed are related to the different values $k_*(h)$ obtained $\leadsto$ OK!!!

Final comments I

1. **Other aspects** related to the HH bifurcation for 4D maps have been also investigated (preprint).

   For example:

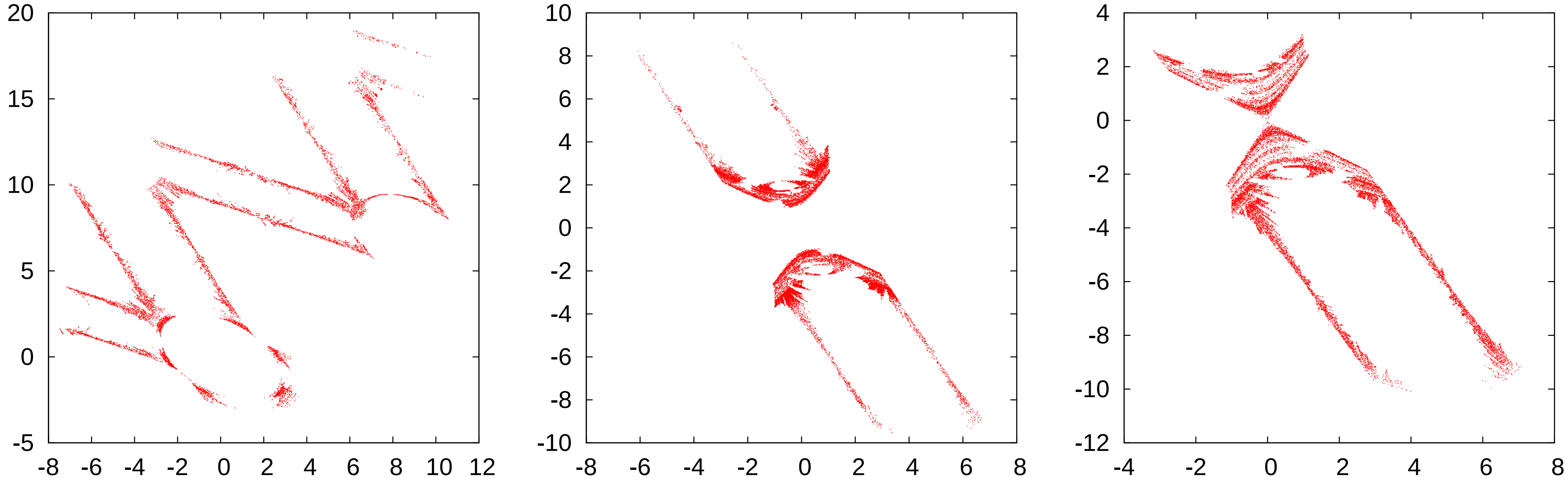
   (a) Structure of the Lyapunov families of invariant curves (analytic results on: the detachment of the Lyapunov families, analysis of the rational and irrational collision angle θ_0 cases, stability of the inv. curves, ...).



Detachment of the Lyapunov families of invariant curves for T:
$\epsilon = -0.1$, -0.4 and -0.5 ($\epsilon_c = -4/9$).

Final comments II

(b) Possible diffusive patterns **through and around** the double resonance.



Left: Positive definite case ($\delta = \epsilon = a_2 = 0.5$ and $a_3 = 1.25$).

Centre/Right: Non-definite case ($\delta = \epsilon = a_2 = 0.5$ and $a_3 = -0.75$).

...but this will be explained in future talks...

2. **Many open questions:** Theorem of splitting for a family of 4D maps? Separatrix return map? Diffusive properties (quantitative data)?

...but this is left for future works...

Thanks for your attention!!